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A BRIEF REVIEW OF THE LAPLACE TRANSFORM USEFUL IN CONTROL FUNCTIONS

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Given a function $f(t)$ its *Laplace transform* is given by

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

The Laplace variable s can be seen as a generalized frequency $s = \sigma + j\omega$.

This is the one-sided Laplace transform, as the lower integration limit starts at 0 and not at $-\infty$. Note that the lower integration limit is in fact 0^-, that is, the integration starts just before 0. This is to include any possible discontinuities in $f(t)$ that may occur exactly at 0; this is the case, for example, with the impulse or delta function $\delta(t)$. In control, it is common for the signals of interest to be assumed to be zero for $t < 0$ and so no signal information is lost when moving into the transform domain. This one-sided Laplace transform is particularly convenient when solving linear ordinary differential equations using Laplace transforms.

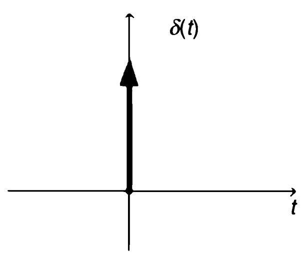
There is one-to-one correspondence between $f(t)$ and $F(s)$. To recover $f(t)$ from $F(s)$ one can use the inverse Laplace transform given by

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\tau - j\omega}^{\tau + j\infty} F(s)e^{st}\, ds$$

However, in control it is common to obtain $f(t)$ from $F(s)$ using partial fraction expansion and properties and tables of common Laplace transforms.

It is not difficult to derive the Laplace transforms of simple functions. To demonstrate, we select some functions common in control:

Impulse (Delta or Dirac) function $\delta(t)$



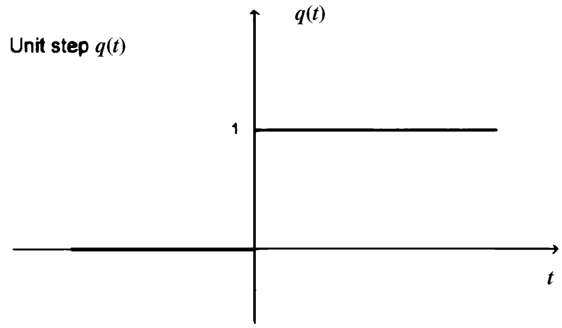
An important property of the delta function is given by

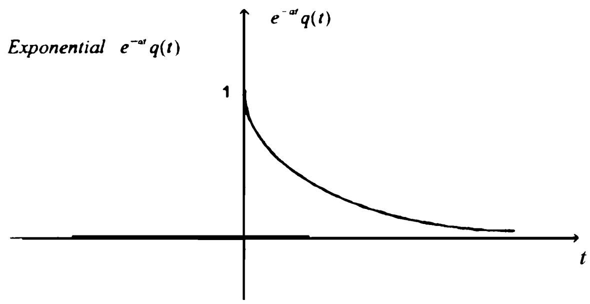
$$f(t) = \int_{-\infty}^{+\infty} \delta(t-\tau)f(\tau)d\tau$$

where in fact the integration range needs to cover only the point where the argument in $\delta(\cdot)$ becomes zero. Then

$$L\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \left. e^{-st}\right|_{t\ 0} = 1$$

Unit Step





It is not difficult to show that $L\{q(t)\} = 1/s$ where $q(t)$ is the *unit step* (zero for $t < 0$ and 1 for $t \leq 0$). Also that $L\{e^{at} \cdot q(t)\} = 1/(s + a)$; note that $e^{at}q(t)$ is zero for $t < 0$ and e^{-at} for $t \leq 0$.

Laplace transform useful in control functions:

$f(t)$	$F(s)$
$\delta(t)$	1
$q(t)$	$1/s$
$e^{-at} \cdot q(t)$	$1/(s + a)$
$\sin \omega t \cdot q(t)$	$\omega/(s^2 + \omega^2)$
$\cos \omega t \cdot q(t)$	$s/(s^2 + \omega^2)$
$t^a \cdot q(t)$	$n!/s^{n+1}$
$t^n e^{-at} \cdot q(t)$	$n!/(s + a)^{n+1}$

Some Properties of Laplace Transform

When solving linear ordinary differential equations with constant coefficients using Laplace transform properties that involve derivatives of the time functions are useful. These include

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0) - \frac{df(0)}{dt}$$

where $F(s) = L\{f(t)\}$. These and other properties are included in the list below:

$f(t)$	$F(s)$
$\dfrac{d^k f^{(k)}}{dt^k} (= f^{(k)}(t))$	$S^k F(s) \sim S^{k-1} f(0) - S^{k-2} f^{(1)}(0) - \cdots - f^{(k-1)}(0)$
$e^{-at} f(t) \cdot q(t)$	$F(s + a)$
Delay $f(t - a) \cdot q(t - a)$	$e^{-as} F(s)$
Convolution $\int f(t-\delta) g(\tau) d\tau$	$F(s) \cdot G(s)$
Final value $\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
Initial value $\lim_{t\to 0} f(t)$	$\lim_{s\to 0} sF(s)$

Partial Fraction Expansion

Write

$$G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad m \le n$$

as the sum of simple terms, namely,

$$G(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n}$$

where p_j , $j = 1, \ldots, n$, are the n roots of the denominator polynomial (poles of $G(s)$)

1. When all n p_j are distinct then

$$c_j = \lim_{s \to p_j} [(s - p_j) G(s)]$$

2. When two poles are complex conjugate, that is, $p_1 = a + jb$ and $p_2 = a - jb$ (also written p_2, p^*_1) then

$$F(s) = \frac{c_1}{s - p_1} + \frac{c_1^*}{s - p_1}$$

from which

$$f(t) = 2\,|c_1| e^{at} \cos\,[bt + \arg(c_1)]$$

Note that it is also common in the case of complex conjugate poles to allow a second-order term in the partial fraction expansion of the form

$$\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

which could then be found directly from the tables in some cases.

3. When a p_j is repeated r times then the expansion must include r terms that correspond to the pole p_j. These r terms are of the form

$$\frac{c_{j1}}{s-p_j}+\frac{c_{j2}}{(s-p_j)^2}+\cdots+\frac{c_{jr}}{(s-p_j)^r}$$

where

$$c_{jr} = \lim_{s\to p_j}\left[(s-p_j)^r G(s)\right]$$

$$C_{j1} = \lim_{s\to p_j}\left(\frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left\{(s-p_j)^r G(s)\right\}\right.$$

Example 1

$$F(s)=\frac{2s^2+3s+3}{(s+1)(s+3)^3}=\frac{1/4}{s+1}+\frac{-1/8}{(s+3)}+\frac{2/3}{(s+3)^2}+\frac{-6}{(s+3)^3}$$

Then

$$f(t)=L^{-1}\{F(s)\}=\left(\frac{1}{4}e^{-t}-\frac{1}{8}e^{-3t}+\frac{2}{3}te^{-3t}-3t^2e^{-3t}\right)q(t)$$

Note that if $F(s)$ represents the transfer function of a system then $f(t)=L^{-1}\{F(s)\}$ is the impulse response of the system.

Example 2

$$F(s)=\frac{2s+1}{s^2+1}=\frac{2s}{s^2+1}+\frac{1}{s^2+1}$$

From the tables,

$$f(t) = (2\cos t + \sin t)q(t)$$

Note that the poles of $F(s)$ are at $\pm j$. Alternatively, using the formulas for poles $p_1 - j$ and $p_2 - j$

$$\frac{1}{s^2+1}=\frac{-1/2j}{s-j}+\frac{+1/2j}{s-(-j)}$$

and

$$L^{-1}\left\{\frac{1}{s^2+1}\right\}=2\cdot 1/2\cdot e^{at}\cos(t+\pi/2)=\sin t\cdot q(t)$$

Similarly for the other term.