

# Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results

Hai Lin, *Member, IEEE*, and Panos J. Antsaklis, *Fellow, IEEE*,

**Abstract**—During the past several years, there have been increasing research activities in the field of stability analysis and switching stabilization for switched systems. This paper aims to briefly survey recent results in this field. First, the stability analysis for switched systems is reviewed. We focus on the stability analysis for switched linear systems under arbitrary switching, and we highlight necessary and sufficient conditions for asymptotic stability. After a brief review of the stability analysis under restricted switching and the multiple Lyapunov function theory, the switching stabilization problem is studied, and a variety of switching stabilization methods found in the literature are outlined. Then the switching stabilizability problem is investigated, that is under what condition it is possible to stabilize a switched system by properly designing switching control laws. Note that the switching stabilizability problem has been one of the most elusive problems in the switched systems literature. A necessary and sufficient condition for asymptotic stabilizability of switched linear systems is described here.

**Keywords:** Switched systems, Stability, Stabilization, Lyapunov function.

## I. INTRODUCTION

A Switched system is a dynamical system that consists of a finite number of subsystems and a logical rule that orchestrates switching between these subsystems. Mathematically, these subsystems are usually described by a collection of indexed differential or difference equations. One convenient way to classify switched systems is based on the dynamics of their subsystems, for example continuous-time or discrete-time, linear or nonlinear and so on.

A continuous-time switched nonlinear system can be modeled as

$$\dot{x}(t) = f_i(x(t), u(t)), \quad t \in \mathbb{R}^+, \quad i \in \mathcal{I} = \{1, \dots, N\}$$

where the state  $x \in \mathbb{R}^n$ , the control  $u \in \mathbb{R}^m$ ,  $\mathbb{R}^+$  denotes non-negative real numbers, the finite set  $\mathcal{I}$  is an index set and stands for the collection of discrete modes. Similarly, we can represent a discrete-time switched system as a collection of difference equations

$$x[k+1] = f_i(x[k], u[k]), \quad k \in \mathbb{Z}^+, \quad i \in \mathcal{I}$$

where  $\mathbb{Z}^+$  stands for non-negative integers.

The logical rule that orchestrates switching between these subsystems generates switching signals, which are usually described as classes of piecewise constant maps,  $\sigma : \mathbb{R}^+ \rightarrow \mathcal{I}$  (or sequences  $\sigma : \mathbb{Z}^+ \rightarrow \mathcal{I}$ ). The logical rules that generates the switching signals constitute the switching logic, and the

index  $i = \sigma(t)$  is called the active mode at the time instant  $t$ . In general, the active mode at  $t$  may depend not only on the time instant  $t$ , but also on the current state  $x(t)$  and/or previous active mode  $\sigma(\tau)$  for  $\tau < t$ . Accordingly, the switching logic is usually classified as time-controlled (depends on time  $t$  only), state-dependent (depends on state  $x(t)$  as well), and with memory (also depends on the history of active modes).

By requesting a switching signal be piecewise constant, we mean that the switching signal  $\sigma(t)$  has finite number of discontinuities on any finite interval of  $\mathbb{R}^+$ . This actually corresponds to no-chattering requirement for the continuous-time switched systems; note that this is not an issue in the discrete-time case. This assumption makes sense when we consider the stability analysis problem, for which the sliding-like motion can be easily identified before hand (by checking the direction of the vector fields along the switching surfaces) and may be incorporated by defining its equivalent dynamics [27] as an additional mode [20]. However, when considering stabilization issues, one may need to deal with sliding motions explicitly, which may arise either on purpose or unintentionally. We will revisit this sliding motion issue later in the discussion of switching stabilization problems.

Properties of this type of model have been studied for the past fifty years when considering engineering systems that contain relays and/or hysteresis. The primary motivation for studying such switched systems comes partly from the fact that switched systems and switched multi-controller systems have numerous applications in the control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields. In addition, there exists a large class of nonlinear systems which can be stabilized by switching control schemes, but cannot be stabilized by any continuous static state feedback control law [11]. Switched systems with all subsystems described by linear differential or difference equations are called *switched linear systems*, and have attracted most of the attention [2], [4], [5], [32], [41]. Recent efforts in switched linear system research typically focus on the analysis of dynamic behaviors, such as stability [20], [32], [41], [45], [47], controllability, reachability [39], [40], [83], [84] and observability [4], [22], [34] etc., and aim to design controllers with guaranteed stability and performance [5], [13], [41], [65], [82], [88].

In this paper, we will focus on stability issues for autonomous switched linear systems, i.e., without continuous-variable control input  $u$ . In particular, we are interested in switched linear systems, the subsystems of which are continuous-time linear time-invariant (LTI) systems

$$\dot{x}(t) = A_i x(t), \quad t \in \mathbb{R}^+, \quad i \in \mathcal{I} \quad (1)$$

H. Lin is with the ECE Dept., National University of Singapore, Singapore 117576. E-mail: e1elh@nus.edu.sg (corresponding author)

P.J. Antsaklis is with the the EE. Dept., University of Notre Dame, Notre Dame, IN 46556, USA. E-mail: antsaklis.1@nd.edu

or a collection of discrete-time LTI systems

$$x[k+1] = A_i x[k], \quad k \in \mathbb{Z}^+, \quad i \in \mathcal{I} \quad (2)$$

where the state  $x \in \mathbb{R}^n$  and  $A_i \in \mathbb{R}^{n \times n}$  for all  $i \in \mathcal{I}$ .

Note that the origin  $x_e = 0$  is an equilibrium (maybe unstable) for the systems described in (1) and (2). Our main concern here is to understand the conditions that can guarantee the stability of the switched linear system.

The stability issues of such switched systems include several interesting phenomena. For example, even when all the subsystems are exponentially stable, the switched systems may have divergent trajectories for certain switching signals [20], [46]. Another remarkable fact is that one may carefully switch between unstable subsystems to make the switched system exponentially stable [20], [46]. As these examples suggest, the stability of switched systems depends not only on the dynamics of each subsystem but also on the properties of switching signals. Therefore, the stability study of switched systems can be roughly divided into two kinds of problems. One is the stability analysis of switched systems under given switching signals (maybe arbitrary, slow switching etc.); the other is the synthesis of stabilizing switching signals for a given collection of dynamical systems.

In the current paper, we will briefly overview some recent results on the stability and stabilizability of switched systems from these two aspects. First, stability analysis results for switched systems are reviewed. In particular, we focus on the stability analysis for switched linear systems under arbitrary switching in Section II, and we highlight necessary and sufficient conditions for asymptotic stability. Since there exist excellent reviews on the stability under restricted switching (like dwell time and average dwell time [32], [46]), multiple Lyapunov functions [20], [59] and piecewise quadratic Lyapunov functions [20], we will review these topics very briefly in Section III. In Section IV, the switching stabilization problem is studied, where a recent necessary and sufficient condition for the switching stabilizability of a switched linear system is highlighted.

The stability issues of switched systems, especially switched linear systems, have drawn a lot of attentions in the recent decade. There have been several excellent survey papers on the stability of switched systems; see for example the survey papers [20], [32], [47], [59], the recent books [41], [45] and the references cited therein. Since their publications, however, this field has seen a large amount of activities and new results, and this paper aims to briefly report and survey these recent results and new discoveries in this field. The authors hope that the current paper provides useful additional results and represents a meaningful complementary resource to previous survey papers [20], [32], [47], [59].

## II. STABILITY ANALYSIS UNDER ARBITRARY SWITCHING

For the stability analysis problem, the first question is whether the switched system is stable when there is no restriction on the switching signals. This problem is usually called stability analysis under arbitrary switching. For this problem, it is necessary to require that all the subsystems are

asymptotically stable. However, even when all the subsystems of a switched system are exponentially stable, it is still possible to construct a divergent trajectory from any initial state for such a switched system. Therefore, in general, the above subsystems' stability assumption is not sufficient to assure stability for the switched systems under arbitrary switching, except for some special cases, such as  $A_i$  being pairwise commutative ( $A_i A_j = A_j A_i$  for all  $i, j \in \mathcal{I}$ ) [63], [94],  $A_i$  symmetric ( $A_i = A_i^T$  for all  $i$ ) [95], or  $A_i$  normal ( $A_i A_i^T = A_i^T A_i$  for all  $i$ ) [97]. On the other hand, if there exists a common Lyapunov function for all the subsystems, then the stability of the switched system is guaranteed under arbitrary switching. This provides us with a possible way to solve this problem, and a lot of efforts have been focused on the common quadratic Lyapunov functions.

### A. Common Quadratic Lyapunov Functions

The existence of a common quadratic Lyapunov function (CQLF) for all its subsystems assures the quadratic stability of the switched system. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and has attracted a lot of research efforts due to its importance in practice. It is known that the conditions for the existence of a CQLF can be expressed as linear matrix inequalities (LMIs) [9]. Namely, there exists a positive definite symmetric matrix  $P$ ,  $P \in \mathbb{R}^{n \times n}$ , such that

$$P A_i + A_i^T P < 0, \quad \forall i \in \mathcal{I}, \quad (3)$$

for the continuous-time case, or

$$A_i^T P A_i - P < 0, \quad \forall i \in \mathcal{I}, \quad (4)$$

for the discrete-time case, hold simultaneously. However, the standard interior point methods for LMIs may become ineffective as the number of modes increases. In [48], an interactive gradient decent algorithm was proposed, which could converge to the CQLF in finite number of steps. In addition, the authors showed that the convergence rate could be improved by introducing some randomness; here the convergence is in the sense of probability one.

While numerical methods to solve these LMIs for a finite number of stable LTI systems have existed for some time, determining algebraic conditions (on the subsystems' state matrices) for the existence of a CQLF remains a challenging task. Since these kind of conditions should be easier to verify, and, more importantly, may give us valuable insights in the stability problem of an arbitrary switching system, there have been various attempts to derive algebraic conditions for the existence of a CQLF [45], [46](Chapter 2).

In [77], [79], Shorten and Narendra considered a second-order switched LTI systems with two modes; they proposed a necessary and sufficient condition for the existence of a common quadratic Lyapunov function. The results in [77], [79] were based on the stability of the matrix pencil formed by the pair of subsystems' state matrices. Given two matrices  $A_1$  and  $A_2$ , the matrix pencil  $\gamma_\alpha(A_1, A_2)$  is defined as the one-parameter family of matrices  $\gamma_\alpha(A_1, A_2) = \alpha A_1 + (1 - \alpha) A_2$ ,  $\alpha \in [0, 1]$ . The matrix pencil  $\gamma_\alpha(A_1, A_2)$  is said to be Hurwitz

if its eigenvalues are in the open left half plane for all  $0 \leq \alpha \leq 1$ . Formally, the results for the pair of second order LTI systems in [77]–[79] can be summarized by the following theorem.

*Theorem 1:* [77]–[79] Let  $A_1, A_2$  be two Hurwitz matrices in  $\mathbb{R}^{2 \times 2}$ . The following conditions are equivalent

- 1) there exist a CQLF for (1) with  $A_1, A_2$  as the two subsystems;
- 2) the matrix pencils  $\gamma_\alpha(A_1, A_2)$  and  $\gamma_\alpha(A_1, A_2^{-1})$  are Hurwitz;
- 3) the matrices  $A_1 A_2$  and  $A_1 A_2^{-1}$  do not have any negative real eigenvalues.

To generalize the above algebraic condition to higher dimensional systems turns out to be difficult. In [42], necessary and sufficient algebraic conditions were derived for the non-existence of a CQLF for an arbitrary switching systems composed of a pair of third-order LTI systems. For a pair of  $n$ -th order LTI systems, a necessary condition for the existence of a CQLF was derived in [76], [79].

*Theorem 2:* [76], [79] Let  $A_1, A_2$  be two Hurwitz matrices in  $\mathbb{R}^{n \times n}$ . A necessary condition for the existence of a CQLF is that the matrix products  $A_1[\alpha A_1 + (1 - \alpha)A_2]$  and  $A_1[\alpha A_1 + (1 - \alpha)A_2]^{-1}$  do not have any negative real eigenvalues for all  $0 \leq \alpha \leq 1$ .

As a special case, consider a switched LTI system consisting of two matrices differing by rank one, and the following necessary and sufficient condition for the existence of a CQLF was obtained in [76].

*Theorem 3:* [76] Let  $A_1, A_2$  be two Hurwitz matrices in  $\mathbb{R}^{n \times n}$  with  $\text{rank}(A_2 - A_1) = 1$ . A necessary and sufficient condition for the existence of a CQLF for the switched system (1) with  $A_1, A_2$  as the two subsystems is that the matrix product  $A_1 A_2$  does not have any negative real eigenvalues. Equivalently, the matrix  $A_1 + \gamma A_2$  is non-singular for all  $\gamma \in [0, +\infty)$ .

So far, our discussion on the existence of a CQLF has been restricted to switched LTI systems consisting of only two modes. However, in general, a switched system may contain more than two subsystems. Obviously, a necessary condition for the existence of a CQLF for a switched systems is that every pair of its subsystems share a CQLF. Actually, the existence of a CQLF for every pair of subsystems may also imply the existence of a CQLF for the switched system in certain special cases, e.g., second order positive systems [30]. Unfortunately, this does not hold in general. The existence of a CQLF for a finite number of second order LTI systems was investigated in [78], and it is interesting to observe that a necessary and sufficient condition for the existence of a CQLF is that a CQLF exists for every 3-tuple of systems. Formally,

*Theorem 4:* Let  $A_1, A_2, \dots, A_N$  be a finite number of Hurwitz matrices in  $\mathbb{R}^{2 \times 2}$  with  $a_{21i} \neq 0$  for all  $i$ . A necessary and sufficient condition for the existence of a CQLF is that a CQLF exists for every 3-tuple of systems  $\{A_i, A_j, A_k\}$ ,  $i \neq j \neq k$ , for all  $i, j, k \in \{1, \dots, N\}$ .

Recently, in [43], A tensor condition was introduced as a necessary condition for the existence of a CQLF for the general case, i.e., a switched system consisting of a finite number of  $n$ -th order LTI systems. Interestingly, the tensor

condition was shown to be necessary and sufficient when the switched system only contains a pair of subsystems. However, for general cases of higher order and more than two modes, necessary and sufficient conditions for the existence of a CQLF for a switched LTI systems are still lacking and this remains an open problem.

Alternatively, Liberzon, Hespanha and Morse proposed a Lie algebraic condition in [46] for switched LTI systems, which is based on the solvability of the Lie algebra generated by the subsystems' state matrices. It was shown that if the Lie algebra generated by the set of state matrices  $A_i$  is solvable, then there exists a CQLF, and the switched linear system is stable under arbitrary switching. The Lie algebraic condition was also extended to switched nonlinear systems [1], [46] to obtain local stability results based on Lyapunov's first method. Most recently, global stability properties for switched nonlinear systems were studied in [58], and a Lie algebraic global stability criterion was derived based on Lie brackets of the nonlinear vector fields. Interested readers may refer to [45], [46], [56] for further details on the Lie algebraic conditions.

## B. Switched Quadratic Lyapunov Functions

It is worth pointing out that the existence of a CQLF is only sufficient for the stability of arbitrary switching systems. There are examples [45] (Chapter 2) of systems that do not have a CQLF, but are exponentially stable under arbitrary switching. In general, the existence of a CQLF is only sufficient for the asymptotic stability of switched linear systems under arbitrary switching signals, and could be rather conservative.

Due to the conservatism of CQLFs, some attentions have been paid to a less conservative class of Lyapunov functions, called switched quadratic Lyapunov functions [17], [23]. Basically, since every subsystem is stable, there exists a positive definite symmetric matrix  $P_i$  that solves the Lyapunov equation for each  $i$ -th subsystem ( $i \in \mathcal{I}$ ). Next, these matrices  $P_i$  are patched together based on the switching signals  $\sigma(t)$  to construct a global Lyapunov function as

$$V(t, x(t)) = x^T(t) P_{\sigma(t)} x(t). \quad (5)$$

Then, the stability checking for the arbitrary switched linear systems can be performed by solving certain linear matrix inequalities (LMIs) [9]. To illustrate the main idea, we outline a result from [23] as follows.

*Theorem 5:* If there exist positive definite symmetric matrices  $P_i \in \mathbb{R}^{n \times n}$  ( $P_i = P_i^T$ ) and matrices  $F_i, G_i \in \mathbb{R}^{n \times n}$  ( $i \in \mathcal{I}$ ), satisfying

$$\begin{bmatrix} A_i F_i^T + F_i A_i^T - P_i & A_i G_i - F_i \\ G_i^T A_i^T - F_i^T & P_j - G_i - G_i^T \end{bmatrix} < 0 \quad (6)$$

for all  $i, j \in \mathcal{I}$ , then the switched linear system (2) is asymptotically stable.  $\square$

With some pre-selections for the auxiliary matrices  $F_i$  and  $G_i$ , the LMI (6) in Theorem 5 can be replaced either by

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0, \quad (7)$$

or [17] by

$$\begin{bmatrix} -P_i & A_i G_i \\ G_i^T A_i^T & P_j - G_i - G_i^T \end{bmatrix} < 0. \quad (8)$$

It is clear that when  $P_i = P_j$  for all  $i, j \in \mathcal{I}$ , the switched quadratic Lyapunov function becomes the CQLF. Therefore, the stability criteria based on the switched quadratic Lyapunov function generalizes the CQLF approach and usually gives us less conservative results. However, it is worth pointing out that the switched quadratic Lyapunov function method is still a sufficient only condition.

### C. Necessary and Sufficient Stability Conditions

In the sequel, we will provide some necessary and sufficient conditions for the asymptotic stability of switched linear systems under arbitrary switching signals [49]. This is a relatively new result, which provides a solution for this long standing problem. It shows that the asymptotic stability problem for switched linear systems with arbitrary switching is equivalent to the robust asymptotic stability problem for polytopic uncertain linear time-variant systems, for which several strong stability conditions exist.

Let us first recall a robust stability result for linear time-variant systems with polytopic uncertainty

$$x[k+1] = A(k)x[k] \quad (9)$$

where  $A(k) \in \mathcal{A} \triangleq \text{Conv}\{A_1, A_2, \dots, A_N\}$ . Here,  $\text{Conv}\{\cdot\}$  stands for convex combination. In other words, the state matrix  $A(k)$  of the above linear time-variant system (9) is constructed by a convex combinations (with time-variant coefficients) of all the subsystems' state matrices of the switched linear system (2).

*Lemma 1:* [3] The linear time-variant system (9) is robustly<sup>1</sup> asymptotically stable if and only if there exists a finite integer  $n$  such that

$$\|A_{i_1} A_{i_2} \cdots A_{i_n}\| < 1,$$

for all  $n$ -tuple  $A_{i_j} \in \{A_1, A_2, \dots, A_N\}$ , where  $j = 1, \dots, n$ .

Here the norm  $\|\cdot\|$  stands for the  $\infty$  norm of a matrix [3]. Based on the above lemma, a necessary and sufficient condition for the asymptotic stability of switched linear systems (2) can be expressed by the following theorem [50].

*Theorem 6:* A switched linear system  $x[k+1] = A_{\sigma(k)}x[k]$ , where  $A_{\sigma(k)} \in \{A_1, A_2, \dots, A_N\}$ , is asymptotically stable under arbitrary switching if and only if there exists a finite integer  $n$  such that

$$\|A_{i_1} A_{i_2} \cdots A_{i_n}\| < 1,$$

for all  $n$ -tuple  $A_{i_j} \in \{A_1, A_2, \dots, A_N\}$ , where  $j = 1, \dots, n$ .

The sufficiency of the above condition is implied by Lemma 1, and the necessity can be shown by contradiction [50]. Notice that this condition coincides with the necessary and sufficient condition for the robust asymptotic stability for polytopic uncertain linear time-variant systems (9). Therefore, we derive the following equivalence relationship between these two problems.

<sup>1</sup>Here the robustness is with respect to the parametric uncertainties.

*Proposition 1:* The following statements are equivalent:

- 1) The switched linear system  $x[k+1] = A_{\sigma(k)}x[k]$  where  $A_{\sigma(k)} \in \{A_1, A_2, \dots, A_N\}$ , is asymptotically stable under arbitrary switching;
- 2) the linear time-variant system  $x[k+1] = A(k)x[k]$ , where  $A(k) \in \mathcal{A} \triangleq \text{Conv}\{A_1, A_2, \dots, A_N\}$ , is robustly asymptotically stable;
- 3) there exists a finite integer  $n$  such that

$$\|A_{i_1} A_{i_2} \cdots A_{i_n}\| < 1,$$

for all  $n$ -tuple  $A_{i_j} \in \{A_1, A_2, \dots, A_N\}$ , where  $j = 1, \dots, n$ .  $\square$

It is quite interesting that the study of robust stability of a polytopic uncertain linear time-variant system, which has infinite number of possible dynamics (modes), is equivalent to considering only a finite number of its vertex dynamics in an arbitrary switching system. Note that this is not a surprising result since this fact has already been implied by the finite vertex stability criteria for robust stability in the literature, e.g., [6], [61]. By explicitly exploring this equivalence relationship, we may obtain some "new" stability criteria for switched linear systems using the existing robust stability results [6], [61]. For example,

*Theorem 7:* The switched linear system  $x[k+1] = A_{\sigma(k)}x[k]$  where  $A_{\sigma(k)} \in \{A_1, A_2, \dots, A_N\}$ , is asymptotically stable under arbitrary switching if and only if there exists an integer  $m \geq n$  and  $L \in \mathbb{R}^{n \times m}$ ,  $\text{rank}(L) = n$  such that for all  $A_i, i \in \mathcal{I}$ , there exists  $\bar{A}_i \in \mathbb{R}^{m \times m}$  with the following properties:

- 1)  $A_i^T L = L \bar{A}_i^T$ ,
- 2) each column of  $\bar{A}_i$  has no more than  $n$  nonzero elements and

$$\|\bar{A}_i\|_\infty = \max_{1 \leq k \leq m} \sum_{l=1}^m |\bar{a}_{kl}| < 1.$$

$\square$

Following similar arguments, the above equivalence also holds for the continuous-time case. In particular, we may derive a necessary and sufficient algebraic condition for arbitrary switching linear system based on results from [61] for uniform asymptotic stability of differential and difference inclusions, namely,

*Theorem 8:* The following statements are equivalent:

- 1) The switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t),$$

where  $A_{\sigma(t)} \in \{A_1, A_2, \dots, A_N\}$ , is asymptotically stable under arbitrary switching;

- 2) the linear time-variant system

$$\dot{x}(t) = A(t)x(t),$$

where  $A(t) \in \mathcal{A} \triangleq \text{Conv}\{A_1, A_2, \dots, A_N\}$ , is robustly asymptotically stable;

- 3) there exist a full column rank matrix  $L \in \mathbb{R}^{m \times n}$ ,  $m \geq n$ , and a family of matrices  $\{\bar{A}_i \in \mathbb{R}^{m \times n} : i \in \mathcal{I}\}$  with

strictly negative row dominating diagonal, i.e., for each  $\bar{A}_i$ ,  $i \in \mathcal{I}$  its elements satisfying

$$\hat{a}_{kk} + \sum_{k \neq l} |\hat{a}_{kl}| < 0, \quad k = 1, \dots, m,$$

such that the matrix relations

$$LA_i = \bar{A}_i L$$

are satisfied.  $\square$

It is interesting to notice that the nice property of  $\bar{A}_i$  ( $i \in \mathcal{I}$ ) implies the existence of a common quadratic Lyapunov function for the higher dimensional switched system. Unfortunately, applying the above theorem is still difficult because, in general, the numerical search for the matrix  $L$  is not simple. However, this equivalence bridges together two research fields. Therefore, existing results in the robust stability area, which has been extensively studied for over two decades, can be directly introduced to study the arbitrarily switching systems and vice versa. For example, it is known in the robust stability literature that the convergence, (global) asymptotic stability, and (global) exponential stability are all equivalent for the polytopic uncertain linear time-variant systems [6]. Hence, these stability concepts are also equivalent for switched linear systems under arbitrary switching. In the next subsection, we will present a converse Lyapunov function for arbitrary switching systems, which is known in the literature of robust stability for linear time-variant systems.

#### D. Converse Lyapunov Theorem

In [19], a converse Lyapunov theorem was derived for the globally uniformly asymptotically stable and locally uniformly exponentially stable continuous-time switched systems with arbitrary switching signals. It was shown that such arbitrary switching system admits a common Lyapunov function.

*Theorem 9:* [19] If the switched system is globally uniformly asymptotically stable and in addition uniformly exponentially stable, the family has a common Lyapunov function.  $\square$

The converse Lyapunov theorem was extended in [55] to switched nonlinear systems that are globally uniformly asymptotically stable with respect to a compact forward invariant set. These converse Lyapunov theorems justify the common Lyapunov function method being pursued. However, they also suggest that *the common Lyapunov function may not necessarily be quadratic*, although most of the available results pertain to the existence of common quadratic Lyapunov functions. Therefore, the study of non-quadratic Lyapunov function, especially polyhedral Lyapunov function, has been attracting more and more attentions.

Based on the equivalence between the asymptotic stability of arbitrary switching linear systems and the robust stability of polytopic uncertain linear time-variant systems, some well established converse Lyapunov theorems can be introduced for arbitrary switching linear systems. For example, the following results were taken from [61].

*Theorem 10:* [61] If the switched linear system (2) is exponentially stable under arbitrary switching, then it has

a strictly convex, homogenous (of second order) common Lyapunov function of a quasi-quadratic form

$$V(x) = x^T L(x)x,$$

where  $L(x) = L^T(x) = L(\tau x)$  for all nonzero  $x \in \mathbb{R}^n$  and  $\tau \in \mathbb{R}$ .  $\square$

Furthermore, we may restrict our search to include only polyhedral Lyapunov functions (also known as piecewise linear Lyapunov function) [8] as the following result pointed out.

*Theorem 11:* [8], [61] If a switched linear system is asymptotically stable under arbitrary switching signals, then there exists a polyhedral Lyapunov function, which is monotonically decreasing along the switched system's trajectories.  $\square$

This converse Lyapunov theorem holds for both discrete-time and continuous-time cases. Compared with previous converse Lyapunov theorems, the above result has the following advantages. First, it shows that one may focus on polyhedral Lyapunov functions without loss of generality. Second, there exist automated computational methods to calculate polyhedral Lyapunov functions. In the sequel, we will briefly review some results for calculating polyhedral Lyapunov functions.

Several methods for automated construction of a common polyhedral Lyapunov function have been proposed in the literature. Early results include [21], where the Lyapunov function construction was reduced to the design of a balanced polytope satisfying some invariance properties. An alternative approach was given by Molchanov and Pyatnitskiy in [61], where algebraic stability conditions based on weighted infinity norms were proposed. A linear programming based method for solving these conditions was given by Polański in [69]. Recently, in [92], Yfoulis and Shorten proposed a numerical approach, called ray-gridding, to calculate polyhedral Lyapunov functions, which was based on uniform partitions of the state-space in terms of ray directions.

Finding conditions to guarantee stability under all possible switching signals is also of practical importance. For example, multiple-controller schemes are often employed to satisfy different performance requirements. When one designs multiple controllers for a plant, a desirable property is that switching between these controllers does not cause instability. The benefit of this property is that there is no need to worry about stability when switching among controllers and one can focus on gaining better performance. Hespanha and Morse [36] showed that it is possible to guarantee such a nice property for multiple controller design in certain cases. Actually, it was shown that a CQLF exists for proper realizations for the plant and the candidate controllers when these controllers are LTI and asymptotically stabilize the LTI plant.

It is noticed that the results presented in this subsection for arbitrary switching systems have been known in the fields of absolute stability and robust stability of differential or difference inclusions. These fields have been studied for decades and contain many interesting results that can be used to study arbitrary switching systems. An interesting line of research in the absolute stability literature is based on identifying the "most unstable" trajectory of a differential or difference inclusion through variational principles [56]. The

basic idea is simply: if the worst case trajectory is stable, then the whole system should be stable as well. Interested readers may refer to, e.g., [37], [56]–[58], [71], for details and developments. Most recently, Teel and colleagues [12], [28] developed theoretic results on the solutions, stability properties and converse Lyapunov theorem for differential inclusions with impulsive effects, called impulsive differential inclusions, which can be seen as parallel extensions of classical results of differential and difference inclusions, see [12], [28] and the references therein.

### III. STABILITY ANALYSIS UNDER RESTRICTED SWITCHING

Switched systems, for example a closed-loop multiple controller system, may fail to preserve stability under arbitrary switching, but may be stable under restricted switching signals. Restricted switching may arise naturally from the physical constraints of the system, e.g., in the automobile gear switching, particular switching sequence/order (from first gear to the second gear etc.) must be followed. Moreover, there are cases when one may have some knowledge about possible switching logic in a switched system, e.g., partitions of the state space and their induced switching rules. This knowledge may imply restrictions on the switching signals. For example, there must exist certain bound on the time interval between two successive switchings, which may be due to the fact that the state trajectories have to spend some finite length of time in traveling from the initial set to certain guard sets, if these two sets are separated. With such kind of *a priori* knowledge about the switching signals, we can derive stronger stability results for a given hybrid system than in the arbitrary switching case where we use, by necessity, worst case arguments.

This section will study the case when the switching signals are restricted, and our problem is to study the stability of the switched systems under these restricted switching signals. With this problem solved, one could provide an answer to the question regarding what restrictions should be put on the switching signals in order to guarantee the stability of switched systems. The restrictions on switching signals may be either time domain restrictions (e.g., dwell-time, average dwell-time switching signals that will be defined below) or state space restrictions (e.g., abstractions from partitions of the state space). Notice that the distinction between time-controlled switching signals (trajectory independent) and trajectory dependent switching signals is significant. In [33], Hespanha showed that when the class of switching signals is time-controlled, i.e., trajectory independent, uniform asymptotic stability of switched linear systems is equivalent to exponential stability. However, this equivalence does not hold for trajectory dependent switching signals. A counter example is given in [33].

#### A. Slow Switching

By studying the example in [20], [47] where divergent trajectories are generated through switching between two stable systems, one may notice that the unboundedness is caused by the failure to absorb the energy increase caused by the

switching. In addition, when there is an unstable subsystem (e.g., controller failure or sensor fault), if one either stays too long at or switches too frequently to the unstable subsystem, the stability may be lost. Therefore, a natural question is what if we restrict the switching signal to some constrained subclasses. Intuitively, if one stays at stable subsystems long enough and switches less frequently, i.e., slow switching, one may trade off the energy increase caused by switching or unstable modes, and maintain stability. These ideas are proved to be reasonable and are captured by concepts like dwell time and average dwell time switching proposed by Morse and Hespanha; see for example [33], [35], [93].

*Definition 1:* A positive constant  $\tau_d \in \mathbb{R}$  is called the dwell time of a switching signal if the time interval between any two consecutive switchings is no smaller than  $\tau_d$ .  $\square$

It can be shown that it is always possible to maintain stability when all the subsystems are stable and switching is slow enough, in the sense that  $\tau_d$  is sufficiently large [62]. Actually, it really does not matter if one occasionally have a smaller dwell time between switching, provided this does not occur too frequently. This concept is captured by the concept of “average dwell-time” in [35].

*Definition 2:* A positive constant  $\tau_a$  is called the average dwell time for a switching signal  $\sigma(t)$  if

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a}$$

holds for all  $t \geq \tau \geq 0$  and some scalar  $N_0 \geq 0$ , where  $N_\sigma(t, \tau)$  denotes the number of mode switches of a given switching signal  $\sigma$  over the interval  $(\tau, t)$ .  $\square$

Here the constant  $\tau_a$  is called the *average dwell time* and  $N_0$  the *chatter bound*. The reason for a switching signal that satisfies

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a}$$

is considered having an average dwell time no less than  $\tau_a$  is because

$$N_\sigma(t, \tau) \leq N_0 + \frac{t - \tau}{\tau_a} \Leftrightarrow \frac{t - \tau}{N_\sigma(t, \tau) - N_0} \geq \tau_a,$$

which means that on average the ‘dwell time’ between any two consecutive switchings is no smaller than  $\tau_a$ . It was shown in [35] that if all the subsystems are exponentially stable then the switched system remains exponentially stable provided that the average dwell time is sufficiently large.

*Theorem 12:* [35] Assume that all subsystems in a switched linear system are exponentially stable. There exists a scalar  $\tau_a^* > 0$  such that the switched system is exponentially stable if the average dwell time is larger than  $\tau_a^*$ .  $\square$

It is clear that switching signals with bounded (fixed) dwell time also have bounded average dwell time by definition. Therefore, the average dwell time scheme characterizes a larger class of stable switching signals than (fixed) dwell time scheme. Interested readers may refer to [32], [33] for further details and a recent review on this topic.

The stability results for slow switching can be extended to the discrete-time switched systems, where the dwell time  $\tau_d$  or average dwell time  $\tau_a$  is counted as the number of sampling periods [94], and similar results can be developed. In addition,

it is possible to extend the discrete-time results to the case where both stable and unstable subsystems coexist. When one considers unstable dynamics, slow switching (i.e., long enough dwell or average dwell time) is not sufficient for stability; it is also required to make sure that the switched system does not spend too much time in the unstable subsystems. The reason to consider unstable subsystems in switched systems is because there are cases where switching to unstable subsystems becomes unavoidable; such is the case, e.g., when a failure occurs or there are packet dropouts in communication. It is interesting to identify conditions under which the stability of the switched systems is still preserved [53], [93], [94].

Although the dwell-time and average dwell time mainly characterize the time-controlled switching signals, the slow switching idea can be generalized to hybrid systems or state-controlled switching signals. In [60], the authors studied the stability analysis problem for a given hybrid automaton (called structured hybrid automaton [60]) via abstracting it into a ‘similar’ switched system. The similarity is in the sense of preserving the average dwell time property. Actually, the authors developed abstraction schemes to guarantee that the derived switched system has no greater average dwell time than the original hybrid automaton. Under the assumption that all the subsystems are stable, the stability of the abstracted switched systems then implies the original hybrid automaton’s stability. In the next subsection, we will explicitly characterize the conditions on the state dependent switching signals and give conditions for the global stability of the switched linear systems.

### B. Multiple Lyapunov Functions

The stability analysis with constrained switching has been usually pursued in the framework of multiple Lyapunov functions (MLF). The basic idea is that multiple Lyapunov or Lyapunov-like functions, which may correspond to each single subsystem or certain region in the state space, are concatenated together to produce a non-traditional Lyapunov function. The non-traditionality is in the sense that the MLF may not be monotonically decreasing along the state trajectories, may have discontinuities and be piecewise differentiable. The reason for considering non-traditional Lyapunov functions is that traditional Lyapunov functions may not exist for switched systems with restricted switching signals. For such cases, one may still construct a collection of Lyapunov-like functions, which only require non-positive Lie-derivatives for certain subsystems in certain regions of the state space, instead of being negative globally. Since, the MLF theory is perhaps the most well studied area in the switched system literature and there already exist several excellent reviews, see e.g., [20], [32], [47], [59], our discussion on this topic will be very brief.

There are several versions of MLF results in the literature. A very intuitive MLF result [20] is illustrate in Figure 1, for which the Lyapunov-like function is decreasing when the corresponding mode is active and does not increase its value at each switching instant.

Actually, one may obtain less conservative results. For example, the switching signals may be restricted in such a

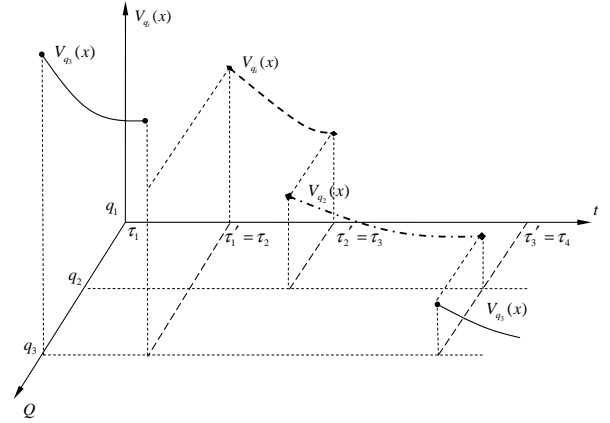


Fig. 1. The switched system is asymptotically stable if the Lyapunov-like functions’ values at the switching instants form a decreasing sequence.

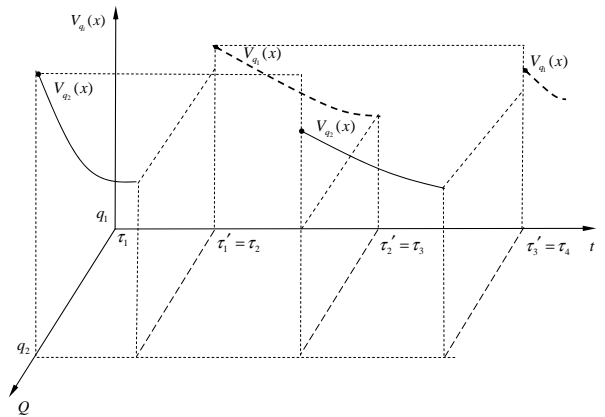


Fig. 2. For every subsystem, its Lyapunov-like function’s value  $V_i$  at the start point of each interval exceeds the value at the start point of the next interval on which the  $i$ -th subsystem is activated, then the switched system is asymptotically stable.

way that, at every time when we exit (switch from) a certain subsystem, its corresponding Lyapunov-like function value is smaller than its value at the previous exiting time, then the switched system is asymptotically stable [10]. In other words, for each subsystem the corresponding Lyapunov-like function values at every exiting instant form a monotonically decreasing sequence. Alternatively, the decreasing tendency is captured by the Lyapunov-like function’s value at the entering instant instead. This case is illustrate in Figure 2.

Furthermore, the Lyapunov-like function may increase its value during a time interval, only if the increment is bounded by certain kind of continuous functions [91] as illustrated in Figure 3. Interested readers may refer to the survey papers [20], [47], [59] and their references. Note that all the arguments for continuous-time hybrid/switched systems can be extended to the discrete-time case without essential differences.

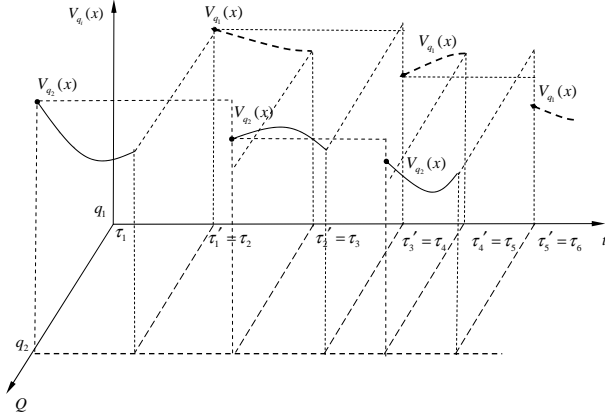


Fig. 3. The switched system can remain stable even when the Lyapunov-like function increases its value during certain period.

### C. Piecewise Quadratic Lyapunov Functions

The critical challenge of applying the MLF theorems in practical switched/hybrid systems is how to construct a proper family of Lyapunov-like functions. Usually this is a hard problem. However, if one focuses on the linear case, piecewise quadratic Lyapunov-like function could be an attractive candidate, since the stability conditions in the MLF theorems can be formulated as LMIs [41], [68], [72], for which efficient software solution packages exist.

Considering the switched LTI system (1), since we do not assume that the subsystem  $\dot{x}(t) = A_i x(t)$ , is stable, there may not exist a quadratic Lyapunov function in a classical sense. However, it is still possible to restrict our search to certain regions of the state space, say  $\Omega_i \subset \mathbb{R}^n$ , and the abstracted energy of the  $i$ -th subsystem could be decreasing along the trajectories inside this region (there is no decreasing requirements outside  $\Omega_i$ ). Suppose that the union of all these regions  $\Omega_i$  covers the whole state space; then we obtain a cluster of Lyapunov-like functions. Broadly speaking, the problem entails searching for Lyapunov-like functions whose associated  $\Omega$ -region cover the state space [18].

Assume that the state space  $\mathbb{R}^n$  has a partition given by  $\{\Omega_1, \dots, \Omega_N\}$ , and these regions  $\Omega_i$  are defined *a priori* as a restriction of the possible switching signals (state-dependent). In this subsection, we present conditions on LMI for the existence of quadratic Lyapunov-like functions of the form of  $V_i(x) = x^T P_i x$ , assigned to each region  $\Omega_i$ .<sup>2</sup> The results in this subsection are mainly based on [67]. A Lyapunov-like function  $V_i(x) = x^T P_i x$  needs to satisfy the following two conditions [67]:

**Condition 1:** There exist constant scalars  $\beta_i \geq \alpha_i > 0$  such that

$$\alpha_i \|x\|^2 \leq V_i(x) \leq \beta_i \|x\|^2$$

hold for all  $x \in \Omega_i$ .

<sup>2</sup>It is clear that the (common) quadratic Lyapunov function is a special case for the piecewise quadratic Lyapunov function by setting  $P_i = P_j$  for all  $i, j \in \mathcal{I}$ .

Consider a quadratic Lyapunov-like function candidate,  $V_i(x) = x^T P_i x$ , and require that

$$\alpha_i x^T I x \leq x^T P_i x \leq \beta_i x^T I x,$$

holds for any  $x \in \Omega_i$ . That is

$$\begin{cases} x^T (\alpha_i I - P_i) x \leq 0 \\ x^T (P_i - \beta_i I) x \leq 0 \end{cases}$$

holds for all  $x \in \Omega_i$ .

**Condition 2:** For all  $x \in \Omega_i$  and  $x \neq 0$ ,  $\dot{V}_i(x) < 0$ .

This negativeness of the Lyapunov-like function's derivative along the trajectories of a subsystem can be represented as:  $\exists P_i, (P_i = P_i^T)$  such that

$$x^T [A_i^T P_i + P_i A_i] x < 0 \quad (10)$$

for  $x \in \Omega_i$ .

**Switching Condition:** In addition, based on the MLF theorem of [20], it is also required that for stability, the Lyapunov-like functions' values at switching instant are non-increasing, which can be expressed by

$$x^T P_j x \leq x^T P_i x$$

for  $x \in \Omega_{i,j} \subseteq \Omega_i \cap \Omega_j$ . The region  $\Omega_{i,j}$  stands for the states where the trajectory passes from region  $\Omega_i$  to  $\Omega_j$ .

Note that all the above matrix inequalities are constrained in a local region, such as  $x \in \Omega_i$  or  $\Omega_{i,j}$ . A technique called  $\mathcal{S}$ -procedure [9] can be applied to replace a constrained matrix inequality condition by a condition without constraints. To employ the  $\mathcal{S}$ -procedure, the regions  $\Omega_i$  and  $\Omega_{i,j}$  need to be expressed or be contained in regions characterized by quadratic forms. This is always possible, and techniques to obtain less conservative quadratic forms to express hyperplanes, polyhedra or more general sets can be found in [9], [73]. For simplicity, we assume here that each region  $\Omega_i$  has a quadratic representation or approximation, that is

$$\Omega_i = \{x \mid x^T Q_i x \geq 0\},$$

and regions  $\Omega_{i,j}$  can be expressed or approximated by

$$\Omega_{i,j} = \{x \mid x^T Q_{i,j} x \geq 0\}.$$

Then the above matrix inequalities can be transformed into unconstrained ones based also on the  $\mathcal{S}$ -procedure, namely

**Theorem 13:** [68] The system (1) is (exponentially) stable if there exist matrices  $P_i$  ( $P_i = P_i^T$ ) and scalars  $\alpha > 0, \beta > 0, \mu_i \geq 0, \nu_i \geq 0, \vartheta_i \geq 0$  and  $\eta_{i,j} \geq 0$ , such that

$$\begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A_i^T P_i + P_i A_i + \vartheta_i Q_i \leq -I \\ P_j + \eta_{i,j} Q_{i,j} \leq P_i \end{cases} \quad (11)$$

are satisfied.  $\square$

If there is a solution to the above LMI problem, the exponential stability is verified. In addition, a bound on the convergence rate can be estimated:

$$\|x(t)\| \leq \sqrt{\frac{\beta}{\alpha}} e^{-\frac{1}{2\beta} t} \|x_0\|$$

where  $x(t)$  is the continuous trajectory with initial state  $x_0$ , and the constants  $\alpha, \beta$  are solutions of the LMI (11). Based



on similar arguments, LMI based sufficient conditions for the discrete-time case can be derived, see e.g., [25].

Notice that the above conditions are all based on MLF theorems, so the results developed in this subsection are sufficient only. To reduce the possible conservativeness of piecewise quadratic Lyapunov functions, a new kind of polynomial Lyapunov functions was introduced and investigated for the stability analysis of switched and hybrid systems [64], [70]. The computation of such polynomial Lyapunov functions can be efficiently performed using convex optimization, based on the sum of squares decomposition of multivariate polynomials. To be more precise, a multivariate polynomial  $p(x)$  is a sum of squares (SOS) if there exist polynomials  $p_1(x), \dots, p_m(x)$  such that  $p(x) = \sum_{i=1}^m p_i^2(x)$ . This in turn is equivalent to the existence of a positive semidefinite matrix  $Q$ , and a properly chosen vector of monomials  $Z(x)$  such that  $p(x) = Z^T(x)QZ(x)$ . For example,  $x \in \mathbb{R}^2$  and  $Z(x)$  of order  $k = 2$  implies  $Z(x) = [1, x_1, x_2, x_1x_2, x_1^2, x_2^2]^T$ . It is obvious that the quadratic Lyapunov function  $x^T Px$  is a special case of SOS. Another advantage that makes the SOS technique attractive is the fact that being a SOS automatically implies the positiveness of the polynomial, which could be very difficult to check otherwise (checking the positiveness of a polynomial belongs to the class of NP-hard problems).

It is also possible to use SOS techniques together with the  $\mathcal{S}$ -procedure to construct piecewise polynomial Lyapunov functions, with each polynomial as a SOS while incorporating the state constraints, so to generalize piecewise quadratic Lyapunov functions. Using the SOS approach, higher degree Lyapunov functions can be constructed, thus reducing the conservatism of searching for only quadratic candidates. Actually, the degree of the polynomials is very crucial for SOS approaches. On one hand, lower order means lower computation complexity (refer to [7] for the discussions and examples of the computational complexity issues), on the other hand higher degree is desirable to reduce the conservativeness of the method. While moving to higher order polynomials, we get more degrees of freedom in choosing the Lyapunov function and improve our chances to construct such Lyapunov function if it exists. There must exist an interesting tradeoff to optimally select the SOS degrees, and this problem needs to be investigated. In addition, another open problem is whether one can always find such a polynomial or piecewise polynomial Lyapunov function provided that the Lyapunov function exists, i.e., whether SOS is universal. If so, can an upper bound on the degree of the polynomials be estimated?

In addition to MLF based arguments, there are alternative methods for stability analysis of switched systems (under state-dependent switching logic), using for example impact maps and surface Lyapunov functions [29] associated with switching surfaces. Interested readers may refer to [29] for details.

#### IV. SWITCHING STABILIZATION

In the previous two sections, we discussed stability properties of switched systems under given switching signals, which may be restricted or arbitrary. The problem studied was under what conditions (either on the subsystems' dynamics and/or on

the switching signals) the switched system is stable. This is a stability analysis problem. Another basic problem for switched systems is the synthesis of stabilizing switching signals for a given collection of dynamical systems, called the switching stabilization problem.

##### A. Quadratic Switching Stabilization

In the switching stabilization literature, most of the work has focused on quadratic stabilization for certain classes of systems. A switched system is called quadratically stabilizable when there exist switching signals which stabilize the switched system along a quadratic Lyapunov function,  $V(x) = x^T Px$ .

It is known that a necessary and sufficient condition for a pair of LTI systems to be quadratically stabilizable is the existence of a stable convex combination of the two subsystems' matrices. Specially,

*Theorem 14:* [26], [86] A switched system that contains two LTI subsystems,  $\dot{x}(t) = A_i x(t)$ ,  $i = 1, 2$ , is quadratically stabilizable if and only if the matrix pencil  $\gamma_\alpha(A_1, A_2)$  contains a stable matrix.  $\square$

A generalization to more than two LTI subsystems was suggested in [67] by using a "min-projection strategy", i.e.,

$$\sigma(t) = \arg \min_{i \in \mathcal{I}} x(t)^T P A_i x(t). \quad (12)$$

*Theorem 15:* [67] If there exist constants  $\alpha_i \in [0, 1]$ , and  $\sum_{i \in \mathcal{I}} \alpha_i = 1$  such that

$$A_\alpha = \sum_{i \in \mathcal{I}} \alpha_i A_i,$$

is stable, then the min-projection strategy (12) quadratically stabilizes the switched system.

However, the existence of a stable convex combination matrix  $A_\alpha$  is only sufficient for switched LTI systems with more than two modes. There are example systems for which no stable convex combination state matrix exists, yet the system is quadratically stabilizable using certain switching signals [46]. A necessary and sufficient condition for the quadratic stabilizability of switched controller systems was derived in [80].

*Theorem 16:* [80] The switched system is quadratically stabilizable if and only if there exists a positive definite real symmetric matrix  $P = P^T > 0$  such that the set of matrices  $\{A_i P + P A_i^T\}$  is strictly complete, i.e., for any  $x \in \mathbb{R}^n \setminus \{0\}$ , there exists  $i \in \mathcal{I}$  such that  $x^T (A_i P + P A_i^T) x < 0$ . In addition, a stabilizing switching signal can be selected as  $\sigma(t) = \min_i \{x^T(t) (A_i P + P A_i^T) x(t)\}$ .  $\square$

Analogously, for the discrete-time case, it is necessary and sufficient for quadratic stabilizability to check whether there exists a positive symmetric matrix  $P$  such that the set of matrices  $\{A_i^T P A_i - P\}$  is strictly complete [80]. Obviously, the existence of a convex combination of state matrices  $A_\alpha$  automatically satisfies the above strict completeness conditions due to convexity, while the inverse statement is not true in general. Unfortunately, checking the strict completeness of a set of matrices is NP hard [80].

Other approaches include [86] and extensions of [86] to the output-dependent switching and discrete-time cases [47],

[96]. For robust stabilization of polytopic uncertain switched systems, a quadratic stabilizing switching law was designed for polytopic uncertain switched linear systems based on LMI techniques in [96].

Quadratic stability means that there exists a positive constant  $\epsilon$  such that  $\dot{V}(x) \leq -\epsilon x^T x$ . All of these methods guarantee stability by using a common quadratic Lyapunov function, which is conservative in the sense that there are switched systems that can be asymptotically or even exponentially stabilized without using a common quadratic Lyapunov function [34]. There have been some results in the literature that propose constructive synthesis methods in switched systems using multiple Lyapunov functions [20]. A stabilizing switching law design based on multiple Lyapunov functions was proposed in [85], where piecewise quadratic Lyapunov functions were employed for two mode switched LTI systems. Exponential stabilization for switched LTI systems was considered by Pettersson in [65], also based on piecewise quadratic Lyapunov functions, and the synthesis problem was formulated as a bilinear matrix inequality (BMI) problem. In the next subsection, we will briefly describe the BMI conditions derived in [65].

### B. Piecewise Quadratic Switching Stabilization

According to Theorem 13, if there exist real matrices  $P_i$  ( $P_i = P_i^T$ ) and scalars  $\alpha > 0$ ,  $\beta > 0$ ,  $\mu_i \geq 0$ ,  $\nu_i \geq 0$ ,  $\vartheta_i \geq 0$  and  $\eta_{i,j} \geq 0$ , satisfying

$$\begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A^T P_i + P_i A + \vartheta_i Q_i \leq -I \\ P_j + \eta_{i,j} Q_{i,j} \leq P_i \end{cases}$$

then the switched linear system (1) is exponentially stable.

Different from the stability analysis problem, the state space partitions  $\Omega_i$  are not given *a priori* any more. Actually, the state partitions  $\Omega_i$ , which induces the state-dependent switching signals, are to be designed. Moreover, the state space cannot be partitioned in an arbitrary way. The partition of the state space should facilitate the search of proper quadratic Lyapunov-like functions, and satisfy the non-increasing conditions when switching occurs. This will be discussed in detail in the sequel.

1) *State Space Partition*: Once again, the purpose of dividing the whole state space  $\mathbb{R}^n$  into pieces, denoted as  $\Omega_i$ , is to facilitate the search for Lyapunov-like functions for one of these subsystems. After successfully obtaining these Lyapunov-like functions associated with each region  $\Omega_i$ , one may patch them together, following the conditions of the above MLF theorem, so to guarantee global stability.

For this purpose, it is necessary to require that these regions  $\Omega_i$  cover the whole state space, i.e., the following *covering property* holds.

$$\Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_N = \mathbb{R}^n.$$

This condition merely says that there are no regions in the state space where none of the subsystems is activated.

Since we will restrict our attention to quadratic Lyapunov-like functions for purpose of computational efficiency, we will

consider regions given (or approximated) by quadratic forms

$$\Omega_i = \{x \in \mathbb{R}^n \mid x^T Q_i x \geq 0\},$$

where  $Q_i \in \mathbb{R}^{n \times n}$  are symmetric matrices, and  $i \in \{1, \dots, N\}$ .

The following lemma gives a sufficient condition for the covering property.

*Lemma 2:* [65] If for every  $x \in \mathbb{R}^n$

$$\sum_{i=1}^N \theta_i x^T Q_i x \geq 0 \quad (13)$$

where  $\theta_i \geq 0$ ,  $i \in \mathcal{I}$ , then  $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n$ .  $\square$

2) *Switching Condition*: In order to guarantee exponential stability we also need to make sure that

- 1) Subsystem  $i$  is active only when  $x(t) \in \Omega_i$ ,
- 2) When switching occurs, it is required to guarantee that the Lyapunov-like function values are not increasing.

To verify the first requirement, we consider the *largest region function strategy* [65], i.e.,

$$\sigma(x(t)) = \arg \left( \max_{i \in \mathcal{I}} x(t)^T Q_i x(t) \right). \quad (14)$$

This is due to the selection of subsystems (at state  $x(t)$ ) corresponding to the largest value of the region function  $x(t)^T Q_i x(t)$ .

Suppose that the covering condition (13) holds, i.e.,

$$\sum_{i=1}^N \theta_i x^T Q_i x \geq 0$$

for some  $\theta_i \geq 0$ ,  $i \in \mathcal{I}$ . Then, based on the largest region function strategy, the state  $x$  with the current active mode  $i$  satisfies  $x^T Q_i x \geq 0$ . This implies that  $x \in \Omega_i$ . So the first condition holds for the largest region function strategy (14).

To satisfy the second energy decreasing condition at switching instants, we need to know in which direction the state trajectory  $x(t)$  is passing through the switching surfaces. However, the switching surface is to be designed, and so such information is lacking in general. The author in [65] makes a compromise and requires that

$$x^T P_i x = x^T P_j x$$

for states at the switching plane, i.e.,  $x \in \Omega_i \cap \Omega_j$ . Assume that the set  $\Omega_i \cap \Omega_j$  can be represented by the following quadratic form

$$\Omega_i \cap \Omega_j = \{x \mid x^T (Q_i - Q_j) x = 0\}.$$

Again, applying the  $\mathcal{S}$ -procedure, we obtain

$$P_i - P_j + \eta_{i,j} (Q_i - Q_j) = 0,$$

for an unknown scalar  $\eta_{i,j}$ , as the switching condition.

3) *Synthesis Condition:* In summary, the above discussion can be encapsulated by the following sufficient conditions for the continuous-time system (1) to be exponentially stabilized.

*Theorem 17:* [65] If there exist real matrices  $P_i$  ( $P_i = P_i^T$ ) and scalars  $\alpha > 0$ ,  $\beta > 0$ ,  $\mu_i \geq 0$ ,  $\nu_i \geq 0$ ,  $\theta_i \geq 0$ ,  $\vartheta_i \geq 0$  and  $\eta_{i,j}$ , solving the optimization problem:

$$\min \beta$$

$$s.t. \begin{cases} \alpha I + \mu_i Q_i \leq P_i \leq \beta I - \nu_i Q_i \\ A^T P_i + P_i A + \vartheta_i Q_i \leq -I \\ P_j = P_i + \eta_{i,j} (Q_i - Q_j) \\ \theta_1 Q_1 + \dots + \theta_N Q_N \geq 0 \end{cases}$$

for all  $i, j \in \{1, \dots, N\}$ , then the switched linear system (1) can be exponentially stabilized (with decay rate  $\frac{1}{2\beta}$ ) by the largest region function strategy (14).  $\square$

4) *Discrete-time Switching Stabilization:* The extension of the synthesis method for continuous-time switched linear systems to discrete-time counterpart is not obvious. The main difficulty is that, unlike the continuous-time case, discrete-time switched systems do not have the nice property that the switching occurs exactly on the switching surface. Instead, the switching happens in a region around the switching surface. As a result, we can not simply capture the switching instants for discrete-time switched systems as the time instants when the state trajectories cross the switching surfaces. Therefore, in order to guarantee the non-increasing requirement at the switching instants for the discrete-time case, we need to include more constraints involving state transitions for the discrete-time switched systems around the switching surfaces. This makes the switching stabilization problem for discrete-time switched systems more challenging.

A piecewise quadratic Lyapunov function based switching stabilization for discrete-time switched linear systems is studied in [51], where the state transitions at switching instants were treated as additional constraints and were incorporated into matrix inequalities via Finsler's Lemma [9]. The main results in [51] can be stated as follows.

*Theorem 18:* If there exist matrices  $P_i$  ( $P_i = P_i^T$ ),  $Q_i$  ( $Q_i = Q_i^T$ ),  $F_i$ ,  $G_i$ ,  $F_{ij}$ ,  $Q_{ij}$ , and scalars  $\nu > 0$ ,  $\alpha_i > 0$ ,  $\beta_i > 0$ ,  $\eta_i \geq 0$ ,  $\rho_i \geq 0$ ,  $\mu_i \geq 0$ ,  $\mu_{ij} \geq 0$ ,  $\theta_i \geq 0$ , solving the optimization problem (15) for all  $i, j \in \{1, \dots, N\}$ ,  $i \neq j$ , then the largest region function strategy implies that the origin of the discrete-time switched system (2) is exponentially stable.  $\square$

Some remarks are in order. First, for both the continuous-time and discrete-time cases, the optimization problem above is a Bilinear Matrix Inequality (BMI) problem, due to the product of unknown scalars and matrices. BMI problems are NP-hard, and not computationally efficient. However, practical algorithms for optimization problems over BMIs exist and typically involve approximations, heuristics, branch-and-bound, or local search. As suggested in [65], one possible way to solve the BMI problem is to grid up the unknown scalars, and then solve a set of LMIs for fixed values of these parameters. It is argued that the gridding of the unknown scalars can be made quite sparsely [65].

Other approaches exist in the literature. A probabilistic algorithm was proposed in [38] for the synthesis of an

asymptotically stabilizing switching law for switched LTI systems along with a piecewise quadratic Lyapunov function. In [15], exponentially stabilizing switching laws were designed based on solving extended LQR optimal problems. Practical stabilization problem for switched nonlinear systems were investigated in [89], [90]. Related to switching stabilization literature as described above, there is work on feedback stabilization of switched systems or piecewise affine systems, where state or output feedback (continuous-variable) control laws are designed, *given a class of switching signals*. Several classes of switching signals are considered, for example arbitrary switching [17], [23], slow switching [14], [31], restricted switching induced by partitions of state space [16], [24], [44], [74], [75] etc. The distinctive feature of feedback stabilization compared with the switching stabilization problem is that the switching signal is no longer a free design variable. Although the continuous control inputs may have indirect effects on switching signals, the design focuses on the continuous feedback control law instead of the switching signals.

### C. Switching Stabilizability

So far, we have only derived sufficient conditions for the existence of stabilizing switching signals for a given collection of linear systems. A more elusive problem has been the necessity part of the switching stabilizability problem, and particularly challenging part has been the problem of necessary and sufficient conditions for switching stabilizability. In [81], Sun proved the following necessary condition for switching stabilizability.

*Theorem 19:* If there exists an asymptotically stabilizing switching signal among a finite number of LTI systems

$$\dot{x}(t) = A_i x(t),$$

where  $i = 1, 2, \dots, N$ , then there exists a subsystem, say  $A_k$ , such that at least one of the eigenvalues of  $A_k + A_k^T$  is a negative real number.  $\square$

This condition can be easily checked, but it is necessary only. A necessary and sufficient condition for asymptotic stabilizability of second-order switched LTI systems was derived in [87] by detailed vector field analysis. However, it was not apparent how to extend the method to either higher dimensions or to the parametric uncertainty case.

Recently, Lin and Antsaklis [52] proposed a necessary and sufficient condition for the existence of a switching control law (in static state feedback form) for asymptotic stabilization of continuous-time switched linear systems. The approach is briefly described below.

For each unstable subsystem,

$$\dot{x}(t) = A_i x(t) \quad (16)$$

it is assumed that there exists a full row rank matrix  $L_i \in \mathbb{R}^{m_i \times n}$ , where  $m_i < n$ , such that the auxiliary system for the  $i$ -th subsystem

$$\dot{\xi}(t) = L_i A_i R_i \xi(t), \quad t \in \mathbb{R}^+ \quad (17)$$

is asymptotically stable. Here  $R_i \in \mathbb{R}^{n \times m_i}$  is a right inverse of  $L_i$  [52].

$$\begin{aligned} & \max \nu \\ & \text{s.t.} \left\{ \begin{array}{l} \alpha_i I + \eta_i Q_i \leq P_i \leq \beta_i I - \rho_i Q_i \\ \begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i + \mu_i Q_i + \nu I & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} \leq 0, \\ \begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i - P_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix} \leq 0 \\ \theta_1 Q_1 + \dots + \theta_N Q_N \geq 0 \end{array} \right. \end{aligned} \quad (15)$$

Intuitively, the above assumption can be interpreted as considering a linear combination of the states of the original system (16) that evolves in an asymptotically stable manner. The auxiliary system evolves in the lower dimensional subspace, to which the original system can be projected for stability. Note that even when all parts of the states of the original system (16) are unstable, there still may exist  $L$  to satisfy the assumption. For example,

*Example 1:* Consider a continuous-time linear system,

$$\dot{x}(t) = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix} x(t)$$

The above continuous-time system is obviously unstable. However, we may select  $L = [1 \ 0]$  and  $R = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  to obtain

$$LAR = [1 \ 0] \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -0.5 < 0.$$

Therefore, the auxiliary system

$$\dot{\xi}(t) = -0.5\xi(t)$$

is asymptotically stable.

It can be shown that there always exist  $L$  and  $R$  satisfying the above assumptions in (17), except for the case when all the eigenvalues of  $A$  equal the same positive real number  $\lambda > 0$  and the geometric multiplicity of the eigenvalue  $\lambda$  equals to  $n$ . The proof of this claim explores the structure of the Jordan canonical form of  $A$  and uses straight-forward computations.

For the case when there does not exist an  $L$  to satisfy the above assumption for a particular subsystem, we simply set  $L$  as the null row vector, which implies that the corresponding subsystem makes no contribution to the stabilization of the switched system. To justify this, note that in this case the matrix  $A$  is similar to the matrix  $\lambda I$  for some positive real number  $\lambda > 0$ . Here  $I$  stands for the identity matrix. If we look at the phase plane of the LTI system,  $\dot{x}(t) = \lambda I x(t)$ , all the field vectors point to infinity along the radial directions. Intuitively speaking, the dynamics are explosive and do nothing but drag all the states to infinity, which we would like to avoid.

The basic idea is that a polyhedral Lyapunov-like function can be constructed for each subsystem by transforming the corresponding polyhedral Lyapunov function of its auxiliary system. Notice that every auxiliary system is asymptotically stable, so such polyhedral Lyapunov functions exist [54], [61], in a lower dimension. Via inserting the level sets of these polyhedral Lyapunov functions from a lower dimensional state space into  $\mathbb{R}^n$ , one obtains their corresponding polyhedral

Lyapunov-like functions in  $\mathbb{R}^n$ . An important observation is that for each subsystem the polyhedral Lyapunov-like function is decreasing for all state values  $x$  in the range space of  $R_i$ .

Assume there is no sliding motion occurring in the switched system. If for all the subsystems, the matrix

$$\begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{bmatrix} \in \mathbb{R}^{\sum_q m_i \times n}, \quad (18)$$

has full row rank and the union of the range space of  $R_i$ 's is the whole state space, then one can patch together these polyhedral Lyapunov-like function and construct an asymptotically stabilizing switching law. This shows the sufficiency part of the above condition. Actually, it is shown that the above two conditions are also necessary for switching stabilizability. The necessity proof is based on the lemma that a switched linear system can be asymptotically stabilized by a static switching signal if there exists a conic partition based switching law. A necessary and sufficient condition for switching stabilizability can now be presented, under the assumption that *there is no sliding motion in the closed-loop switched system*.

*Theorem 20:* [52] Assume that there is no sliding motion in the closed-loop switched system. The continuous-time switched linear system can be globally asymptotically stabilized, if and only if

- 1) there exist matrices  $L_i$ , which satisfy (17) for each subsystem, such that the matrix (18) has  $n$  linear independent row vectors,
- 2) Let  $\Omega^i$  stand for conic cones induced through the intersection of these polyhedral Lyapunov-like functions' level sets, and  $\Omega^i$  be required to be contained in the range space of  $R_i$ . These induced conic cones cover the whole state space, i.e.,

$$\bigcup_{i \in \mathcal{I}} \Omega^i = \mathbb{R}^n.$$

□

So far, all the arguments are under the assumption that no sliding motion is generated by the switched systems. However, sliding motions may occur through the proposed conic partition based switching laws. It is also possible that the generated sliding motion causes instability in the closed-loop switched system. Therefore, it is important to explicitly consider sliding motions. Similar issues arise in the methods for switching stabilization based on piecewise quadratic Lyapunov functions, where special care needs to be taken, see e.g., [66].

To explicitly deal with possible sliding motions, a necessary and sufficient condition for the occurrence of unstable sliding motions was identified in [52]. To avoid generating unstable sliding motions, we need to introduce an additional requirement to the above theorem:

*Theorem 21:* [52] The continuous-time switched linear system can be globally asymptotically stabilized, if and only if

- 1) there exist matrices  $L_i$ , which satisfy (17) for each subsystem and the rank condition, i.e., (18) has full row rank;
- 2) the union of  $\Omega^i$  cover the whole state space;
- 3) along the switching surface ( $\Omega^i \cap \Omega^j \neq \emptyset$  for  $i \neq j \in \mathcal{I}$ ), there exists a row vector  $L_{ij}$  such that

$$L_{ij}[\theta A_i + (1 - \theta)A_j]R_{ij} < 0,$$

for  $\theta \in [0, 1]$ . Here,  $R_{ij}$  is selected such that  $L_{ij}R_{ij} \neq 0$  and  $\Omega^i \cap \Omega^j$  is contained in the range space of  $R_{ij}$ .  $\square$

Note that the first two conditions are exactly the same as in Theorem 20, while the third condition is added to exclude possible unstable sliding motions. It is shown in [52] that this additional requirement of common  $L_{ij}$  and  $R_{ij}$  on the switching surface is not conservative, in the sense that it excludes exactly the unstable sliding motions; and, clearly, a switched system is stabilizable only when it can be done so without unstable sliding motions.

It is very interesting to note that if a switched linear system can be asymptotically stabilized by a static state feedback switching law without sliding motion, then one can always implement it in a conic partition based switching law. However, it is not known yet whether a stabilizable switched system can always be stabilized by a switching law in a static state feedback form.

Although the conditions given in [52] were proved to be necessary and sufficient, the checking of the necessity is not easy, as it requires to parameterize all  $L_q$  and  $R_q$  that satisfy (17). The calculation of such  $L_q$  and  $R_q$  for a given subsystem could be tedious, and systematic approaches need to be developed for parameterizations of such generalized similarity matrices. Fortunately, it is always possible to restrict the search to the vector case, i.e.,  $m_q = 1$ ,  $L_q \in \mathbb{R}^{1 \times n}$  and  $R_q \in \mathbb{R}^{n \times 1}$ . This makes it possible to formulate the determination of  $L_q$  and  $R_q$  into an optimization problem. Nevertheless, the properties of such generalized similarity transformations and the parameterizations of such  $L_q$  and  $R_q$  need further study. Interested readers may find further details in [52].

## V. CONCLUDING REMARKS

In this paper, we gave a, by necessity, brief overview of the most recent developments in the field of stability and stabilizability of switched linear systems. Especially, several recent results are highlighted in this survey. First, necessary and sufficient conditions for the asymptotic stability of switched linear systems under arbitrary switching were explored. Secondly, a necessary and sufficient condition for

the switching stabilizability of continuous-time switched linear systems was outlined.

The past decade has seen a lot of research activities in the field of switched systems. This survey is far from a complete review of stability and stabilizability of switched systems. There are results that are not mentioned here either due to space limitation or because the authors were not aware of them, and we apologize for these omissions.

## ACKNOWLEDGEMENTS

The insightful comments and suggestions from the AE, Prof. Hespanha, and anonymous reviewers are highly appreciated. The financial support from Singapore Ministry of Education's AcRF Tier 1 funding is gratefully acknowledged.

## REFERENCES

- [1] A. A. Agrachev and D. Liberzon. Lie-algebraic stability criteria for switched systems. *SIAM J. Control Optim.*, 40:253–270, 2001.
- [2] E. Asarin, O. Bournez, T. Dang, O. Maler, and A. Pnueli. Effective synthesis of switching controllers for linear systems. In P. J. Antsaklis, editor, *Proceedings of the IEEE: Special issue on hybrid systems*, volume 88, pages 1011–1025. IEEE Press, 2000.
- [3] P. Bauer, K. Premaratne, and J. Duran. A necessary and sufficient condition for robust asymptotic stability of time-variant discrete systems. *IEEE Trans. Automat. Contr.*, 38(9):1427–1430, 1993.
- [4] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Trans. Automat. Contr.*, 45(10):1864–1876, 2000.
- [5] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.
- [6] A. Bhaya and F. Mota. Equivalence of stability concepts for discrete time-varying systems. *Int. J. Robust and Nonlin. Contr.*, 4:725–740, 1994.
- [7] P. Biswas, P. Grieder, J. Lofberg, and M. Morari. A survey on stability analysis of discrete-time piecewise affine systems. In *Proceedings of 16th IFAC World Congress, July 2005*.
- [8] F. Blanchini. Nonquadratic Lyapunov functions for robust control. *Automatica*, 31(3):451–461, 1995.
- [9] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.
- [10] M. S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Automat. Contr.*, 43(4):475–482, 1998.
- [11] R. W. Brockett. Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman, and H. J. Sussmann, editors, *Differential Geometric Control Theory*, pages 181–191. Boston, MA: Birkhuser, 1983.
- [12] C. Cai, A. R. Teel, and R. Goebel. Smooth Lyapunov functions for hybrid systems part i: Existence is equivalent to robustness. *IEEE Trans. Automat. Contr.*, 52(7):1264–1277, 2007.
- [13] C.G. Cassandras, D.L. Pepyne, and Y. Wardi. Optimal control of a class of hybrid systems. *IEEE Trans. Automat. Contr.*, 46(3):398–415, 2001.
- [14] D. Cheng, L. Guo, Y. Lin, and Y. Wang. Stabilization of switched linear systems. *IEEE Trans. Automat. Contr.*, 50(5):661–666, 2005.
- [15] D. Corona, A. Giua, and C. Seatzu. Stabilization of switched systems via optimal control. In *Proc. 16th IFAC World Congress*, 2005.
- [16] F.A. Cuzzola and M. Morari. An LMI approach for  $\mathcal{H}_\infty$  analysis and control of discrete-time piecewise affine systems. *Inter. J. Contr.*, 75(16-17):1293–1301, 2002.
- [17] J. Daafouz, R. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Trans. Automat. Contr.*, 47(11):1883–1887, 2002.
- [18] J. M. Davoren and A. Nerode. Logics for hybrid systems. In P. J. Antsaklis, editor, *Proceedings of the IEEE: Special issue on hybrid systems*, volume 88, pages 985–1010. IEEE Press, 2000.
- [19] W. Dayawansa and C. F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Trans. Automat. Contr.*, 44(4):751–760, 1999.

- [20] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. In P. J. Antsaklis, editor, *Proceedings of the IEEE: Special issue on hybrid systems*, volume 88, pages 1069–1082. IEEE Press, 2000.
- [21] K. Drayton and C. Tong. Constructive stability and asymptotic stability of dynamical systems. *IEEE Trans. Circuits and Systems*, CAS-27:1121–1130, 1980.
- [22] M. Egerstedt and M. Babaali. On observability and reachability in a class of discrete-time switched linear systems. In *Proc. 2005 American Control Conf.*, pages 1179–1180, 2005.
- [23] L. Fang, H. Lin, and P. J. Antsaklis. Stabilization and performance analysis for a class of switched systems. In *Proc. 43rd IEEE Conf. Decision Control*, pages 3265–3270, 2004.
- [24] G. Feng. Controller design and analysis of uncertain piecewise linear systems. *IEEE Trans. Circuits and Systems I*, 49(2):224–232, 2002.
- [25] G. Feng. Stability analysis of piecewise discrete-time linear systems. *IEEE Trans. Automat. Contr.*, 47(7):1108–1112, 2002.
- [26] E. Feron. Quadratic stabilizability of switched systems via state and output feedback. Technical Report CICS-P-468, MIT, 1996.
- [27] A. F. Filippov. *Differential Equations with Discontinuous Right Hand Sides*. Norwell, MA: Kluwer Academic, 1988.
- [28] R. Goebel and A.R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42(4):573–587, 2006.
- [29] J.M. Goncalves, A. Megretski, and M.A. Dahleh. Global analysis of piecewise linear systems using impact maps and surface Lyapunov functions. *IEEE Trans. Automat. Contr.*, 48(12):2089–2106, 2003.
- [30] L. Gurvits, R. Shorten, and O. Mason. On the stability of switched positive linear systems. *IEEE Trans. Automat. Contr.*, 52(6):1099–1103, 2007.
- [31] A. Hassibi, S. P. Boyd, and J. P. How. Control of asynchronous dynamical systems with rate constraints on events. In *Proc. 38th IEEE Conf. Decision Control*, pages 1345–1351, 1999.
- [32] J. P. Hespanha. Stabilization through hybrid control. In H. Unbehauen, editor, *Encyclopedia of Life Support Systems (EOLSS)*, volume Control Systems, Robotics, and Automation. Oxford, UK, 2004.
- [33] J. P. Hespanha. Uniform stability of switched linear systems: Extensions of Lasalle’s invariance principle. *IEEE Trans. Automat. Contr.*, 49(4):470–482, 2004.
- [34] J. P. Hespanha, D. Liberzon, D. Angeli, and E. D. Sontag. Nonlinear norm-observability notions and stability of switched systems. *IEEE Trans. Automat. Contr.*, 52(2):154–168, 2005.
- [35] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proc. 38th IEEE Conf. Decision Control*, pages 2655–2660, 1999.
- [36] J. P. Hespanha and A. S. Morse. Switching between stabilizing controllers. *Automatica*, 38(11):1905–1907, 2002.
- [37] Z. Huang, C. Xiang, H. Lin, and T. H. Lee. A stability criterion for arbitrarily switched second-order LTI systems. In *Proc. 2007 IEEE 2007 IEEE Conf. Control and Automation*, pages 951–956, 2007.
- [38] H. Ishii, T. Basar, and R. Tempo. Randomized algorithms for synthesis of switching rules for multimodal systems. *IEEE Trans. Automat. Contr.*, 50(6):754–767, 2005.
- [39] Z. Ji, L. Wang, and X. Guo. Design of switching sequences for controllability realization of switched linear systems. *Automatica*, 43(4):662–668, 2007.
- [40] Z. Ji, G. Feng, and X. Guo. A constructive approach to reachability realization of discrete-time switched linear systems. *Syst. Contr. Lett.*, 56(11-12):669–677, 2007.
- [41] M. Johansson. *Piecewise Linear Control Systems - A Computational Approach*, volume 284 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 2002.
- [42] C. King and R. Shorten. A singularity test for the existence of common quadratic Lyapunov functions for pairs of stable lti systems. In *Proc. 2004 American Contr. Conf.*, pages 3881–3884, 2004.
- [43] T. Laffey and H. Smigoc. Tensor conditions for the existence of a common solution to the Lyapunov equation. *Linear Algebra and its Applications*, 420:672–685, 2007.
- [44] M. Lazar, M. Heemels, S. Weiland, A. Bemporad, and O. Pastravanu. Infinity norms as Lyapunov functions for model predictive control of constrained PWA systems. In *Hybrid Systems: Computation and Control*, volume 3414 of *Lecture Notes in Computer Science*, pages 417–432. Springer, 2005.
- [45] D. Liberzon. *Switching in Systems and Control*. Birkhauser, Boston, 2003.
- [46] D. Liberzon, J. P. Hespanha, and A. S. Morse. Stability of switched linear systems: A Lie-algebraic condition. *Syst. Contr. Lett.*, 37(3):117–122, 1999.
- [47] D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE Contr. Syst. Magazine*, 19(5):59–70, 1999.
- [48] D. Liberzon and R. Tempo. Common Lyapunov functions and gradient algorithms. *IEEE Trans. Automat. Contr.*, 49(6):990–994, 2004.
- [49] H. Lin and P. J. Antsaklis. Persistent disturbance attenuation properties for networked control systems. In *Proc. 43rd IEEE Conf. Decision Control*, pages 953–958, 2004.
- [50] H. Lin and P. J. Antsaklis. Stability and persistent disturbance attenuation properties for networked control systems: Switched system approach. *Inter. J. Contr.*, 78(18):1447–1458, 2005.
- [51] H. Lin and P. J. Antsaklis. Switching stabilization and  $l_2$  gain performance controller synthesis for discrete-time switched linear systems. In *Proc. 45th IEEE Conf. Decision Control*, pages 2673–2678, 2006.
- [52] H. Lin and P. J. Antsaklis. Switching stabilizability for continuous-time uncertain switched linear systems. *IEEE Trans. Automat. Contr.*, 52(4):633–646, 2007.
- [53] H. Lin, G. Zhai, and P. J. Antsaklis. Robust stability and disturbance attenuation analysis of a class of networked control systems. In *Proc. 42nd IEEE Conf. Decision Control*, pages 1182–1187, 2003.
- [54] D. Liu and A. Molchanov. Criteria for robust absolute stability of time-varying nonlinear continuous-time systems. *Automatica*, 38(4):627–637, 2002.
- [55] J. L. Mancilla-Aguilar and R. A. García. A converse Lyapunov theorem for nonlinear switched systems. *Systems & Control Letters*, 41(1):67–71, 2000.
- [56] M. Margaliot. Stability analysis of switched systems using variational principles: An introduction. *Automatica*, 42(12):2059–2077, 2006.
- [57] M. Margaliot and G. Langholz. Necessary and sufficient conditions for absolute stability: the case of second-order systems. *IEEE Trans. Circuits Syst.-I*, 50(2):227–234, 2003.
- [58] M. Margaliot and D. Liberzon. Lie-algebraic stability conditions for nonlinear switched systems and differential inclusions. 55(1):8–16, 2006.
- [59] A. N. Michel. Recent trends in the stability analysis of hybrid dynamical systems. *IEEE Trans. Circuits Syst. I*, 46(1):120–134, 1999.
- [60] S. Mitra, D. Liberzon, and N. Lynch. Verifying average dwell time of hybrid systems. *submitted to ACM Transactions in Embedded Computing Systems*, 2006.
- [61] A. P. Molchanov and E. Pyatnitskiy. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems & Control Letters*, 13:59–64, 1989.
- [62] A. S. Morse. Supervisory control of families of linear set-point controllers - part 1: Exact matching. *IEEE Trans. Automat. Contr.*, 41(10):1413–1431, 1996.
- [63] K. S. Narendra and J. Balakrishnan. A common Lyapunov function for stable LTI systems with commuting A-matrices. *IEEE Trans. Automat. Contr.*, 39(12):2469–2471, 1994.
- [64] A. Papachristodoulou and S. Prajna. A tutorial on sum of squares techniques for systems analysis. In *Proc. 2005 American Control Conf.*, 2005.
- [65] S. Pettersson. Synthesis of switched linear systems. In *Proc. 42nd IEEE Conf. Decision Control*, pages 5283–5288, 2003.
- [66] S. Pettersson. Synthesis of switched linear systems handling sliding motions. In *Proc. 2005 International Symposium on Intelligent Control*, pages 18–23, 2005.
- [67] S. Pettersson and B. Lennartson. Stabilization of hybrid systems using a min-projection strategy. In *Proc. 2001 American Contr. Conf.*, pages 223–228, 2001.
- [68] S. Pettersson and B. Lennartson. Hybrid system stability and robustness verification using linear matrix inequalities. *Inter. J. Contr.*, 75(16-17):1335–1355, 2002.
- [69] A. Polański. On absolute stability analysis by polyhedral Lyapunov functions. *Automatica*, 36(4):573–578, 2000.
- [70] S. Prajna and A. Papachristodoulou. Analysis of switched and hybrid systems - beyond piecewise quadratic methods. In *Proc. 42nd IEEE Conf. Decision Control*, pages 2779–2784, 2003.
- [71] E. S. Pyatnitskiy and L. B. Rapoport. Criteria of asymptotic stability of differential inclusions and periodic motions of time-varying nonlinear control systems. *IEEE Trans. Circuits Syst.-I*, 43(3):219–229, 1996.
- [72] A. Rantzer and M. Johansson. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Automat. Contr.*, 43(4):555–559, 1998. Special issue on Hybrid Systems, P.J. Antsaklis and A.N. Nerode Eds.

- [73] A. Rantzer and M. Johansson. Piecewise linear quadratic optimal control. *IEEE Trans. Automat. Contr.*, 45(4):629–637, 2000.
- [74] L. Rodrigues and S. Boyd. Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization. *Systems & Control Letters*, 54(9):835–853, 2005.
- [75] L. Rodrigues and J. P. How. Observer-based control of piecewise-affine systems. *Inter. J. Contr.*, 76(5):459–477, 2003.
- [76] R. Shorten, O. Mason, F. O. Cairbre, and P. Curran. A unifying framework for the SISO circle criterion and other quadratic stability criteria. *Inter. J. Contr.*, 77(1):1–8, 2004.
- [77] R. Shorten and K. Narendra. Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order linear time-invariant systems. In *Proc. 1999 American Contr. Conf.*, pages 1410–1414, 1999.
- [78] R. Shorten and K. Narendra. Necessary and sufficient conditions for the existence of a CQLF for a finite number of stable LTI systems. *International Journal of Adaptive Control and Signal Processing*, 16(10):709–728, 2002.
- [79] R. Shorten, K. Narendra, and O. Mason. A result on common quadratic Lyapunov functions. *IEEE Trans. Automat. Contr.*, 48(1):110 – 113, 2003.
- [80] E. Skafidas, R. J. Evans, A. V. Savkin, and I. R. Petersen. Stability results for switched controller systems. *Automatica*, 35(4):553–564, 1999.
- [81] Z. Sun. Stabilizability and insensitivity of switched linear systems. *IEEE Trans. Automat. Contr.*, 49(7):1133–1137, 2004.
- [82] Z. Sun and S. S. Ge. Analysis and synthesis of switched linear control systems. *Automatica*, 41(2):181–195, 2005.
- [83] Z. Sun and S. S. Ge. *Switched linear systems: Control and design*. Springer-Verlag, 2005.
- [84] Z. Sun, S. S. Ge, and T. H. Lee. Controllability and reachability criteria for switched linear systems. *Automatica*, 38(5):775–786, 2002.
- [85] M. A. Wicks and R. A. DeCarlo. Solution of coupled Lyapunov equations for the stabilization of multi-modal linear systems. In *Proc. 1997 American Contr. Conf.*, pages 1709–1713, 1997.
- [86] M. A. Wicks, P. Peleties, and R. A. DeCarlo. Switched controller design for the quadratic stabilization of a pair of unstable linear systems. *European J. Control*, 4:140–147, 1998.
- [87] X. Xu and P. J. Antsaklis. Stabilization of second-order LTI switched systems. *Inter. J. Contr.*, 73(14):1261–1279, 2000.
- [88] X. Xu and P. J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Trans. Automat. Contr.*, 49(1):2–16, 2004.
- [89] X. Xu and G. Zhai. Practical stability and stabilization of hybrid and switched systems. *IEEE Trans. Automat. Contr.*, 50(11):1897–1903, 2005.
- [90] X. Xu, G. Zhai, and S. He. Stabilizability and practical stabilizability of continuous-time switched systems: A unified view. In *Proc. 2007 American Control Conf.*, pages 663–668, 2007.
- [91] H. Ye, A. N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Trans. Automat. Contr.*, 43(4):461–474, 1998.
- [92] C. Yfoulis and R. Shorten. A numerical technique for stability analysis of linear switched systems. *Inter. J. Contr.*, 77(11):1019–1039, 2004.
- [93] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel. Disturbance attenuation properties of time-controlled switched systems. *J. Franklin Institute*, 338:765–779, 2001.
- [94] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel. Qualitative analysis of discrete-time switched systems. In *Proc. 2002 American Contr. Conf.*, volume 3, pages 1880–1885, 2002.
- [95] G. Zhai and H. Lin. Controller failure time analysis for symmetric  $\mathcal{H}_\infty$  control systems. *Inter. J. Contr.*, 77(6):598–605, 2004.
- [96] G. Zhai, H. Lin, and P. J. Antsaklis. Quadratic stabilizability of switched linear systems with polytopic uncertainties. *Inter. J. Contr.*, 76(7):747–753, 2003.
- [97] G. Zhai, X. Xu, H. Lin, and A. N. Michel. Analysis and design of switched normal systems. *Nonlinear Analysis, Theory, Methods and Applications*, 65(12):2248–2259, 2006.