Quadratic Stabilizability of Switched Linear Systems with Polytopic Uncertainties

Guisheng ZHAI[1*] Hai LIN[2] and Panos J. ANTSAKLIS[2]

[1]Faculty of Systems Engineering, Wakayama University
930 Sakaedani, Wakayama 640-8510, Japan
Email: zhai@sys.wakayama-u.ac.jp
Tel:+81(73)457-8187, Fax:+81(73)457-8201

[2]Department of Electrical Engineering, University of Notre Dame
Notre Dame, IN 46556, USA
Email: {hlin1, antsaklis.1}@nd.edu
Tel:+1(574)631-5792, Fax:+1(574)631-4393

Abstract: In this paper, we consider quadratic stabilizability via state feedback for both continuous-time and discrete-time switched linear systems that are composed of polytopic uncertain subsystems. By state feedback, we mean that the switchings among subsystems are dependent on system states. For continuous-time switched linear systems, we show that if there exists a common positive definite matrix for stability of all convex combinations of the extreme points which belong to different subsystem matrices, then the switched system is quadratically stabilizable via state feedback. For discrete-time switched linear systems, we derive a quadratic stabilizability condition expressed as matrix inequalities with respect to a family of nonnegative scalars and a common positive definite matrix. For both continuous-time and discrete-time switched systems, we propose the switching rules by using the obtained common positive definite matrix.

Keywords: Continuous-time switched system, discrete-time switched system, quadratic stabilizability, switching rule, state feedback, matrix inequality.

*Corresponding author.

1 Introduction

By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time or discrete-time subsystems and a rule orchestrating the switching between the subsystems. In the last two decades, there has been increasing interest in stability analysis and control design for switched systems (for example, Branicky, 1994 & 1998; Liberzon and Morse, 1999; DeCarlo *et al.*, 2000; Zhai, 2001 & 2003; Zhai *et al.*, 2001a & 2002a; Pettersson and Lennartson, 2002). The motivation for studying switched systems is from the fact that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors (Dayawansa and Martin, 1999; Pettersson and Lennartson, 2002), and that the methods of intelligent control design are based on the idea of switching between different controllers (Morse, 1996; Liberzon and Morse, 1999; Hu *et al.*, 2000 & 2002; Zhai *et al.*, 2002b). For recent progress and perspectives in the field of switched systems, see the survey papers Liberzon and Morse (1999), DeCarlo *et al.* (2000) and the references cited therein.

As also pointed out in Feron (1996), switched systems have been studied from various viewpoints. One viewpoint is that the switching signal is an exogenous variable, and then the problem is to investigate whether there exists a switching signal such that the switched system has desired performance (stability, certain disturbance attenuation level, etc.). Another viewpoint, which is of interest here, is that the switching signal is available to control engineers, and thus it may be used for control purposes. In the last decade, many practical methods have been proposed to use available switchings between various modes for control purposes. In particular, a theoretical framework based on Lyapunov stability theory has been established in the existing works (for example, Wicks *et al.*, 1994; Branicky, 1994 & 1998).

This paper also assumes that the switching signal can be designed by control engineers, and the performance index under consideration is quadratic stability of the switched systems. More precisely, we aim to investigate whether there exists a switching signal such that the switched system is quadratically stable. It is well known that quadratic stability requires for uncertain systems a quadratic Lyapunov function which guarantees asymptotical stability for all uncertainties under consideration, and is thus a kind of robust stability with very good property, yet usually needs more restrictive condition. However, for complex real systems with multiple control specifications, control engineers usually consider quadratic stabilization so that more design freedom can be gained. For detailed study on quadratic stability and stabilization, see for example Barmish (1985) and Khargonekar *et al.* (1990).

There are a few existing results concerning quadratic stabilization of switched linear systems that are composed of several unstable linear time-invariant subsystems. In Wicks *et al.* (1994), it has been shown that the existence of a stable convex combination of the subsystem matrices implies the existence of a state-dependent switching rule that stabilizes the switched system along with a quadratic Lyapunov function that proves it. It has been proved in Feron (1996) that when the number of subsystems is two, the existence of a stable convex combination of the subsystem matrices is necessary and sufficient for quadratic stabilizability of the switched system by state-dependent switching. An extension to output-dependent switching for quadratic stability has also been made with a robust detectability condition in Feron (1996). In Zhai (2001), the results in Wicks *et al.* (1994) and Feron (1996) have been extended to the case of discrete-time switched linear systems, by giving a quadratic stabilizability condition as a nonnegative combination of subsystems' Lyapunov inequalities. In that context, a significant difference between continuous-time systems and discrete-time ones has been pointed out. More precisely, we can easily find a stable convex combination condition of subsystem matrices as in Wicks *et al.* (1994) for quadratic stabilizability of continuous-time switched systems, but for discrete-time switched systems we can not derive such a combination condition without involving a Lyapunov matrix.

Motivated by the results in Wicks *et al.* (1994), Feron (1996) and Zhai (2001), we consider in this paper quadratic stabilizability of switched linear systems that are composed of several uncertain subsystems of polytopic type. It is well known that polytopic uncertainties exist in many real systems, and most uncertain control systems can be approximated by systems with polytopic uncertainties. Although there have been many existing results on switched systems and quadratic stability/stabilization, to the best of our knowledge, there is no existing result concerning quadratic stabilizability of switched systems with polytopic uncertainties.

Both continuous-time systems and discrete-time ones will be dealt with here. The reasons of considering discrete-time switched systems have been enumerated in the recent paper (Zhai *et al.*, 2002a). For example, a multimodal dynamical system may be composed of several discrete-time dynamical subsystems due to its physical structure, and even when all subsystems are of continuous-time, the case of considering sampled-data control for the entire system can be dealt with in the framework of discrete-time switched systems; see Hu and Michel (2000), Rubensson and Lennartson (2000) for detailed discussions. Furthermore, we find that the extension from continuous-time switched systems to discrete-time ones is not obvious in most cases, and the results may be quite different, as also pointed out in Zhai (2001) and will be seen later in this paper.

The contribution of the present paper is that, under the assumption that no subsystem is quadratically stable (otherwise, the switching problem will be trivial by always choosing the stable subsystem), we derive conditions under which the switched system is quadratically stabilizable by appropriate state-dependent switching (*state feedback*). In the case of continuous-time switched system, the condition comes up with a requirement of a common positive definite matrix for stability of all convex combinations of the extreme points which belong to different subsystem matrices. An example is used to demonstrate the result. In the case of discrete-time switched systems, we express the quadratic stabilizability condition as matrix inequalities with respect to a family of nonnegative scalar variables and a common positive definite matrix. However, in discrete-time switched system case we can not regard the condition as a requirement of a common positive definite matrix for stability of all convex combinations of the extreme points. Furthermore, the stabilizability condition for discrete-time switched systems has to deal with interference among the extreme points that belong to the same subsystem matrices. These two significant differences make quadratic stabilization of discrete-time switched systems much more difficult compared with continuous-time ones.

2 Quadratic Stabilizability for Continuous-Time Case

In this section, we consider the continuous-time switched linear system

$$\dot{x}(t) = A_{\sigma(x,t)}x(t)\,, \tag{2.1}$$

where $x(t) \in \Re^n$ is the state, $\sigma(x,t)$ is a switching rule defined by $\sigma(x,t) : \Re^n \times \Re^+ \to \{1,2\}$, and $\Re^+$ denotes nonnegative real numbers. Therefore, the switched system is composed of two continuous-time subsystems

$$\mathbf{CS_1}: \qquad \dot{x}(t) = A_1 x(t)\,, \tag{2.2}$$

and

$$\mathbf{CS_2}: \qquad \dot{x}(t) = A_2 x(t)\,. \tag{2.3}$$

Here, we assume that both $\mathbf{CS_1}$ and $\mathbf{CS_2}$ are uncertain systems of polytopic type described as

$$A_i = \sum_{j=1}^{N_i} \mu_{ij} A_{ij}\,, \qquad i = 1,2 \tag{2.4}$$

where $\mu_i = (\mu_{i1}, \mu_{i2}, \cdots, \mu_{iN_i})$ belongs to

$$\left\{ \mu_i : \sum_{j=1}^{N_i} \mu_{ij} = 1\,, \mu_{ij} \geq 0 \right\}\,, \tag{2.5}$$

and $A_{ij}, j = 1, 2, \cdots, N_i$ are constant matrices denoting the extreme points of the polytope A_i, and N_i is the number of the extreme points.

If $\mathbf{CS_1}$ or $\mathbf{CS_2}$ is quadratically stable, we can always activate the stable subsystem so that the entire switched system is quadratically stable. Therefore, to make the switching problem nontrivial, we make the following assumption.

Assumption 1: Both $\mathbf{CS_1}$ and $\mathbf{CS_2}$ are quadratically unstable, i.e., there does not exist $P_1 > 0$ such that

$$A_{1i}^T P_1 + P_1 A_{1i} < 0\,, \quad i = 1, 2, \cdots, N_1\,, \tag{2.6}$$

and there does not exists $P_2 > 0$ such that

$$A_{2j}^T P_2 + P_2 A_{2j} < 0\,, \quad j = 1, 2, \cdots, N_2\,. \tag{2.7}$$

This assumption is obviously true when one of A_{1i}'s and one of A_{2j}'s are unstable. Furthermore, even if all the matrices A_{1i}'s and A_{2j}'s are stable, there usually does not exist a common Lyapunov matrix for them. Narendra and Balakrishnan (1994) showed that when all A_{1i}'s (or A_{2j}'s) are stable and commutative pairwise, there exists a common Lyapunov matrix. Another simple case is that all A_{1i}'s (or A_{2j}'s) are stable and symmetric (Zhai, 2003). For most real control systems, this assumption is a reasonable one.

Now, we need the definition of quadratic stabilizability via state feedback for the switched system (2.1).

Definition 1: The system (2.1) is said to be *quadratically stabilizable via state feedback* if there exist a positive definite function $V(x) = x^T P x$, a positive number ϵ and a switching rule $\sigma(x, t)$ depending on x such that

$$\frac{d}{dt} V(x) < -\epsilon x^T x \tag{2.8}$$

holds for all trajectories of the system (2.1).

Then, our problem in this section is to find a state feedback (state-dependent switching rule) $\sigma(x, t)$ such that the switched system (2.1) is quadratically stable. We state and prove the following main result.

Theorem 1: *The switched system* (2.1) *is quadratically stabilizable via state feedback if there exist constant scalars λ_{ij}'s ($i = 1, 2; j = 1, 2, \cdots, N_i$) satisfying* $0 \le \lambda_{ij} \le 1$ *and* $P > 0$ *such that*

$$[\lambda_{ij} A_{1i} + (1 - \lambda_{ij}) A_{2j}]^T P + P\, [\lambda_{ij} A_{1i} + (1 - \lambda_{ij}) A_{2j}] < 0 \tag{2.9}$$

holds for all $i = 1, 2; j = 1, 2, \cdots, N_i$.

Proof: For the benefit of notation simplicity, we only give the proof in the case of $N_1 = N_2 = 2$. The extension from $N_1 = N_2 = 2$ to general case is very obvious.

From (2.9), we know that there always exists a positive scalar ϵ such that

$$[\lambda_{ij} A_{1i} + (1 - \lambda_{ij}) A_{2j}]^T P + P\, [\lambda_{ij} A_{1i} + (1 - \lambda_{ij}) A_{2j}] < -\epsilon I\,. \tag{2.10}$$

Then, for any $x \neq 0$, we obtain

$$\begin{aligned}
&x^T\left[\lambda_{11}A_{11}+(1-\lambda_{11})A_{21}\right]^T Px + x^T P\left[\lambda_{11}A_{11}+(1-\lambda_{11})A_{21}\right]x < -\epsilon x^T x\\
&x^T\left[\lambda_{12}A_{11}+(1-\lambda_{12})A_{22}\right]^T Px + x^T P\left[\lambda_{12}A_{11}+(1-\lambda_{12})A_{22}\right]x < -\epsilon x^T x\\
&x^T\left[\lambda_{21}A_{12}+(1-\lambda_{21})A_{21}\right]^T Px + x^T P\left[\lambda_{21}A_{12}+(1-\lambda_{21})A_{21}\right]x < -\epsilon x^T x\\
&x^T\left[\lambda_{22}A_{12}+(1-\lambda_{22})A_{22}\right]^T Px + x^T P\left[\lambda_{22}A_{12}+(1-\lambda_{22})A_{22}\right]x < -\epsilon x^T x\,,
\end{aligned} \tag{2.11}$$

which can be rewritten as

$$\lambda_{11}x^T\left(A_{11}^T P + PA_{11}\right)x + (1-\lambda_{11})x^T\left(A_{21}^T P + PA_{21}\right)x < -\epsilon x^T x \tag{2.12}$$

$$\lambda_{12}x^T\left(A_{11}^T P + PA_{11}\right)x + (1-\lambda_{12})x^T\left(A_{22}^T P + PA_{22}\right)x < -\epsilon x^T x \tag{2.13}$$

$$\lambda_{21}x^T\left(A_{12}^T P + PA_{12}\right)x + (1-\lambda_{21})x^T\left(A_{21}^T P + PA_{21}\right)x < -\epsilon x^T x \tag{2.14}$$

$$\lambda_{22}x^T\left(A_{12}^T P + PA_{12}\right)x + (1-\lambda_{22})x^T\left(A_{22}^T P + PA_{22}\right)x < -\epsilon x^T x\,. \tag{2.15}$$

It is very easy to verify from the above inequalities that either

$$x^T\left(A_{11}^T P + PA_{11}\right)x < -\epsilon x^T x\,, \quad x^T\left(A_{12}^T P + PA_{12}\right)x < -\epsilon x^T x \tag{2.16}$$

or

$$x^T\left(A_{21}^T P + PA_{21}\right)x < -\epsilon x^T x\,, \quad x^T\left(A_{22}^T P + PA_{22}\right)x < -\epsilon x^T x \tag{2.17}$$

is true. For example, if (2.16) is not true with $x^T\left(A_{11}^T P + PA_{11}\right)x \geq -\epsilon x^T x$, then we get $x^T\left(A_{21}^T P + PA_{21}\right)x < -\epsilon x^T x$ from (2.12) and $x^T\left(A_{22}^T P + PA_{22}\right)x < -\epsilon x^T x$ from (2.13). The same is true for other cases.

Now, we define the *switching rule* as

$$\sigma(x,t) \in \left\{ i \middle| x^T\left(A_{ij}^T P + PA_{ij}\right)x < -\epsilon x^T x\,, \quad j = 1,2 \right\}. \tag{2.18}$$

Then, based on the above discussion, we get

$$x^T\left(A_{\sigma 1}^T P + PA_{\sigma 1}\right)x < -\epsilon x^T x\,, \quad x^T\left(A_{\sigma 2}^T P + PA_{\sigma 2}\right)x < -\epsilon x^T x\,, \tag{2.19}$$

and thus

$$x^T\left(A_{\sigma}^T P + PA_{\sigma}\right)x < -\epsilon x^T x \tag{2.20}$$

since A_σ is a linear convex combination of $A_{\sigma 1}$ and $A_{\sigma 2}$. Clearly, (2.20) implies that the inequality

$$\frac{d}{dt}V(x) = x^T\left(A_{\sigma}^T P + PA_{\sigma}\right)x < -\epsilon x^T x \tag{2.21}$$

is true for all trajectories of (2.1) and thus the switched system is quadratically stable. ■

Remark 1: In fact, the condition (2.9) requires a common positive definite matrix for stability of all convex combinations of the extreme points which belong to different subsystem matrices. When there is no uncertainty in (2.1), i.e., $N_1 = N_2 = 1$, (2.9) shrinks to a single matrix inequality

$$[\lambda_{11}A_{11} + (1-\lambda_{11})A_{21}]^T P + P\,[\lambda_{11}A_{11} + (1-\lambda_{11})A_{21}] < 0\,, \tag{2.22}$$

which means that $\lambda_{11}A_{11} + (1-\lambda_{11})A_{21}$ is Hurwitz stable. In Feron (1996), it has been pointed out that in this case the switched system is quadratically stabilizable if and only if there exists a positive scalar $\lambda_{11} < 1$ such that $\lambda_{11}A_{11} + (1-\lambda_{11})A_{21}$ is Hurwitz stable. In this sense, Theorem 1 is the extension of Feron (1996).

Remark 2: Although we made Assumption 1 so as to make our switching problem nontrivial, Theorem 1 has covered the case where Assumption 1 is not true. That is, when $\mathbf{CS_1}$ is quadratically stable, we can choose $\lambda_{ij} = 1$ for all i, j in (2.9). On the contrary, when $\mathbf{CS_2}$ is quadratically stable, we can choose $\lambda_{ij} = 0$ for all i, j in (2.9).

Remark 3: Although we have assumed tacitly in the switched system (2.1) that the number of subsystems is two, Theorem 1 can be extended to the case of more than two subsystems in an obvious way. In that case, if the number of subsystems is M and the number of the extreme points of each A_i is N_i, then the condition for quadratic stabilizability is described as $\Pi_{j=1}^{M} N_j$ matrix inequalities with respect to a family of nonnegative scalars and a common positive definite matrix. With the increase of subsystem number or extreme point number, the computation complexity increases and the possibility of quadratic stabilizability decreases.

Remark 4: Obviously, the condition (2.9) consists of bilinear matrix inequalities (BMIs) with respect to λ_{ij}'s and $P > 0$. As is well known, it is not an easy task to solve BMI. Though in some special case we can convert a BMI into LMIs (Boyd *et al.*, 1994) which can be easily solved by many existing softwares (for example, LMI Toolbox (Gahinet *et al.*, 1995) in Matlab), there is not a globally effective method for general BMI presently. For numerical computation method of (2.9) in real control problems, we suggest trying the branch and bound methods proposed in Goh *et al.* (1994) or the homotopy-based algorithm in Zhai *et al.* (2001b). For example, the algorithm proposed in Zhai *et al.* (2001b) can be outlined as follows. We first find an initial solution of

$$[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}]^T P + P\,[\lambda_{ij}A_{1i} + (1-\lambda_{ij})A_{2j}] < \beta I \tag{2.23}$$

for some positive scalar $\beta = \beta_0$, which is always possible if we fix certain $P > 0$, λ_{ij}'s and use β_0 large enough. Then, we deform the initial solution and β gradually until β reaches a nonpositive scalar. At each step of the deformation, we solve the above matrix inequality by fixing P or λ_{ij}'s .

In the end of this section, we give an example to demonstrate our result.

Example: Consider the switched linear system (2.1) composed of two subsystems where

$$A_{11} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.24}$$

and

$$A_{21} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}. \tag{2.25}$$

Since both A_{11} and A_{12} are unstable, there is not $P_1 > 0$ satisfying (2.6). Similarly, since both A_{21} and A_{22} are unstable, there is not $P_2 > 0$ satisfying (2.7). Therefore, both $\mathbf{CS_1}$ and $\mathbf{CS_2}$ are quadratically unstable.

It is easy to observe that

$$\begin{aligned} \tfrac{1}{2}A_{11} + \tfrac{1}{2}A_{21} = -I_2\,, &\qquad \tfrac{1}{2}A_{12} + \tfrac{1}{2}A_{22} = -I_2 \\ \tfrac{1}{3}A_{11} + \tfrac{2}{3}A_{22} = -I_2\,, &\qquad \tfrac{2}{3}A_{12} + \tfrac{1}{3}A_{21} = -I_2\,, \end{aligned} \tag{2.26}$$

which implies that $\lambda_{11} = \lambda_{22} = \frac{1}{2}$, $\lambda_{12} = \frac{1}{3}$, $\lambda_{21} = \frac{2}{3}$ together with a positive definite matrix P satisfy the condition (2.9) in Theorem 1. Therefore, the switched linear system with (2.24) and (2.25) is quadratically stabilizable via state feedback.

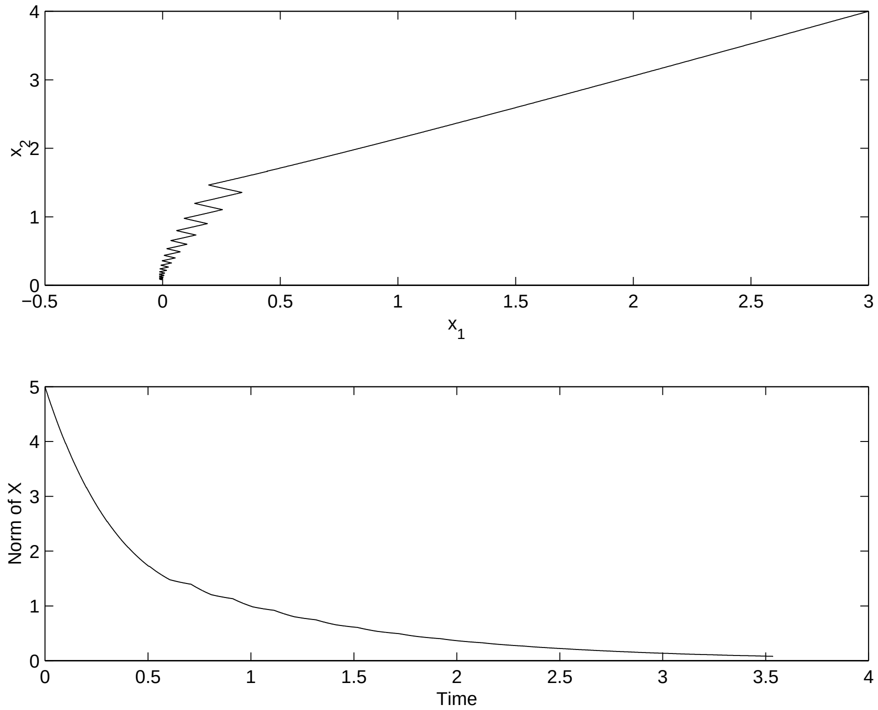

Figure 1. The state and the state's norm of the switched system in the example

Let us investigate the system state trajectory using the two specific subsystems

$$\begin{aligned} A_1 = 0.2A_{11} + 0.8A_{12} &= \begin{bmatrix} -1.0 & 1.2 \\ 1.2 & -1.0 \end{bmatrix}, \\ A_2 = 0.4A_{21} + 0.6A_{22} &= \begin{bmatrix} -1.0 & -1.4 \\ -1.4 & -1.0 \end{bmatrix}, \end{aligned} \tag{2.27}$$

which belong to $\mathbf{CS_1}$ and $\mathbf{CS_2}$, respectively, and are both unstable.

We suppose that the initial state is $x_0 = [3 \;\; 4]^T$, and set $P = I_2$. Then, since

$$\begin{aligned} x_0^T(A_{11}^T P + PA_{11})x_0 &= 46, & x_0^T(A_{12}^T P + PA_{12})x_0 &= -2, \\ x_0^T(A_{21}^T P + PA_{21})x_0 &= -146, & x_0^T(A_{22}^T P + PA_{22})x_0 &= -98, \end{aligned} \tag{2.28}$$

we choose $\sigma(x_0, 0) = 2$ according to the switching rule (2.18).

In the same way, we choose the subsystem mode for every time instant using the switching rule (2.18) and evolve the system forth on. Although both A_1 and A_2 are unstable here, Figure 1 shows that both the system state and the norm of the state converge to zero very quickly under the switching rule we proposed.

3 Quadratic Stabilizability for Discrete-Time Case

In this section, we consider the discrete-time switched linear system

$$x[k+1] = A_{\sigma(x,k)}x[k], \tag{3.1}$$

where $x[k] \in \Re^n$ is the state, $\sigma(x,k)$ is a switching rule defined by $\sigma(x,k) : \Re^n \times \mathcal{N}^+ \to \{1, 2\}$, and $\mathcal{N}^+$ denotes nonnegative integers. Therefore, the switched system is composed of two discrete-time subsystems

$$\mathbf{DS_1}: \quad x[k+1] = A_1 x[k], \tag{3.2}$$

and

$$\mathbf{DS_2}: \quad x[k+1] = A_2 x[k]. \tag{3.3}$$

As in Section 2, we assume here that both $\mathbf{DS_1}$ and $\mathbf{DS_2}$ are uncertain systems of polytopic type described as (2.4) and (2.5), and make the following assumption.

Assumption 2: Both $\mathbf{DS_1}$ and $\mathbf{DS_2}$ are quadratically unstable, i.e., there does not exist $P_1 > 0$ such that

$$A_{1i}^T P_1 A_{1i} - P_1 < 0, \quad i = 1, 2, \cdots, N_1, \tag{3.4}$$

and there does not exist $P_2 > 0$ such that

$$A_{2j}^T P_2 A_{2j} - P_2 < 0, \quad j = 1, 2, \cdots, N_2. \tag{3.5}$$

Definition 2: The system (3.1) is said to be *quadratically stabilizable via state feedback* if there exist a positive definite function $V(x[k]) = x^T[k]Px[k]$, a positive number ϵ and a switching rule $\sigma(x, k)$ depending on x such that

$$V(x[k+1]) - V(x[k]) < -\epsilon x^T[k]x[k] \tag{3.6}$$

for all trajectories of the system (3.1).

Then, the problem in this section is to find a state feedback (state-dependent switching rule) $\sigma(x, k)$ such that the switched system (3.1) is quadratically stable.

Theorem 2: *The switched system* (3.1) *is quadratically stabilizable via state feedback if there exist constant scalars* λ_{ij}*'s* ($i = 1, 2; j = 1, 2, \cdots, N_i$) *satisfying* $0 \leq \lambda_{ij} \leq 1$ *and* $P > 0$ *such that*

$$\lambda_{ij}(A_{1i}^T PA_{1i} - P) + (1 - \lambda_{ij})(A_{2j}^T PA_{2j} - P) < 0\,, \quad i = 1, 2; \quad j = 1, 2, \cdots, N_i \tag{3.7}$$

$$A_{1k}^T PA_{1l} + A_{1l}^T PA_{1k} - 2P \leq 0\,, \qquad k, l = 1, 2, \cdots, N_1, \quad k \neq l \tag{3.8}$$

$$A_{2u}^T PA_{2v} + A_{2v}^T PA_{2u} - 2P \leq 0\,, \qquad u, v = 1, 2, \cdots, N_2, \quad u \neq v\,. \tag{3.9}$$

Proof: As in the proof of Theorem 1, we only give the proof in the case of $N_1 = N_2 = 2$.

From (3.7), we know that there always exists a positive scalar ϵ such that

$$\lambda_{ij}(A_{1i}^T PA_{1i} - P) + (1 - \lambda_{ij})(A_{2j}^T PA_{2j} - P) < -\epsilon I\,. \tag{3.10}$$

Then, for any $x[k] \neq 0$, we obtain from (3.10) that

$$\lambda_{ij}x^T[k]\left(A_{1i}^T PA_{1i} - P\right)x[k] + (1 - \lambda_{ij})x^T[k]\left(A_{2j}^T PA_{2j} - P\right)x[k] < -\epsilon x^T[k]x[k]\,. \tag{3.11}$$

It is very easy to verify that either

$$\begin{cases} x^T[k]\left(A_{11}^T PA_{11} - P\right)x[k] < -\epsilon x^T[k]x[k] \\ x^T[k]\left(A_{12}^T PA_{12} - P\right)x[k] < -\epsilon x^T[k]x[k] \end{cases} \tag{3.12}$$

or

$$\begin{cases} x^T[k]\left(A_{21}^T PA_{21} - P\right)x[k] < -\epsilon x^T[k]x[k] \\ x^T[k]\left(A_{22}^T PA_{22} - P\right)x[k] < -\epsilon x^T[k]x[k] \end{cases} \tag{3.13}$$

is true. From (3.8) and (3.9), we obtain that

$$x^T[k](A_{11}^T PA_{12} - P)x[k] \leq 0\,, \quad x^T[k](A_{21}^T PA_{22} - P)x[k] \leq 0\,. \tag{3.14}$$

Now, we define the *switching rule* as

$$\sigma(x,k) \in \left\{ i \middle| x^T[k] \left(A_{ij}^T P A_{ij} - P \right) x[k] < -\epsilon x^T[k] x[k] \,, \quad j = 1,2 \right\} . \tag{3.15}$$

Thus, we obtain from (3.12)-(3.15) that

$$x^T[k] \left(A_\sigma^T P A_\sigma - P \right) x[k] < -\epsilon x^T[k] x[k] \,. \tag{3.16}$$

This implies that the inequality

$$V(x[k+1]) - V(x[k]) = x^T[k] \left(A_\sigma^T P A_\sigma - P \right) x[k] < -\epsilon x^T[k] x[k] \tag{3.17}$$

is true for all trajectories of (3.1) and thus the switched system is quadratically stable. ■

Remark 5: Although we can regard (3.7) in Theorem 2 as a parallel condition of (2.9) in Theorem 1, there is essential difference between them. (2.9) represents a stable convex combination of A_{1i} and A_{2j}, but (3.7) does not. From this observation we know that the extension from continuous-time switched systems to discrete-time ones is far from trivial.

Remark 6: From Theorem 2, we see that discrete-time switched systems are much more difficult to deal with than continuous-time ones. The reason is that the Lyapunov matrix inequality $A^T PA - P < 0$ for the discrete-time system $x[k+1] = Ax[k]$ is not linear with respect to A. When there is no switching, we can convert $A^T PA - P < 0$ into $\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} > 0$, which is linear with respect to both A and P. By this change, we can deal with polytopic perturbations in A. However, when there is switching, we can not make such manipulation for $x^T(A^T PA - P)x < 0$ $(x \neq 0)$. For this reason, we have to take care of $x^T(A_{1k}^T PA_{1l} - P)x$ $(k \neq l)$ and $x^T(A_{2u}^T PA_{2v} - P)x$ $(u \neq v)$ for any $x \neq 0$. It is known from (3.14) that (3.8) and (3.9) are proposed for this purpose, i.e., to deal with the interference among the extreme points that belong to the same subsystem matrices.

4 Concluding Remarks

In this paper, we have considered quadratic stabilizability via state feedback for both continuous-time and discrete-time switched linear systems that are composed of polytopic uncertain subsystems. For continuous-time switched linear systems, we have shown that if there exists a common positive definite matrix for stability of all convex combinations of the extreme points which belong to different subsystem matrices, then the switched system is quadratically stabilizable via state feedback. For discrete-time switched linear systems, we have derived a quadratic stabilizability condition expressed as matrix inequalities with respect to a family of nonnegative scalars and a common positive definite matrix. For both continuous-time and discrete-time switched systems, we have given the switching rules by using the common positive definite matrix.

There are several important issues which should be studied in future work. First, to solve the proposed matrix inequalities (2.9) for continuous-time switched linear systems, one may first consider how to find the stable convex combinations of the extreme points, as we did in the example. This problem is still open for general case, but some geometric method dealing with interval matrices of constant matrices may be effective. For discrete-time switched linear systems, the stabilizability conditions (3.8) and (3.9) are quite conservative, and thus need to be relaxed. An extension to the present paper is to consider quadratic stabilizability via output feedback (output-dependent switching) for uncertain switched systems. To the best of our knowledge, there are very few results concerning output-dependent switching, though it is a practical problem in real control systems. For this issue, some of the ideas in Feron (1996) and Zhai (2001) should be useful.

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