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Optimal Control of Switched Autonomous Systems[1]

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Abstract

In this paper, optimal control problems for switched autonomous systems are studied. In particular, we focus on problems in which a prespecified sequence of active subsystems is given and propose an approach to finding the optimal switching instants. The approach derives the derivatives of the cost with respect to the switching instants and uses nonlinear optimization techniques to locate the optimal switching instants. The approach is then applied to general quadratic problems for switched linear autonomous systems and to reachability problems. Examples illustrate the results.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. Examples of such systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

Recently, many results for optimal control of switched systems have appeared in the literature (e.g., [2, 6, 7, 8, 10]). Most of them consider problems which seek for the solution of both the optimal continuous input and the optimal switching sequence. Approaches to such problems include ones based on discretization of the time and state space [6, 7] and ones that are not based on discretization [8, 10]. Many of these approaches find approximations to local optimal solutions.

In this paper, we focus on optimal control problems for switched autonomous systems where each subsystem is autonomous (i.e., with no continuous input). In particular, we focus on problems in which a prespecified sequence of active subsystems is given. General autonomous subsystems and general performance costs are considered. For such problems the cost is a function of the switching instants. We propose to use constrained nonlinear optimization techniques to locate open-loop local optimal switching instants for such general problems. To apply nonlinear optimization techniques, we need to first determine the values of the derivatives of the cost with respect to the switching instants. An approach similar to that in [10] is proposed in this paper for their derivations. One of the main results of the paper is Theorem 3.1 which gives us the expressions of the derivatives. Note here the approach provides us with accurate values of the derivatives as opposed to the approximate values in [10]. The approach is then applied to general quadratic problems for switched linear autonomous systems. The computation of the derivatives can be further simplified by utilizing the special structure of such problems. Finally, we apply the optimal control approach to reachability problems. Using the approach, the reachability switching instants can be determined if a final state is reachable from an initial state.

Similar problems have also been looked into by other researchers. Giua *et al* in [4, 5] present closed-loop global optimal solutions to a special class of problems, i.e., infinite horizon problems for switched linear autonomous systems. However, we should indicate that our approach has the following advantages. First, our approach can deal with finite horizon problems with general subsystems and costs as opposed to infinite horizon problems with linear subsystems and quadratic costs in [4, 5]. Moreover, our approach can be applied to reachability problems, while the approach in [4, 5] fits better for stability problems. In view of these, we believe our results are new and contribute to the understanding and the solution of optimal control problems of switched systems.

2 Problem Formulation

We consider the following *switched autonomous systems*, i.e., switched systems which consist of autonomous subsystems (i.e., without continuous input)

$$\dot{x} = f_i(x,t), \quad f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad i \in I = \{1, \cdots, M\}. \quad (2.1)$$

The state trajectory evolution of such a system can be controlled by choosing appropriate switching sequences. A *switching sequence* in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), (t_2, i_2), \cdots, (t_K, i_K)), \quad (2.2)$$

with $0 \leq K < \infty$, $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f$, and $i_k \in I$, $k = 0, 1, \cdots, K$. σ tells us that subsystem i_k is active in $[t_k, t_{k+1})$. Note that the continuous state of a switched system has no discontinuities at the switching instants.

In the following, we assume without loss of generality that a prespecified sequence of active subsystems is given as $(1, 2, \cdots, K, K+1)$, i.e., subsystem k is active in $[t_{k-1}, t_k)$. We can always do this by relabeling the subsystem indices and even expanding the collection of subsystems (i.e., two subsystems may actually refer to the same actual subsystem). We consider the following optimal control problem.

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Problem 2.1 (Optimal Control Problem)
Consider a switched autonomous system with subsystems $f_i(x,t)$, $i \in I$. Assume that a prespecified sequence of active subsystems $(1, 2, \cdots, K, K+1)$ is given. Find optimal switching instants $t_1, \cdots, t_K$ ($t_0 \le t_1 \le \cdots \le t_K \le t_f$) such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and the cost

$$J(t_1, \cdots, t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x,t)\ dt \tag{2.3}$$

is minimized. Here t_0, t_f are given. □

We assume that f_i's, L, and ψ are smooth enough. Under these assumptions, we observe that a small disturbance of $t_1, \cdots, t_K$ will only cause a small disturbance of J value. Furthermore, it can be shown that J is a continuously differentiable function of $t_1, \cdots, t_K$.

2.1 An Algorithm

Note that Problem 2.1 is actually a constrained multivariable optimization problem

$$\begin{array}{c} \min_{\hat{t}} J(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{2.4}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f\}$. The following algorithm can be adopted to solve such a nonlinear optimization problem.

Algorithm 2.1

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.
(2). Find $J(\hat{t}^j)$, $\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J}{\partial \hat{t}^2}(\hat{t}^j)$.
(3). Use the gradient projection method or the constrained Newton's method [1] to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$. Set $j = j + 1$.
(4). Repeat steps (2), (3), (4) until a prespecified termination condition is satisfied (e.g. $\|\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)\|_2 < \epsilon$ where ϵ is a given small number). □

3 Differentiations of the Cost Function

In order to apply Algorithm 2.1, the values of $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$ (step (2)) need to be found. Now we propose an approach to finding these values.

Assume we have a nominal $\hat{t} = (t_1, \cdots, t_K)^T$ and the corresponding nominal $x(t)$. The corresponding cost J can be obtained using (2.3). Since x_0 and t_0 are fixed, J is not a function of them. Next we define the value function at the k-th switching instant as

$$J^k\big(x(t_k), t_k, \cdots, t_K\big) = \psi\big(x(t_f)\big) + \int_{t_k}^{t_f} L(x,t)\ dt. \tag{3.1}$$

Unlike J, J^k's for $k \ge 1$ are functions of t_k and of the initial state $x(t_k)$ which depends on the trajectory before t_k. In the sequel, we denote $\frac{\partial J^k}{\partial x}$ for a function J^k as a row vector J^k_x, $\frac{\partial^2 J^k}{\partial x^2}$ as an $n \times n$ matrix J^k_{xx} and so on.

3.1 Single Switching

Let us first consider the case of a single switching. Given a nominal t_1 and a corresponding nominal trajectory $x(t)$, we denote by $\hat{x}(t)$ the state trajectory after a variation dt_1 has taken place. In the sequel, we adopt the following notational convention. We write f, f_x and f_t with a superscript $1-$ (resp. $1+$) whenever the corresponding active vector field at t_1- (resp. t_1+) is used for evaluation at $(x(t_1), t_1)$. Examples of this convention are $f^{1-} \triangleq f_1\big(x(t_1), t_1\big)$, $f^{1+} \triangleq f_2\big(x(t_1), t_1\big)$, $f^{1-}_t \triangleq \frac{\partial f_1}{\partial t}\big(x(t_1), t_1\big)$, and $f^{1+} \triangleq \frac{\partial f_2}{\partial t}\big(x(t_1), t_1\big)$, etc. Also, we simply write $J^1 \triangleq J^1\big(x(t_1), t_1\big)$, $L^1 \triangleq L\big(x(t_1), t_1\big)$, $J^1_x \triangleq J^1_x\big(x(t_1), t_1\big)$, $L^1_x \triangleq L_x\big(x(t_1), t_1\big), \cdots$ (be careful to distinguish the values J^1, L^1, J^1_x, and $L^1_x, \cdots$ from the functions $J^1\big(x(t_1), t_1\big)$, $L(x,t)$, $J^1_x\big(x(t_1), t_1\big)$, and $L_x(x,t), \cdots$).

It is not difficult to see that

$$J(t_1) = \int_{t_0}^{t_1} L(x,t)\ dt + J^1\big(x(t_1), t_1\big). \tag{3.2}$$

For a small variation dt_1 of t_1, we have

$$J(t_1 + dt_1) = \int_{t_0}^{t_1 + dt_1} L(\hat{x}, t)\ dt + J^1\big(\hat{x}(t_1 + dt_1), t_1 + dt_1\big). \tag{3.3}$$

There are two terms in (3.3). Let us consider the second order Taylor expansion of each term. In the following derivations we denote

$$\begin{aligned} dx(t_1) &\triangleq \hat{x}(t_1 + dt_1) - x(t_1) \\ &= f^{1-} dt_1 + \tfrac{1}{2}(f^{1-}_t + f^{1-}_x f^{1-}) dt_1^2 + o(dt_1^2). \end{aligned} \tag{3.4}$$

Note in (3.4), $o(dt_1^2)$ is a column vector with each element being $o(dt_1^2)$. Consider the first term in (3.3), for either $dt_1 \ge 0$ or $dt_1 < 0$, we have (see [9] for details)

$$\begin{aligned} &\textstyle\int_{t_0}^{t_1+dt_1} L(\hat{x}, t)\ dt = \int_{t_0}^{t_1} L(x,t)\ dt + L^1 dt_1 \\ &\quad + \tfrac{1}{2} dt_1 L^1_x dx(t_1) + \tfrac{1}{2} L^1_t dt_1^2 + (\text{higher order terms}). \end{aligned} \tag{3.5}$$

For the second term in (3.3), we have

$$\begin{aligned} &J^1\big(\hat{x}(t_1 + dt_1), t_1 + dt_1\big) = J^1 + J^1_x dx(t_1) + J^1_{t_1} dt_1 \\ &\quad + \tfrac{1}{2}\big(dx(t_1)\big)^T J^1_{xx} dx(t_1) + \tfrac{1}{2} J^1_{t_1 t_1} dt_1^2 \\ &\quad + dt_1 J^1_{t_1 x} dx(t_1) + (\text{higher order terms}). \end{aligned} \tag{3.6}$$

Now we substitute (3.4) into (3.5) and (3.6) and sum them to obtain the second order expansion of (3.3) with respect to dt_1,

$$\begin{aligned} &J(t_1 + dt_1) = J(t_1) + (L^1 + J^1_{t_1} + J^1_x f^{1-}) dt_1 \\ &\quad + \tfrac{1}{2}\big(L^1_x f^{1-} + L^1_t + J^1_x (f^{1-}_t + f^{1-}_x f^{1-}) \\ &\quad + (f^{1-})^T J^1_{xx} f^{1-} + J^1_{t_1 t_1} + 2 J^1_{t_1 x} f^{1-}\big) dt_1^2 + o(dt_1^2) \\ &\triangleq J(t_1) + J_{t_1} dt_1 + \tfrac{1}{2} J_{t_1 t_1} dt_1^2 + o(dt_1^2). \end{aligned} \tag{3.7}$$

Note that the following dynamic programming equation holds for $J^1\big(x(t_1), t_1\big)$

$$J^1_{t_1} = -J^1_x f^{1+} - L^1. \tag{3.8}$$

(3.8) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (3.8) holds for any trajectory that is not necessarily optimal (for more details see [3]).

By differentiating (3.8), we obtain

$$J^1_{t_1 x} = -(f^{1+})^T J^1_{xx} - J^1_x f^{1+}_x - L^1_x, \tag{3.9}$$

$$\begin{aligned} J^1_{t_1 t_1} &= -J^1_{t_1 x} f^{1+} - J^1_x f^{1+}_t - L^1_t = (f^{1+})^T J^1_{xx} f^{1+} \\ &\quad + (J^1_x f^{1+}_x + L^1_x) f^{1+} - J^1_x f^{1+}_t - L^1_t. \end{aligned} \tag{3.10}$$

By substituting (3.8), (3.9) and (3.10) into (3.7), we can write J_{t_1} and $J_{t_1 t_1}$ in the following form

$$J_{t_1} = J^1_x (f^{1-} - f^{1+}), \tag{3.11}$$

$$\begin{aligned} J_{t_1 t_1} &= J^1_x (f^{1-}_t - f^{1+}_t) - (J^1_x f^{1+}_x + L^1_x)(f^{1-} \\ &\quad - f^{1+}) + J^1_x (f^{1-}_x - f^{1+}_x) f^{1-} + (f^{1-} \\ &\quad - f^{1+})^T J^1_{xx} (f^{1-} - f^{1+}). \end{aligned} \tag{3.12}$$

3.2 Two or More Switchings

Now consider the case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2 ($t_0 \le t_1 \le t_2 \le t_f$). The cost then is

$$J(t_1,t_2) = \int_{t_0}^{t_1} L(x,t)\,dt + J^1\big(x(t_1),t_1,t_2\big) \quad (3.13)$$

$$= \int_{t_0}^{t_2} L(x,t)\,dt + J^2\big(x(t_2),t_2\big). \quad (3.14)$$

Using (3.13), by holding t_2 fixed, J_{t_1}, $J_{t_1t_1}$ can be derived similarly to that in subsection 3.1. In the same manner, J_{t_2}, $J_{t_2t_2}$ can be derived using (3.14). However, we need additional information to derive $J_{t_1t_2}$. Arguments from the calculus of variations are used in the followings to derive $J_{t_1t_2}$. Let us first define the important notion of incremental change.

Definition 3.1 (Incremental Change) *Given variations dt_1 and dt_2, we define* the incremental change $\delta x(t), \min\{t_1, t_1+dt_1\} \le t \le \max\{t_2, t_2+dt_2\}$ *as:*

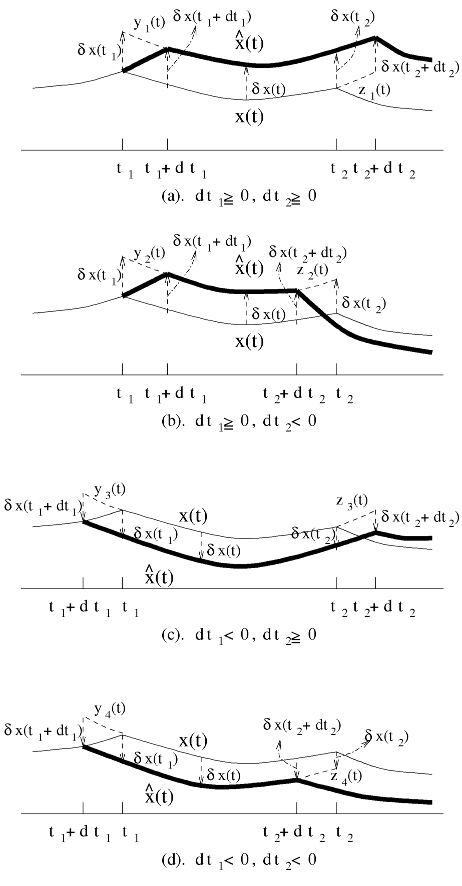

Figure 1: The incremental change $\delta x(t)$.

Case 1: *$dt_1 \ge 0, dt_2 \ge 0$ (see figure 1(a)).*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1+dt_1, t_2] \\ y_1(t) - x(t), & t \in [t_1, t_1+dt_1] \\ \hat{x}(t) - z_1(t), & t \in [t_2, t_2+dt_2] \end{cases} \quad (3.15)$$

where $y_1(t)$ is the solution of

$$\begin{cases} \dot{y}_1(t) = f_2\big(y_1(t),t\big), & t \in [t_1, t_1+dt_1] \\ y_1(t_1+dt_1) = \hat{x}(t_1+dt_1) \end{cases} \quad (3.16)$$

and $z_1(t)$ is the solution of

$$\begin{cases} \dot{z}_1(t) = f_2\big(z_1(t),t\big), & t \in [t_2, t_2+dt_2] \\ z_1(t_2) = x(t_2). \end{cases} \quad (3.17)$$

Case 2: *$dt_1 \ge 0, dt_2 < 0$ (see figure 1(b).)*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1+dt_1, t_2+dt_2] \\ y_2(t) - x(t), & t \in [t_1, t_1+dt_1] \\ z_2(t) - x(t), & t \in [t_2+dt_2, t_2] \end{cases} \quad (3.18)$$

where $y_2(t)$ is the solution of

$$\begin{cases} \dot{y}_2(t) = f_2\big(y_2(t),t\big), & t \in [t_1, t_1+dt_1] \\ y_2(t_1+dt_1) = \hat{x}(t_1+dt_1) \end{cases} \quad (3.19)$$

and $z_2(t)$ is the solution of

$$\begin{cases} \dot{z}_2(t) = f_2\big(z_2(t),t\big), & t \in [t_2+dt_2, t_2] \\ z_2(t_2+dt_2) = \hat{x}(t_2+dt_2). \end{cases} \quad (3.20)$$

Case 3: *$dt_1 < 0, dt_2 \ge 0$ (see figure 1(c).)*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1, t_2] \\ \hat{x}(t) - y_3(t), & t \in [t_1+dt_1, t_1] \\ \hat{x}(t) - z_3(t), & t \in [t_2, t_2+dt_2] \end{cases} \quad (3.21)$$

where $y_3(t)$ is the solution of

$$\begin{cases} \dot{y}_3(t) = f_2\big(y_3(t),t\big), & t \in [t_1+dt_1, t_1] \\ y_3(t_1) = x(t_1) \end{cases} \quad (3.22)$$

and $z_3(t)$ is the solution of

$$\begin{cases} \dot{z}_3(t) = f_2\big(z_3(t),t\big), & t \in [t_2, t_2+dt_2] \\ z_3(t_2) = x(t_2). \end{cases} \quad (3.23)$$

Case 4: *$dt_1 < 0, dt_2 < 0$ (see figure 1(d).)*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1, t_2+dt_2] \\ \hat{x}(t) - y_4(t), & t \in [t_1+dt_1, t_1] \\ z_4(t) - x(t), & t \in [t_2+dt_2, t_2] \end{cases} \quad (3.24)$$

where $y_4(t)$ is the solution of

$$\begin{cases} \dot{y}_4(t) = f_2\big(y_4(t),t\big), & t \in [t_1+dt_1, t_1] \\ y_4(t_1) = x(t_1) \end{cases} \quad (3.25)$$

and $z_4(t)$ is the solution of

$$\begin{cases} \dot{z}_4(t) = f_2\big(z_4(t),t\big), & t \in [t_2+dt_2, t_2] \\ z_4(t_2+dt_2) = \hat{x}(t_2+dt_2). \end{cases} \quad (3.26)$$

□

Remark 3.1 $\delta x(t)$ defines the difference between $\hat{x}(t)$ and $x(t)$ in the interval where subsystem 2 is active. Moreover, by extending $\hat{x}$ and x under subsystem 2 dynamics to $\min\{t_1, t_1+dt_1\} \le t \le \max\{t_2, t_2+dt_2\}$ where at least one of $\hat{x}(t)$ and $x(t)$ evolves along subsystem 2, we can also define $\delta x(t)$ in that interval. □

Lemma 3.1 *The expressions of $\delta x(t_2)$, $\delta x(t_2+dt_2)$, and $dx(t_2)$ (i.e., $\hat{x}(t_2+dt_2) - x(t_2)$) are*

$$\delta x(t_2) = A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1), \quad (3.27)$$

$$\begin{aligned} \delta x(t_2+dt_2) &= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 \\ &+ f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1dt_2 \\ &+ (\textit{terms in } dt_1^2, dt_2^2 \textit{ and higher order terms}), \end{aligned} \quad (3.28)$$

$$\begin{aligned} dx(t_2) &= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 \\ &+ f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1dt_2 + f^{2-}dt_2 \\ &+ (\textit{terms in } dt_1^2, dt_2^2 \textit{ and higher order terms}), \end{aligned} \quad (3.29)$$

where $A(t_2,t_1)$ is the state transition matrix for the variational time-varying equation $\dot{y}(t) = \frac{\partial f(x(t),t)}{\partial x} y(t)$ for $y(t)$ from t_1 to t_2; here f is the corresponding active subsystem vector field (here it is f_2) in $[t_1,t_2]$ and $x(t)$ is the current nominal state trajectory.

Proof: See [9]. □

Remark 3.2 In the expression of $dx(t_2)$, we deliberately express the term $f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2$ explicitly because it will contribute to the coefficient of dt_1dt_2 as can be seen below. □

Equipped with Lemma 3.1, we are ready to derive the coefficient for dt_1dt_2 in the expansion of

$$\begin{aligned}J(t_1+dt_1,t_2+dt_2) = \int_{t_0}^{t_2+dt_2} L\big(\hat{x}(t),t\big)\ dt \\ +J^2\big(\hat{x}(t_2+dt_2),t_2+dt_2\big).\end{aligned} \tag{3.30}$$

For the first term in (3.30), we have

Lemma 3.2 *The contribution of* $\int_{t_0}^{t_2+dt_2} L(\hat{x},t)\ dt$ *to the coefficient of* dt_1dt_2 *is*

$$L_x^2 A(t_2,t_1)(f^{1-}-f^{1+}). \tag{3.31}$$

Proof: See [9]. □

For the second term in (3.30), similar to the single switching case, we can obtain its Taylor expansion as

$$\begin{aligned}J^2\big(\hat{x}(t_2+dt_2),t_2+dt_2\big) = J^2 + J_x^2 dx(t_2) \\ +J_{t_2}^2 dt_2 + \tfrac{1}{2}\big(dx(t_2)\big)^T J_{xx}^2 dx(t_2) + \tfrac{1}{2}J_{t_2t_2}^2 dt_2^2 \\ +dt_2 J_{t_2x}^2 dx(t_2) + (\text{higher order terms}).\end{aligned} \tag{3.32}$$

In (3.32), the terms that will possibly contribute to the coefficient of dt_1dt_2 are those containing $dx(t_2)$. They are $J_x^2 dx(t_2)$, $\frac{1}{2}\big(dx(t_2)\big)^T J_{xx}^2 dx(t_2)$, and $dt_2 J_{t_2x}^2 dx(t_2)$. Substituting the expression of $dx(t_2)$ into these terms and summing them, we obtain the contribution of the second term to the coefficient of dt_1dt_2 as

$$(J_x^2 f_x^{2-} + (f^{2-})^T J_{xx}^2 + J_{t_2x}^2)A(t_2,t_1)(f^{1-}-f^{1+}). \tag{3.33}$$

Summing (3.31) and (3.33) and also substituting into the sum the expression of $J_{t_2x}^2$ which can be obtained similarly to the expression of $J_{t_1x}^1$ in (3.9), we conclude that the coefficient of dt_1dt_2 is

$$\begin{aligned}J_{t_1t_2} = \big(J_x^2(f_x^{2-}-f_x^{2+}) + (f^{2-} \\ -f^{2+})^T J_{xx}^2\big)A(t_2,t_1)(f^{1-}-f^{1+}).\end{aligned} \tag{3.34}$$

Remark 3.3 The above result still holds even when $t_1 = t_2$. □

The above derivations can similarly be extended to the case of K switchings as follows.

Theorem 3.1 *For a switched system with* K *switchings,*

$$\begin{aligned}&J(t_1+dt_1,t_2+dt_2,\cdots,t_K+dt_K) \\ &= J(t_1,t_2,\cdots,t_K) + \textstyle\sum_{k=1}^K J_{t_k}dt_k + \frac{1}{2}\sum_{k=1}^K J_{t_kt_k}dt_k^2 \\ &+ \textstyle\sum_{1\le k<l\le K} J_{t_kt_l}dt_kdt_l + (\textit{higher order terms})\end{aligned} \tag{3.35}$$

where

$$J_{t_k} = J_x^k(f^{k-}-f^{k+}), \tag{3.36}$$

$$\begin{aligned}J_{t_kt_k} = J_x^k(f_t^{k-}-f_t^{k+}) - (J_x^k f_x^{k+} + L_x^k)(f^{k-} \\ -f^{k+}) + J_x^k(f_x^{k-}-f_x^{k+})f^{k-} \\ +(f^{k-}-f^{k+})^T J_{xx}^k(f^{k-}-f^{k+})\end{aligned} \tag{3.37}$$

$$\begin{aligned}J_{t_kt_l} = \big(J_x^l(f_x^{l-}-f_x^{l+}) + (f^{l-} \\ -f^{l+})^T J_{xx}^l\big)A(t_l,t_k)(f^{k-}-f^{k+}).\end{aligned} \tag{3.38}$$

□

3.3 Computation of $A(t_l,t_k)$, J_x^k, and J_{xx}^k

In order to use Theorem 3.1 to compute the values of J_{t_k}, $J_{t_kt_k}$ and $J_{t_kt_l}$, numerical methods need to be used to compute the values of $A(t_l,t_k)$, J_x^k and J_{xx}^k.

First note that $A(t_l,t_k)$ is the state transition matrix for $\dot{y}(t) = \frac{\partial f(x,t)}{\partial x}y(t)$ where f is the vector field of the corresponding active subsystem at each time instant (i.e., $f = f_j$ for $t \in [t_{j-1},t_j)$, $j = k+1,\cdots,l$). To find its value, we can first find the solution $y^{(1)}(t)$,$\cdots$,$y^{(n)}(t)$ corresponding to initial conditions $y^{(1)}(t_k) = e_1$, $\cdots$, $y^{(n)}(t_k) = e_n$ respectively, where e_j is the unit column vector with all 0's except that the j-th element being 1, $j = 1,2,\cdots,n$. From linear systems theory, $A(t_l,t_k)$ is equal to the square matrix whose j-th column is $y^{(j)}(t_l)$, i.e.

$$A(t_l,t_k) = [y^{(1)}(t_l),\cdots,y^{(n)}(t_l)]. \tag{3.39}$$

To obtain the value of J_x^k, note that

$$J^k\big(x(t_k),t_k\big) = \psi\big(x(t_f)\big) + \int_{t_k}^{t_f} L\big(x(t),t\big)\ dt. \tag{3.40}$$

If $x(t_k)$ has a variation $\delta x(t_k)$, then it can be shown that

$$\begin{aligned}&J^k\big(x(t_k)+\delta x(t_k),t_k\big) = J^k\big(x(t_k),t_k\big) \\ &+\Big(\psi_x\big(x(t_f)\big)A(t_f,t_k) + \textstyle\int_{t_k}^{t_f} L_x(x,t)A(t,t_k)\ dt\Big)\delta x(t_k) \\ &+\big(\text{higher order terms in } \delta x(t_k)\big).\end{aligned} \tag{3.41}$$

Hence

$$J_x^k = \psi_x\big(x(t_f)\big)A(t_f,t_k) + \int_{t_k}^{t_f} L_x(x,t)A(t,t_k)\ dt. \tag{3.42}$$

Now if we apply the similar procedure by varying $x(t_k)$ as in (3.42) to $J_x^k\big(x(t_k),t_k\big)$, we can obtain

$$\begin{aligned}J_{xx}^k = A^T(t_f,t_k)\psi_{xx}\big(x(t_f)\big)A(t_f,t_k) \\ +\textstyle\int_{t_k}^{t_f} A^T(t,t_k)L_{xx}(x,t)A(t,t_k)\ dt.\end{aligned} \tag{3.43}$$

From the above discussions, we find that $A(t_l,t_k)$ can be obtained by solving ODEs $\dot{y}(t) = \frac{\partial f(x,t)}{\partial x}y(t)$ along with initial conditions $y^{(j)}(t_k) = e_j$. $A(t_f,t_k)$ can be obtained in the same fashion. J_x^k and J_{xx}^k are in integral forms (3.42) and (3.43) which can easily be rewritten as

$$J_x^k = \psi_x\big(x(t_f)\big)A(t_f,t_k) + \eta_1(t_f), \tag{3.44}$$

$$J_{xx}^k = A^T(t_f,t_k)\psi_{xx}\big(x(t_f)\big)A(t_f,t_k) + \eta_2(t_f). \tag{3.45}$$

with η_1 and η_2 satisfying the following initial value ODEs

$$\dot{\eta}_1 = L_x(x,t)A(t,t_k),\ \eta_1(t_k) = 0_{1\times n}, \tag{3.46}$$

$$\dot{\eta}_2 = A^T(t,t_k)L_{xx}(x,t)A(t,t_k),\ \eta_2(t_k) = 0_{n\times n} \tag{3.47}$$

Remark 3.4 (Computational Cost) The above method we propose reduces the computation of $A(t_l,t_k)$ to solving initial value ODEs for any $k<l$ and the computation of J_x^k and J_{xx}^k to solving initial value ODEs (3.46)-(3.47) for all k. Hence we altogether need to solve $\frac{(K-1)K}{2} + K = \frac{K(K+1)}{2}$ sets of initial value ODEs. With today's powerful ODE solvers (e.g., `ode45` function in MATLAB), these equations can be solved efficiently and accurately. □

4 General Quadratic Problems for Switched Linear Autonomous Systems

In this section, we apply the approach in Section 3 to a special class of problems, namely, general quadratic problems for switched autonomous linear systems.

Problem 4.1 *Consider a switched system with linear autonomous subsystems $\dot{x} = A_i x, i \in I$. Given a prespecified sequence of active subsystems $(1, 2, \cdots, K, K+1)$, find optimal switching instants $t_1, \cdots, t_K$ ($t_0 \le t_1 \le \cdots \le t_K \le t_f$) such that the cost*

$$\begin{aligned} J = &\tfrac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + W_f \\ &+ \int_{t_0}^{t_f} \left(\tfrac{1}{2}\big(x(t)\big)^T Q x(t) + M x(t) + W\right) dt \end{aligned} \tag{4.1}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; Q_f, M_f, W_f, Q, M, W are matrices of appropriate dimensions with $Q_f \ge 0$, $Q \ge 0$. □

For Problem 4.1, we can observe that for any $k < l$

$$A(t_l, t_k) = e^{A_l(t_l - t_{l-1})} \cdots e^{A_{k+1}(t_{k+1} - t_k)}. \tag{4.2}$$

The computation of J_x^k and J_{xx}^k is discussed next. Assume a nominal $\hat{t}$ is given. If for any $x \in \mathbb{R}^n$ and any $t \in [t_0, t_f]$ we denote by $\tilde{J}(x,t)$ the cost incurred if the system starts from the state x at time instant t and evolves according to the portion of the switching sequence generated by $\hat{t}$ in $[t, t_f]$, i.e.,

$$\begin{aligned} \tilde{J}(x,t) = &\tfrac{1}{2}\big(x(t_f)\big)^T Q_f x(t_f) + M_f x(t_f) + W_f \\ &+ \int_t^{t_f} \left(\tfrac{1}{2}\big(x(\tau)\big)^T Q x(\tau) + M x(\tau) + W\right) d\tau \end{aligned} \tag{4.3}$$

where $x(t) = x$. Dynamic programming approach similar to (3.8) can be applied to $\tilde{J}(x,t)$ to obtain

$$\tilde{J}(x,t) = \frac{1}{2}x^T P(t) x + S(t) x + T(t) \tag{4.4}$$

where $P(t) = P^T(t)$ and

$$-\dot{P} = PA + A^T P + Q, \quad P(t_f) = Q_f, \tag{4.5}$$

$$-\dot{S} = SA + M, \quad S(t_f) = M_f, \tag{4.6}$$

$$-\dot{T} = W, \quad T(t_f) = W_f, \tag{4.7}$$

where $A = A(t)$ equals the A_i of the corresponding active subsystem at each time instant t.

Note that if $\hat{t}$ is fixed, we have

$$J^k\big(x(t_k), t_k, \cdots, t_K\big) = \tilde{J}\big(x(t_k), t_k\big), \tag{4.8}$$

$$J_x^k = \tilde{J}_x\big(x(t_k), t_k\big) = \big(x(t_k)\big)^T P(t_k) + S(t_k), \tag{4.9}$$

$$J_{xx}^k = \tilde{J}_{xx}\big(x(t_k), t_k\big) = P(t_k). \tag{4.10}$$

Remark 4.1 (Computational Cost) $A(t_l, t_k)$'s can be computed using (4.2) without solving ODEs. The computation of J_x^k and J_{xx}^k using (4.9) and (4.10) relies on the values of $P(t_k)$'s and $S(t_k)$'s which are easy to obtain by solving the initial value ODEs (4.5)-(4.7) once. Therefore, due to the special structure of the problem, the computation of $A(t_l, t_k)$, J_x^k and J_{xx}^k is simplified. □

5 Reachability Problems

The above optimal control approach can also be applied to the following class of reachability problems.

Problem 5.1 (Reachability Problem) *Given a switched autonomous system, does there exist a switching sequence such that the state trajectory x departs from $x(t_0) = x_0$ and meets x_f at some t_f? Here t_0, x_0, x_f are given; t_f is not given.* □

Note that x_f is reachable from x_0, if and only if the optimal control problem with $J = \frac{1}{2}\|x(t_f) - x_f\|_2^2$ achieves minimum at $J = 0$. Here t_0, x_0, x_f are given. In particular, if a prespecified sequence of active subsystems is given, we can minimize J with respect to the switching instants and the final time t_f. For example, assume subsystem k being active in $[t_{k-1}, t_k)$ (subsystem $K+1$ in $[t_k, t_{K+1}]$ with $t_{K+1} = t_f$), the reachability problem can be formulated as an optimal control problem which seeks for optimal values of $t_1, \cdots, t_K, t_f$ such that

$$J(t_1, \cdots, t_K, t_f) = \frac{1}{2}\|x(t_f) - x_f\|_2^2 \tag{5.1}$$

is minimized. In this case, ideally the minimum cost should be 0 if x_f is reachable from x_0 by the given order of active subsystems. In practice, if the optimal value of J is found to be smaller than a predefined small tolerance $\epsilon > 0$, then we regard x_f as reachable from x_0 and regard the corresponding optimal $t_1, \cdots, t_K, t_f$ as the reachability switching instants.

To minimize $J(t_1, \cdots, t_K, t_f)$ with respect to $(t_1, \cdots, t_K, t_f)$, we can use Algorithm 2.1. To apply the algorithm, the derivatives of J first need to be computed. J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$ can be obtained using the expressions in Theorem 3.1. However, we note here since $t_{K+1} = t_f$ is free, we also need to derive J_{t_f}, $J_{t_f t_f}$ and $J_{t_k t_f}$. These values can be obtained following the idea of the derivation in Section 3 (see [9] for details). It is not difficult to show that

$$J_{t_f} = \big(x(t_f) - x_f\big)^T f^{(K+1)-}, \tag{5.2}$$

$$\begin{aligned} J_{t_f t_f} = &\big(x(t_f) - x_f\big)^T \big(f_t^{(K+1)-} \\ &+ f_x^{(K+1)-} f^{(K+1)-}\big) + (f^{(K+1)-})^T f^{(K+1)-}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} J_{t_k t_f} = &\Big(\big(x(t_f) - x_f\big)^T f_x^{(K+1)-} \\ &+ (f^{(K+1)-})^T\Big) A(t_f, t_k)(f^{k-} - f^{k+}). \end{aligned} \tag{5.4}$$

6 Examples

In this section, we present two examples to illustrate the effectiveness of the approach developed in this paper.

Example 6.1 Consider a switched autonomous system consisting of

$$\text{subsystem 1:} \begin{cases} \dot{x}_1 = x_1 + 0.5\sin x_2 \\ \dot{x}_2 = -0.5\cos x_1 - x_2 \end{cases} \tag{6.1}$$

$$\text{subsystem 2:} \begin{cases} \dot{x}_1 = 0.3\sin x_1 + 0.5 x_2 \\ \dot{x}_2 = -0.5 x_1 + 0.3\cos x_2 \end{cases} \tag{6.2}$$

$$\text{subsystem 3:} \begin{cases} \dot{x}_1 = -x_1 - 0.5\cos x_2 \\ \dot{x}_2 = 0.5\sin x_1 + x_2 \end{cases} \tag{6.3}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 2 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). Find optimal switching instants t_1, t_2 such that the cost $J = \frac{1}{2}x_1^2(3) + \frac{1}{2}x_2^2(3) + \frac{1}{2}\int_0^3 x_1^2(t) + x_2^2(t)\, dt$ is minimized. Here $x_1(0) = 1$ and $x_2(0) = 3$.

For this problem, choose initial nominal $t_1 = 1$, $t_2 = 1.5$. By using the Algorithm 2.1 (using constrained Newton's method) along with Theorem 3.1, after 9 iterations we find the optimal $t_1 = 0.5466$, $t_2 = 2.0337$ and the corresponding optimal cost 9.9933. The corresponding state trajectory is shown in Figure 2. □

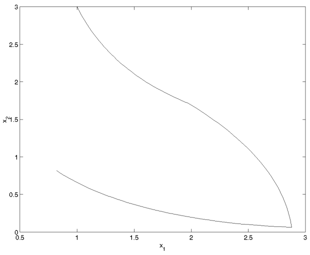

Figure 2: The state trajectory for Example 6.1.

Example 6.2 (A Reachability Problem) Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = A_1 x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x, \quad (6.4)$$

$$\text{subsystem 2: } \dot{x} = A_2 x = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x. \quad (6.5)$$

Assume that at $t_0 = 0$, the system state departs from the initial condition $x_1(0) = 1$ and $x_2(0) = 1$ and evolves following the dynamics of subsystem 1. Also assume that the system switches once at t_1 from subsystem 1 to 2. Find a t_1 and a t_f $(0 \le t_1 \le t_f)$ such that the system state arrives at $[e^3,\ e^3]^T$ at t_f.

This problem can be posed as an optimal control problem with unknown t_f and cost $J = \frac{1}{2}\big((x_1(t_f) - e^3)^2 + (x_2(t_f) - e^3)^2\big)$. Choose initial nominal $t_1 = 0.7$, $t_f = 1.7$. J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ and $J_{t_1 t_f}$ can be derived using the formulae (3.36)-(3.37) and (5.2)-(5.4). By using Algorithm 2.1 with the constrained Newton's method, after 8 iterations we find the optimal are $t_1 = 1.0000$, $t_f = 2.0000$ and the corresponding optimal cost 6.3109×10^{-29}. The corresponding state trajectory is shown in Figure 3.

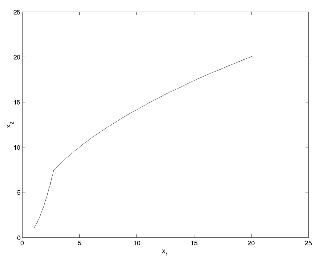

Figure 3: The state trajectory for Example 6.2.

For this example we can verify the correctness of (5.2)-(5.4). For example, the expression of $J_{t_1 t_f}$ can be derived from (5.4) as (here $K = 1$)

$$J_{t_1 t_f} = \Big(\big(x(t_f) - x_f\big)^T A_2 + \big(A_2 x(t_f)\big)^T\Big) A(t_f, t_1)(A_1 - A_2)x(t_1). \quad (6.6)$$

We can substitute $x(t_1) = [e^{t_1},\ e^{2t_1}]^T$, $x(t_f) = [e^{2t_f - t_1},\ e^{t_f + t_1}]^T$, $x_f^T = [e^3,\ e^3]$, $A(t_f, t_1) = e^{A_2(t_f - t_1)}$, and A_1, A_2 into (6.6) and obtain $J_{t_1 t_f} = -4e^{4t_f - 2t_1} + 2e^{2t_f - t_1 + 3} + 2e^{2t_f + 2t_1} - e^{t_f + t_1 + 3}$.

The correctness of $J_{t_1 t_f}$ can be verified by directly differentiating the expression $J = \frac{1}{2}\big((e^{2t_f - t_1} - e^3)^2 + (e^{t_f + t_1} - e^3)^2\big)$ and obtain the same $J_{t_f t_f}$. Similarly, we can also verify the correctness of the expressions of J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ by direct differentiations of J. □

7 Conclusion

In this paper, we proposed an approach for solving optimal control problems for switched autonomous systems with prespecified sequences of active subsystems. In particular, we derived the derivatives of the cost with respect to the switching instants and use nonlinear optimization techniques to locate the optimal switching instants. It was also shown that the computational duty can be eased for general quadratic problems for switched linear autonomous systems. Finally reachability problems were also studied using the optimal control techniques. A more detailed version of this paper can be found [9]. Further research topics include the search for optimal switching sequences when the active subsystems are not prespecified, and the application of the approach to hybrid systems with state discontinuities at the switching instants.

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