Random Walks, Resistor Networks, and Synchronization in a Noisy Environment in Weighted Complex Networks

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Real-Life Networks

- **Infrastructures:** transportation nw-s (airports, highways, roads, rail, water) energy transport nw-s (electric power, petroleum, natural gas)

- **Communications:** telephone, internet, www, etc.

- **Biology:** protein-gene interactions, protein-protein interactions, metabolic nw-s, cell-signaling nw-s, the food web, etc.

- **Social Systems:** acquaintance (friendship) nw-s, terrorist nw-s, collaboration networks, epidemic networks, the sex-web

- **Geology:** river networks

- **Quantum Gravity:** wormhole networks in space-time [Anderson&DeWitt (1986); Requardt (2000)]

"A particularly noteworthy phenomenon is the appearance of translocal bridges or short cuts connecting widely separated regions of ordinary space-time and which we expect to become relevant in various of the notorious quantum riddles."

Networks & Dynamics on Networks

- Structure of networks
  (prototypical models: Erdős-Rényi ‘60, Watts-Strogatz ‘98, Barabási-Albert ‘99, …)
- Dynamics and interacting particle-systems on static networks (Scalettar 1991, …)
- Co-evolving network structures – coupled to the dynamics on the network

Collective dynamics on the network

Examples:

- Internet (packet traffic/flux in search or routing)
- Load-balancing schemes (job allocation among processors)
- Electric power grid (voltage and phase fluctuations)
- High-performance or grid-computing networks (task-completion landscapes in distributed computing)
- Information flow, opinion dynamics in social networks
- Coupled nonlinear chaotic oscillators (neuron networks)

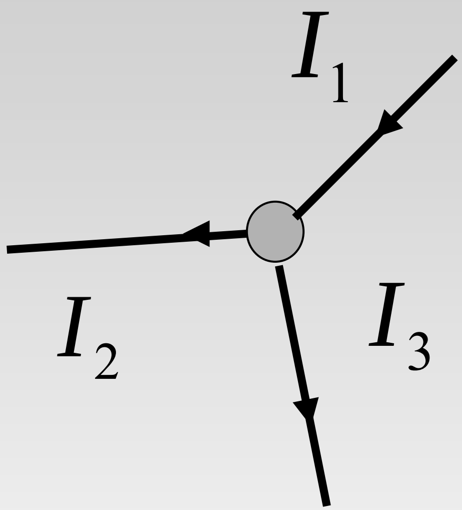
Overview

- ❖Resistor networks and random walks
  Doyle & Snell (1984); Chandra et al. (1989);
  Tetali (1990); Wu (2004);
  Lopez et al. (2005); GK et al. (2005); Gallos (2007)

- ❖ Synchronization in a noisy environment in networks
  (the Edwards-Wilkinson process on networks)
  GK et al. (2003, 2005); Guclu et al. (2006, 2007)

- ❖ Optimizing synchronization on weighted networks
  Zhou et al. (2006); Motter et al. (2004); GK (2007)

Resistor networks



G. Kirchhoff (1847)

$$I_1 + I_2 + I_3 = 0$$

G. Ohm (1827)

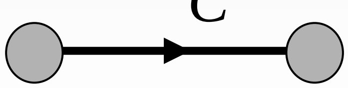
$$V = RI$$

$$(I = CV)$$

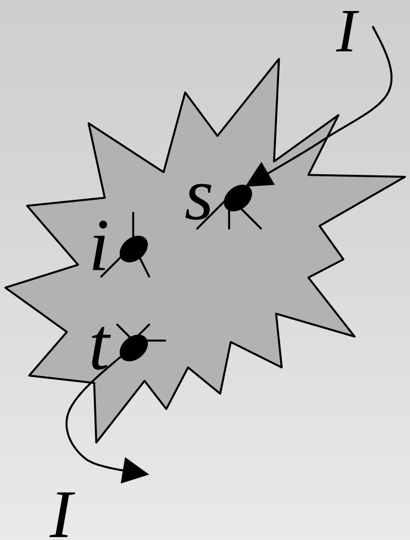
link resistance

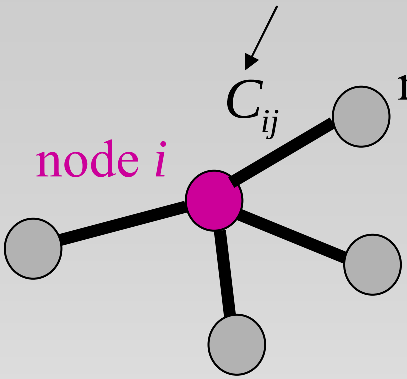
$$R = \frac{1}{C}$$

link conductance



For an arbitrary network:





$$\sum_j C_{ij}(V_i - V_j) = I(\delta_{is} - \delta_{it})$$

weighted degree: $$C_i = \sum_l C_{il}$$

Laplacian: $$\Gamma_{ij} \equiv \delta_{ij} C_i - C_{ij}$$

$$\sum_j \Gamma_{ij} V_j = I(\delta_{is} - \delta_{it})$$

Formally inverting Γ:

$$\sum_j \Gamma_{ij} V_j = I(\delta_{is} - \delta_{it})$$

$$\Gamma \boldsymbol{\psi}_k = \boldsymbol{\psi}_k \lambda_k$$

$$(k = 0, 1, 2, \ldots, N-1)$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_{N-1}$$

$$\boldsymbol{\psi}_0 = N^{-1/2}(1,1,\ldots,1)$$

pseudoinverse or Green's function:

$$G_{ij} \equiv \hat{\Gamma}^{-1}_{ij} = \sum_{k=1}^{N-1} \frac{1}{\lambda_k} \psi_{ki} \psi_{kj}$$

$$\hat{V}_i \equiv V_i - \overline{V} = I(G_{is} - G_{it})$$

$$\overline{V} = \sum_l V_l$$

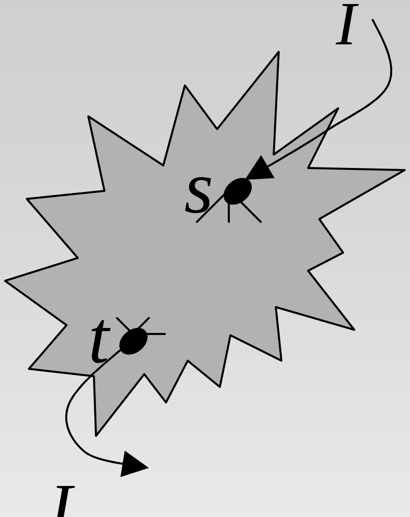
Inversion or exact numerical diag: $O(N^3)$ routines.
Conceptually useful, and can also be practically employed following exact numerical diagonalization [up to $O(10^4)$ nodes on sparse networks]

$$V_i - V_j = \hat{V}_i - \hat{V}_j = I(G_{is} - G_{it} - G_{js} + G_{jt})$$

Effective two-point resistance:

specifically, for i=s, j=t:

$$V_s - V_t = I\underbrace{(G_{ss} - G_{st} - G_{ts} + G_{tt})}_{R_{st}}$$



$$R_{st} = G_{ss} + G_{tt} - 2G_{ts}$$

$$R_{st} = \sum_{k=1}^{N-1} \frac{1}{\lambda_k}(\psi_{ks}^2 + \psi_{kt}^2 + \psi_{ks}\psi_{kt}) = \sum_{k=1}^{N-1} \frac{1}{\lambda_k}(\psi_{ks} - \psi_{kt})^2$$

$$\frac{2}{\lambda_{N-1}} \le R_{st} \le \frac{2}{\lambda_1}$$

A Global Observable: the Average System Resistance

$$\overline{R} = \frac{1}{N(N-1)} \sum_{s \neq t} R_{st} = \frac{2}{N-1} \sum_{k=1}^{N-1} \frac{1}{\lambda_k}$$

$$\max\left(\frac{2}{(N-1)\lambda_1}, \frac{2}{\lambda_{N-1}} \right) < \overline{R} \leq R_{\max} \leq \frac{2}{\lambda_1}$$

$$\overline{R} = \frac{2}{N-1} \sum_{k=1}^{N-1} \frac{1}{\lambda_k} \xrightarrow[N \to \infty]{} 2 \int \frac{\rho(\lambda) d\lambda}{\lambda}$$

$\rho(\lambda)$: density of states

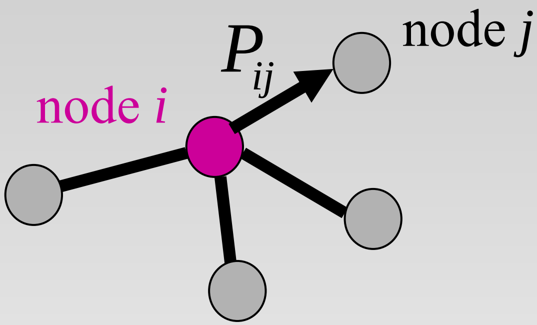
From the theory of disordered systems:

- replica method, Bray & Rodgers (1988); Kim and Kahng (2006)
- effective medium approximation, Monasson (1999); Dorogovtsev et al. (2003)

Random Walks on Networks

Doyle & Snell (1984)
Tetali (1990)

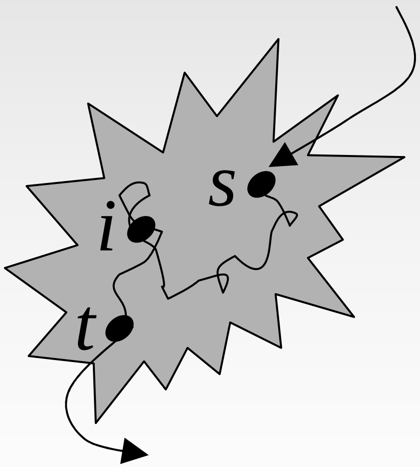
symmetric weighted edges: $C_{ij} = C_{ji}$



$prob\{i \rightarrow j\}$:

$\sum_j P_{ij} = 1$

$$P_{ij} \equiv \frac{C_{ij}}{\sum_l C_{il}} = \frac{C_{ij}}{C_i}$$



expected # of visits to node ***i***, starting at node ***s***, before reaching node ***t*** : $(i \neq s,t)$

$$E_i^{st} = \sum_j E_j^{st} P_{ji}$$

$$(E_t^{st} \equiv 0)$$

$$P_{ij} C_i = P_{ji} C_j$$

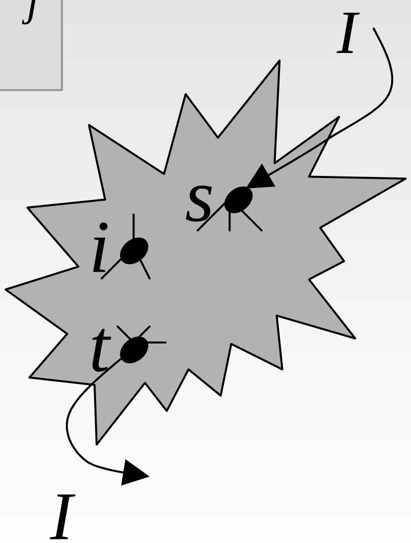
$$E_i^{st} = \sum_j E_j^{st} P_{ij} \frac{C_i}{C_j} \quad \Longrightarrow \quad \frac{E_i^{st}}{C_i} = \sum_j P_{ij} \frac{E_j^{st}}{C_j}$$

RWs and Resistor Networks

Doyle & Snell (1984)
Tetali (1990)

recall for resistor networks: $(i \neq s,t)$

$$\sum_j C_{ij}(V_i - V_j) = 0 \quad \Rightarrow \quad V_i = \sum_j P_{ij} V_j$$



E_i^{st}/C_i and V_i obey the same harmonic equation

$$E_i^{st} = C_i(V_i - V_t)$$ with $I = 1$ (unit current)

$$I_{ij} = C_{ij}(V_i - V_j) = E_i^{st} P_{ij} - E_j^{st} P_{ji}$$

RW node betweenness ("load")

expected # of visits to node ***i***, starting at node ***s***, before reaching node ***t*** :

$$E_i^{st} = C_i(V_i - V_t) \qquad (I=1)$$

$$E_i^{st} = C_i(V_i - V_t) = C_i(G_{is} - G_{it} - G_{ts} + G_{tt})$$

RW node betweness:

$$b_i \equiv \frac{1}{N(N-1)}\sum_{s\neq t} E_i^{st} = \frac{1}{2N(N-1)}\sum_{s\neq t}(E_i^{st} + E_i^{ts}) =$$

$$\frac{C_i}{2N(N-1)}\sum_{s\neq t}(G_{tt} + G_{ss} - 2G_{ts}) = \frac{C_i}{2}\overline{R}$$

local load:

$$b_i = \frac{C_i}{2}\overline{R}$$

global average load:

$$\overline{b} = \frac{1}{2N}\left(\sum_i C_i\right)\overline{R}$$

First-Passage and Commute Times in RWs

Chandra et al. (1989)
Tetali (1990)

$$E_i^{st} = C_i(V_i - V_t) = C_i(G_{is} - G_{it} - G_{ts} + G_{tt})$$

expected first-passage time:

$$\tau^{st} = \sum_i E_i^{st} = \sum_i C_i(G_{is} - G_{it} - G_{ts} + G_{tt})$$

expected commute time:

$$\tau^{st} + \tau^{ts} = \sum_i C_i(G_{ss} + G_{tt} - 2G_{st}) = (\sum_i C_i) R_{st}$$

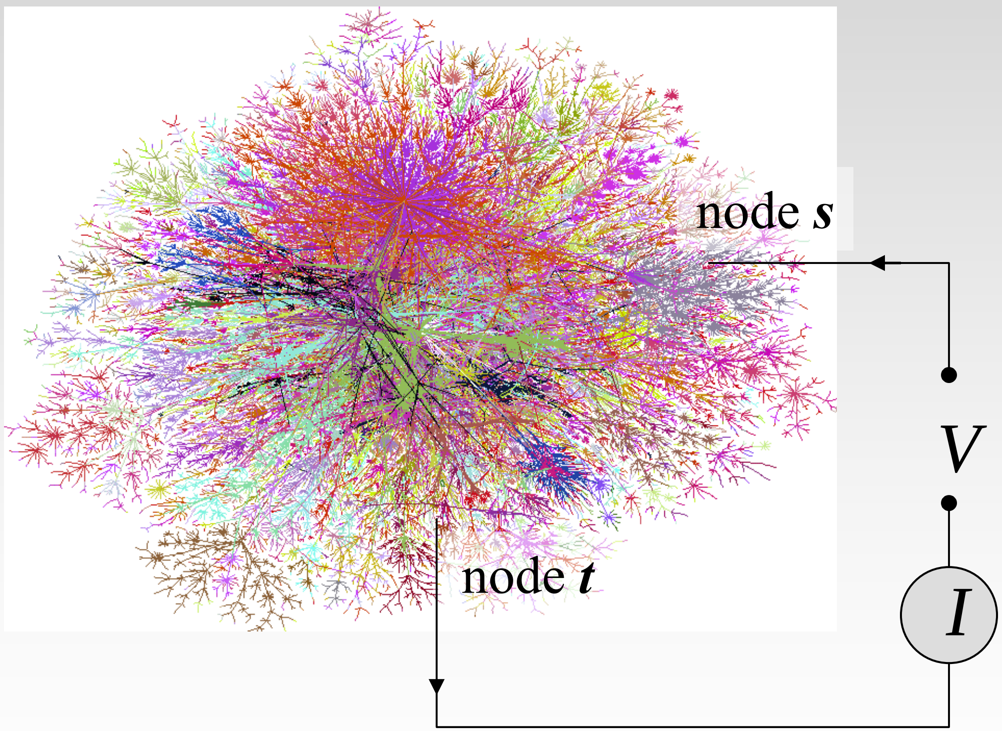
average expected first-passage time:
(averaged over all pairs of nodes
in the graph)

$$\overline{\tau} = \frac{\sum_i C_i}{2} \overline{R}$$

(specific realization of Little's law:)

$$\overline{\tau} = N\overline{b}$$

Transport and Flow in Complex Networks: Prototypical Resistor Networks



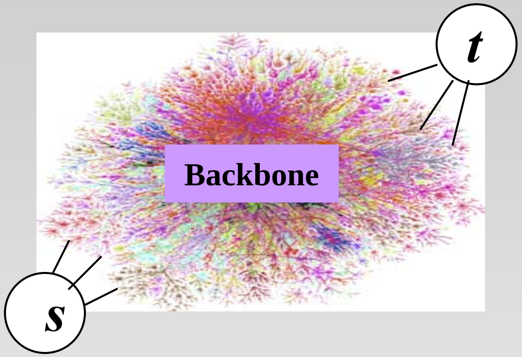
each link is an Ohmic resistor
(can be identical or weighted)

- for SF networks:
  Andrade et al., *PRL* (2005)
  López et al., *PRL* (2005)
  Gallos et al. (2007)
- for SW networks:
  GK et al., *PLA* (2005)

Effective Resistance in SF Networks

López et al., *PRL* (2005)

- anomalous transport
- scaling



scale-free degree distribution

$$P(k) \sim 1/k^{\gamma}$$

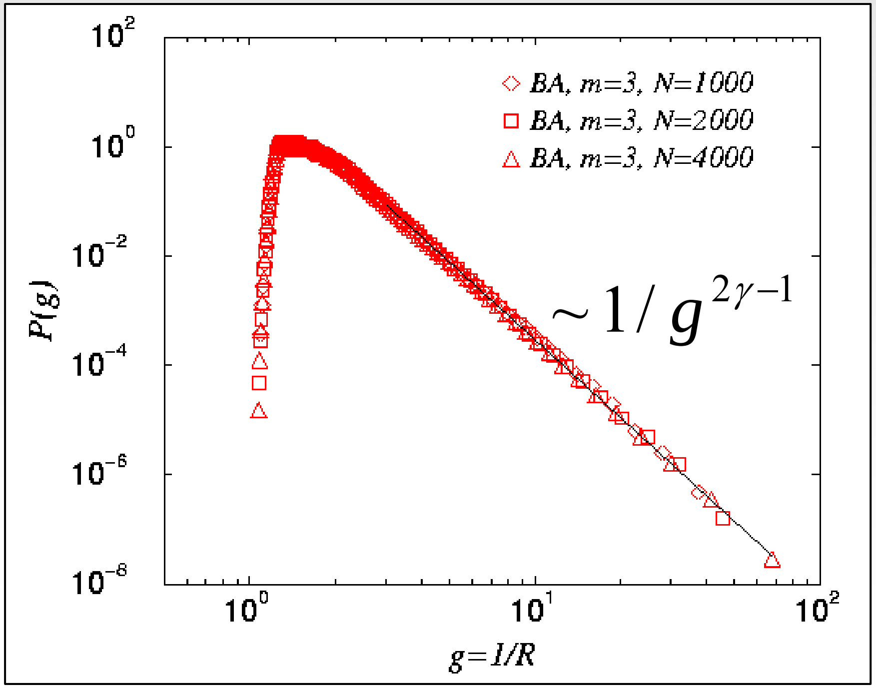
$$R_{st} \approx R_{sB} + R_B + R_{Bt}$$

$$R_{sB} \sim 1/k_s \,, \quad R_{tB} \sim 1/k_t \,, \quad R_B << R_{sB}, R_{tB}$$

$$R_{st} \approx R_{sB} + R_{Bt} \sim \frac{k_s + k_t}{k_s k_t}$$

$$g_{st} = \frac{1}{R_{st}} \sim \frac{k_s k_t}{k_s + k_t} = k_s f\left(\frac{k_t}{k_s}\right)$$

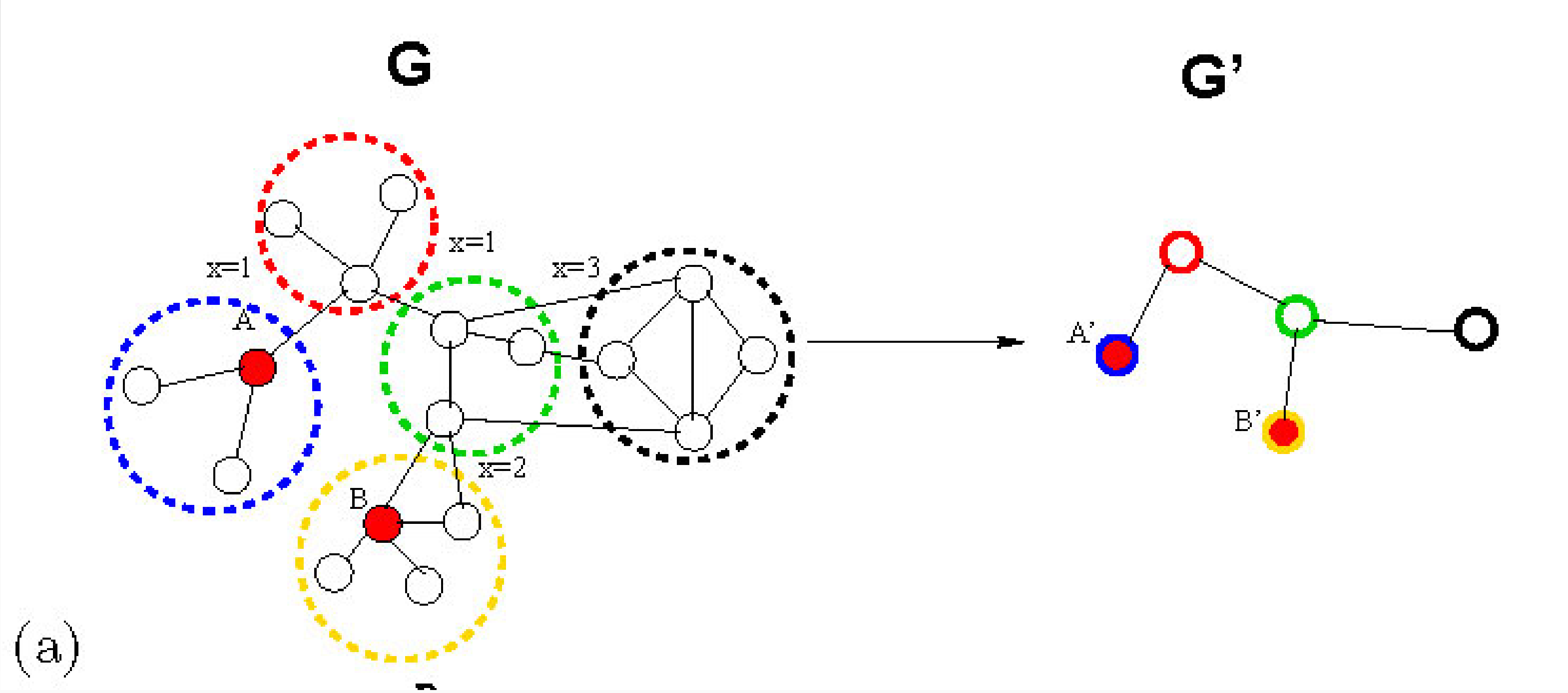
$$f(x) = \frac{x}{1+x}$$



Scaling Approach to Transport

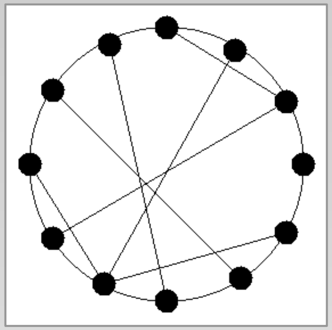
(renormalization)

Gallos et al. (2007)
cond-mat/0702151



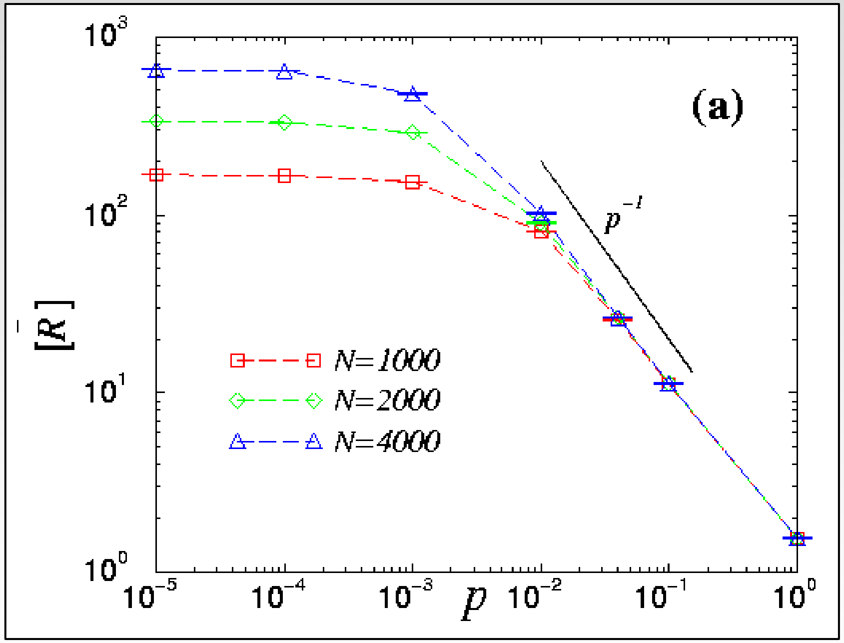
$$\Rightarrow R(l_{st}; k_s, k_t)$$

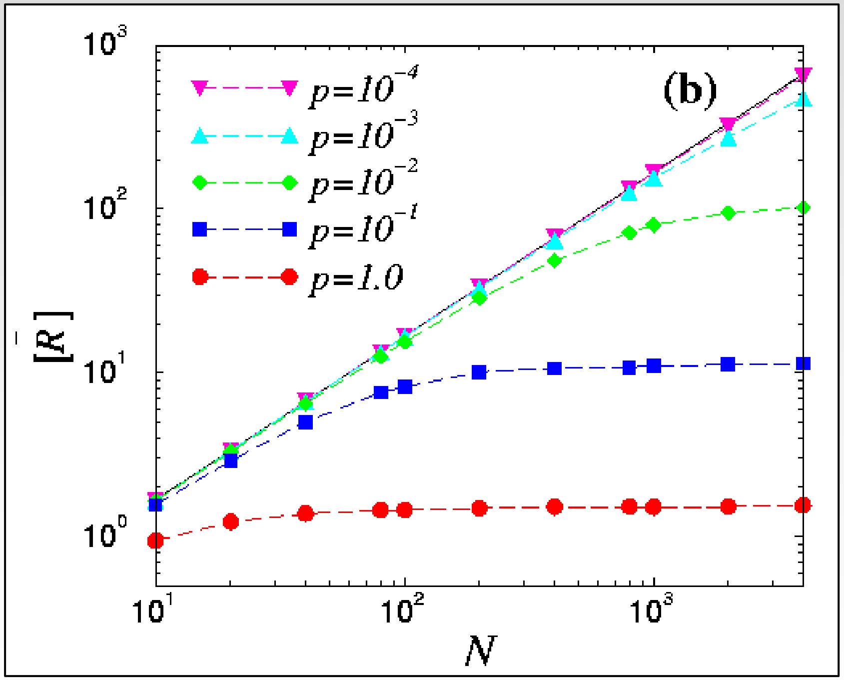
Finite-Size Effects in SW Resistor Networks



p is the density of random links

Kozma et al.(2004,2005)
GK et al. (2005)





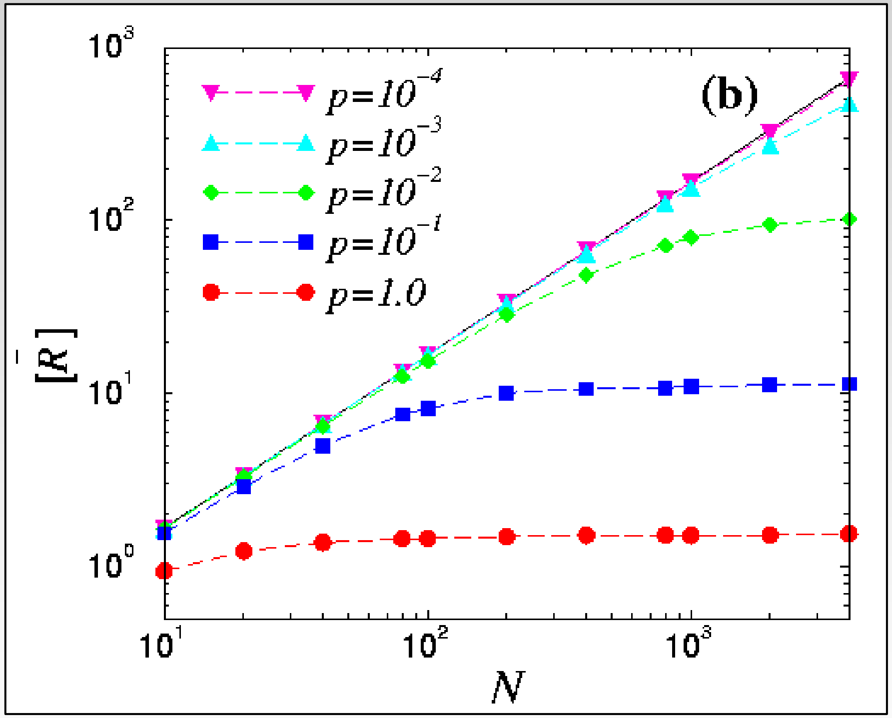
$$[\bar{R}] \sim p^{-1}$$

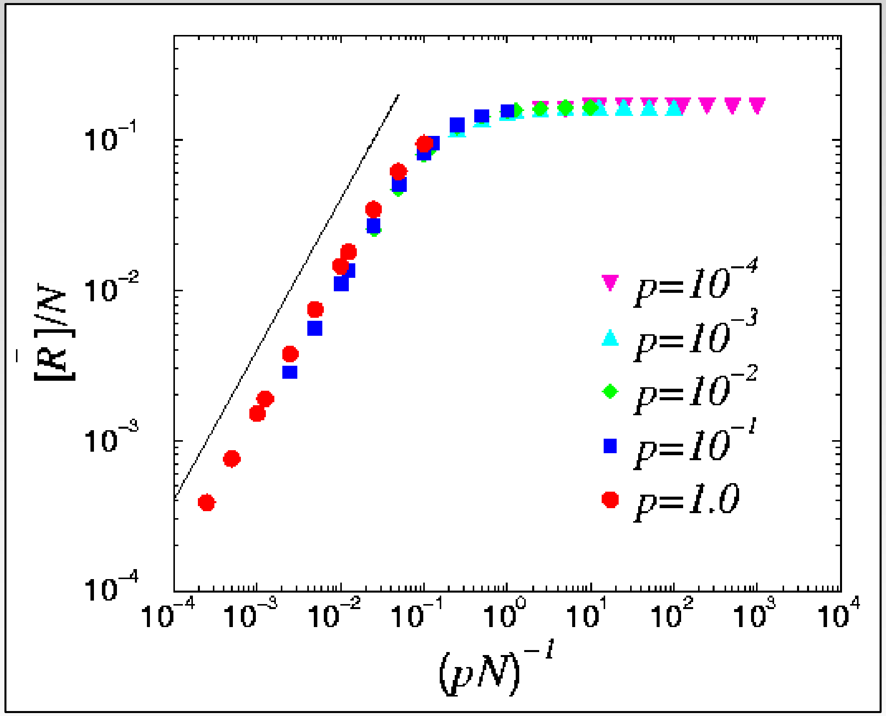
In contrast: 1d regular ring (p=0): $\bar{R} \approx N/6$

Finite-Size Scaling

two length scales: $N, \quad \xi \sim p^{-1}$

$[\bar{R}]/N$ vs ξ / N





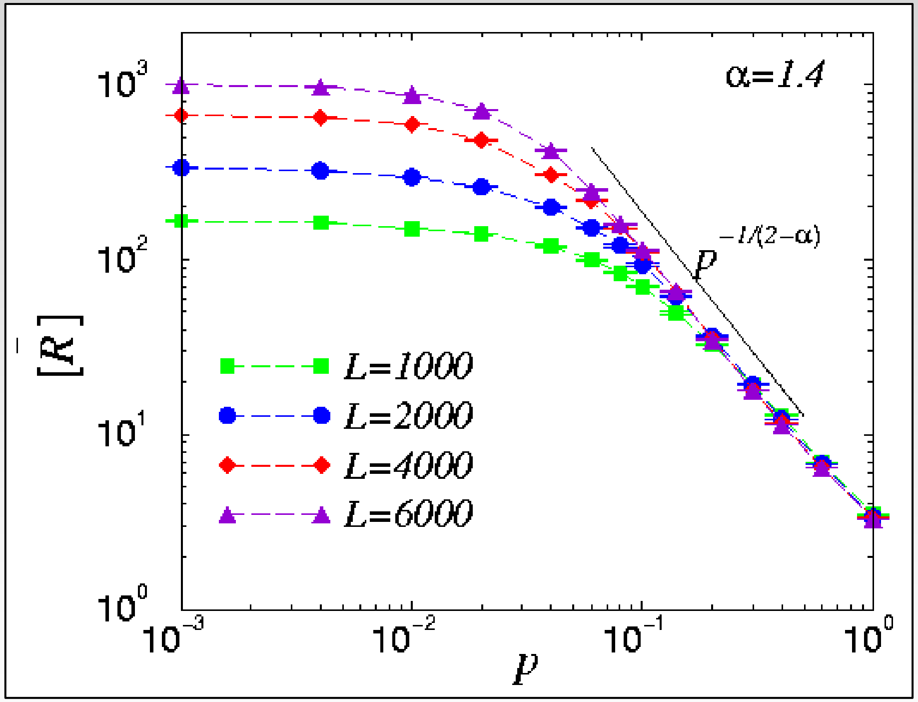
$$[\bar{R}] = N\ f(\xi / N)$$

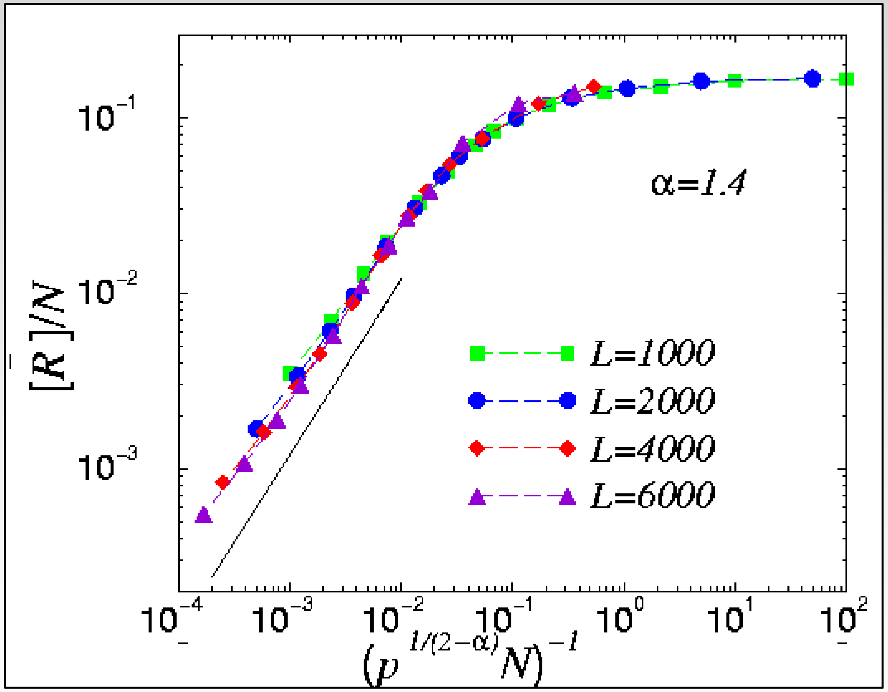
$$\xi \sim p^{-1}$$

$$f(x) \sim \begin{cases} x & \text{for} \quad x << 1 \\ const. & \text{for} \quad x >> 1 \end{cases}$$

Distance-dependent link probability

$$A_{ij} = \begin{cases} 1 & \text{with probability} \quad pc/|i-j|^{\alpha} \\ 0 & \text{with probability} \quad 1 - pc/|i-j|^{\alpha} \end{cases}$$





$$[\bar{R}] \sim p^{-1/(2-\alpha)} \qquad 1 < \alpha < 2$$

$$[\bar{R}] = N\, f(\xi / N)$$

$$\xi \sim p^{-1(2-\alpha)}$$

Application of Resistor Networks

Analyzing community structures in social networks:

- Newman (2003), Newman & Girvan (2004)
- Wu & Huberman (2004)
- discovering connection subgraphs, Faloutsos (2004)

Other applications:

- protein binding networks, Maslov et al. (2006)
- page ranking with "diodes" (directed electrical network)
  (6 billion nodes, 2 days on 60 CPUs), Kaul et al. (2007)

Synchronization in Networks

$$\partial_t \boldsymbol{x}^i = \boldsymbol{F}(\boldsymbol{x}^i) - \sigma \sum_j \Gamma_{ij} \boldsymbol{H}(\boldsymbol{x}^j) + \boldsymbol{\eta}^i(t)$$

on SF networks: Zhou and Kurth, '06

$$\partial_t \boldsymbol{x}^i = \boldsymbol{F}(\boldsymbol{x}^i) - \sigma \sum_j \Gamma_{ij} \boldsymbol{H}(\boldsymbol{x}^j)$$

on SW networks: Barahona & Pecorra, '02
on SF networks: Zhou et al., '06; Motter et al., '04; Nishikawa & Motter, '06

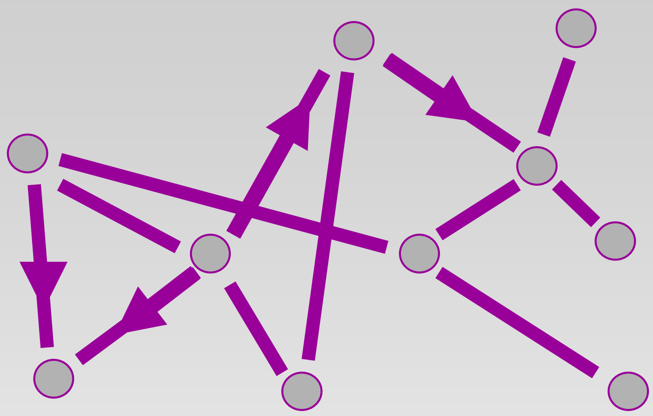
eigenvalues of the coupling matrix Γ : $\{0 = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{N-1}\}$

synchronizablity: $\alpha_1 < \sigma\lambda_i < \alpha_2 , \quad i \neq 0$

eigenratio: $\mathcal{R} \equiv \dfrac{\lambda_{N-1}}{\lambda_1}$

smaller $\mathcal{R}$ → "better" synchronization

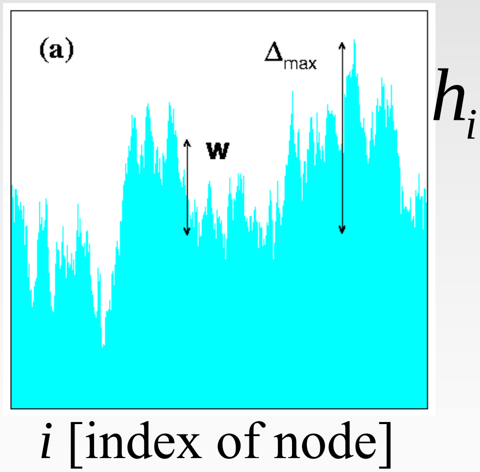
Task Completion Networks



GK et al., '03,
Kozma et al., '04, '05
Guclu et al., '04, '07

- manufacturing supply chains (Zhang et al., 2003)
- e-commerce networks (Nagurney et al., 2005)
- distributed computer networks

task-completion landscapes →



$$\langle w^2 \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} (h_i - \bar{h})^2 \right\rangle$$

❖ understanding back-log formation and worst-case delays in networked processing systems

Synchronization in Networks in a Noisy Environment: the Edwards-Wilkinson Process on a Network

$$\partial_t h_i(t) = -\sum_j C_{ij}(h_i - h_j) + \eta_i(t)$$

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\delta_{ij}\delta(t-t')$$

global observable:
(spread/width of the synchronization landscape)

$$\left\langle w(t)^2 \right\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N}(h_i - \bar{h})^2 \right\rangle$$

$$\partial_t h_i(t) = -\sum_j \Gamma_{ij} h_j + \eta_i(t)$$

$$\Gamma_{ij} \equiv \delta_{ij} C_i - C_{ij}$$

$$\bar{h} = \sum_l h_l$$

$$\left\langle w(t)^2 \right\rangle = \frac{1}{N}\sum_{l\neq 0}\frac{1}{\lambda_l}(1 - e^{-2\lambda_l t})$$

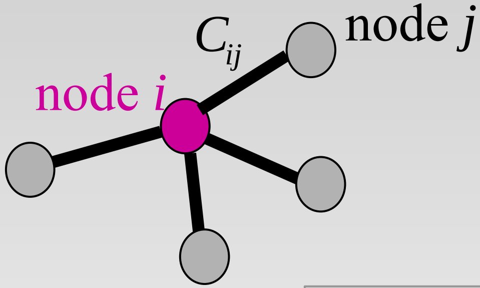
(starting from a flat landscape)

$$\left\langle w^2 \right\rangle = \left\langle w(\infty)^2 \right\rangle = \frac{1}{N}\sum_{l\neq 0}\frac{1}{\lambda_l}$$

steady-state width

Connection between the two-point resistance and the steady-state fluctuations

for *any* graph:



$$\partial_t h_i(t) = -\sum_j C_{ij}(h_i - h_j) + \eta_i(t)$$

$$\sum_j C_{ij}(V_i - V_j) = I(\delta_{is} - \delta_{it})$$

$$\Gamma_{ij} = \delta_{ij} C_i - C_{ij}$$

$$G_{ij} \equiv \hat{\Gamma}^{-1}_{ij}$$

$$\left\langle (h_s - \overline{h})(h_t - \overline{h}) \right\rangle = G_{st}$$

$$\left\langle (h_s - h_t)^2 \right\rangle = G_{ss} + G_{tt} - 2G_{st}$$

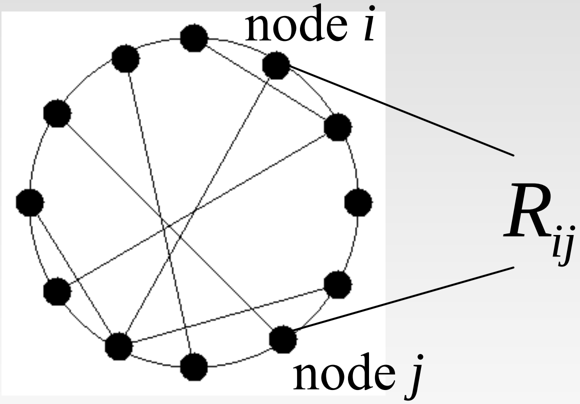
$$\hat{V}_i = I(G_{is} - G_{it})$$

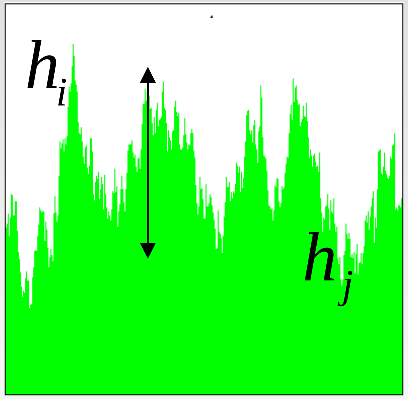
$$R_{st} = G_{ss} + G_{tt} - 2G_{st}$$

Connection between the two-point resistance and the steady-state fluctuations

$$R_{ij} = G_{ii} + G_{jj} - 2G_{ij} = \langle (h_i - h_j)^2 \rangle$$

(for *any* graph)





$$\overline{R} = \frac{2}{N(N-1)} \sum_{i<j} R_{ij} = \frac{N}{N-1} 2\langle w^2 \rangle \approx 2\langle w^2 \rangle$$

GK et al., (2005)

Optimizing Synchronization in Uncorrelated Weighted Scale-Free Networks

$$C_{ij} \propto A_{ij}(k_i k_j)^{\beta}$$

$$P(k) \propto k^{-\gamma}, \quad 2 < \gamma \leq 3$$

$$k_{\min} = m$$

"cost" constraint:

$$\sum_{i,j} C_{ij} = 2C_{\text{tot}} = \textit{fixed} = N\bar{k}$$

$$\Rightarrow \quad C_{ij} = N\bar{k}\,\frac{A_{ij}(k_i k_j)^{\beta}}{\sum_{l,n} A_{ln}(k_l k_n)^{\beta}}$$

task:

minimize the "width" $< w^2 >$ with respect to β, subject to the above constraint

Uncorrelated Scale-Free Networks

$$P(k) = (\gamma - 1)m^{\gamma - 1}k^{-\gamma}, \quad 2 < \gamma \le 3$$

$$k_{\min} = m$$

probability that an edge emanating from a node with degree *k'* connects to a node with degree *k*: $P(k|k')$

uncorrelated networks: $$P(k \mid k') = \frac{kP(k)}{\langle k \rangle}$$ (independent of *k'*)

Mean-field approximation

$$\partial_t h_i(t) = -\sum_j C_{ij}(h_i - h_j) + \eta_i(t) \approx -C_i(h_i - \bar{h}) + \eta_i$$

$$C_i \equiv \sum_j C_{ij}$$

$$\left\langle (h_i - \bar{h})^2 \right\rangle = G_{ii} \approx \frac{1}{C_i}$$

$$\left\langle (h_i - \bar{h})(h_j - \bar{h}) \right\rangle = G_{ij} << G_{ii}$$

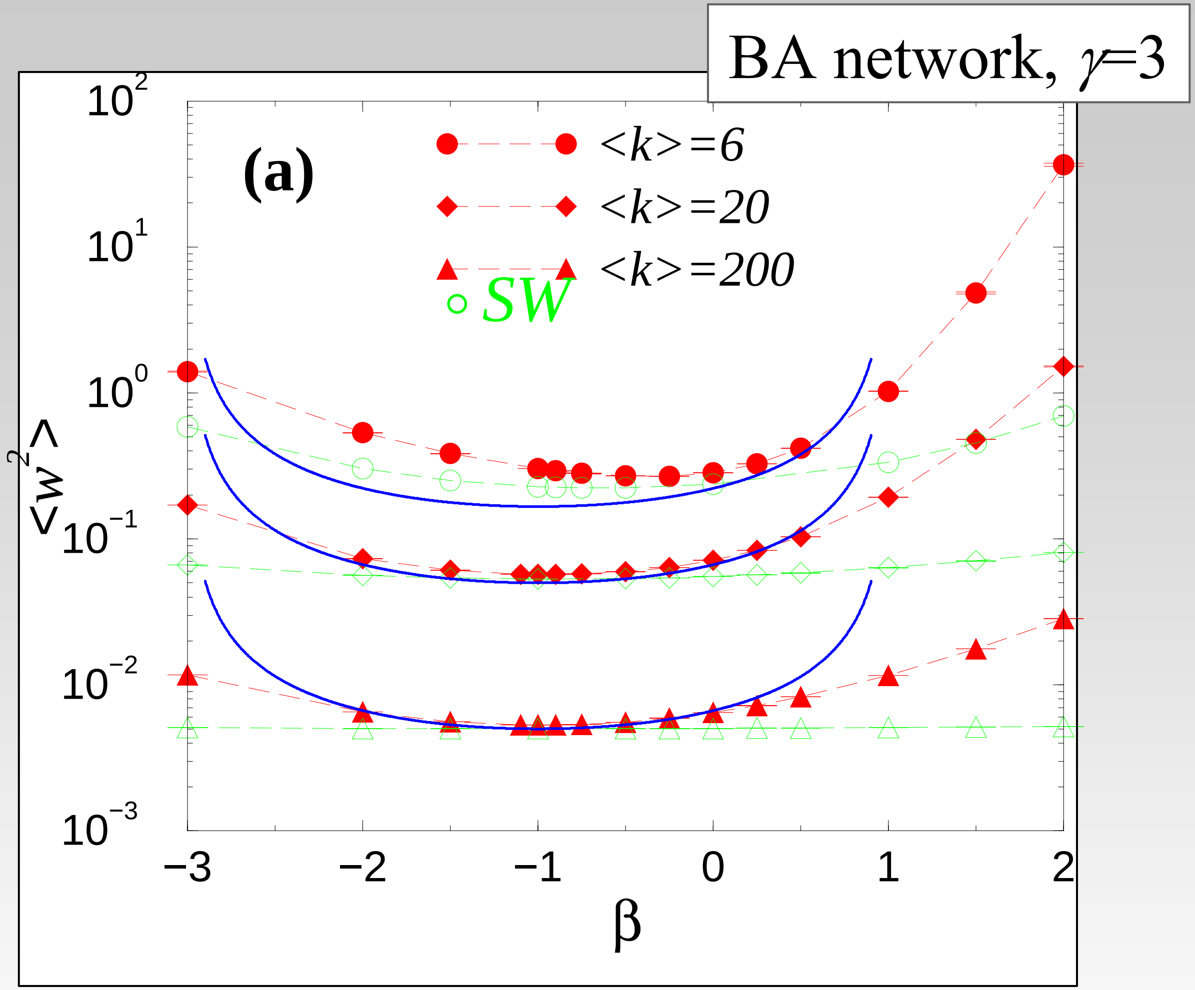
$$\left\langle w^2 \right\rangle = \frac{1}{N}\sum_i \langle (h_i - \bar{h})^2 \rangle \approx \frac{1}{N}\sum_i \frac{1}{C_i} \approx \int \frac{P(k)dk}{C(k)} \approx$$

$$\approx \frac{(\gamma-1)^2}{\langle k \rangle}\frac{1}{(\gamma-2-\beta)(\gamma+\beta)}$$

$\langle w^2 \rangle$ is min. at $\beta^* = -1$

$$\langle w^2 \rangle_{\min} = \frac{1}{\langle k \rangle}$$

$$(-\gamma < \beta < \gamma - 2)$$



$$-1 < \beta^* < 0$$

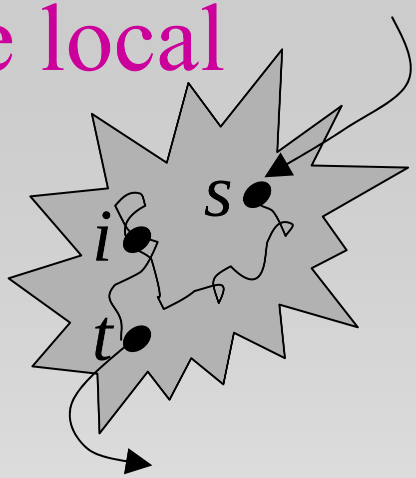
$$-1 \approx \beta^*, \quad 1 << m << N$$

also:

$$R_{ij} = G_{ss} + G_{tt} - 2G_{st} \approx G_{ss} + G_{tt} \sim \frac{k_s^{1+\beta} + k_t^{1+\beta}}{(k_s k_t)^{1+\beta}}$$

$\overline{R} \cong 2\langle w^2 \rangle$ is minimum at $\beta^* \approx -1$, in the equivalent resistor network

Connection with congestion-aware local "routing" schemes



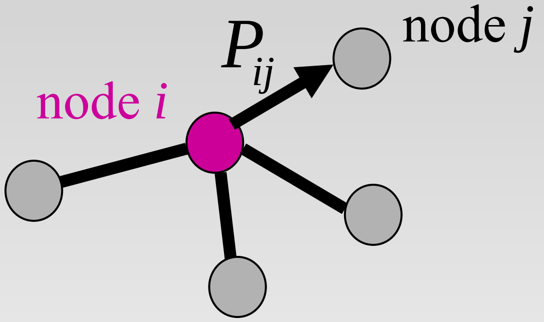
Guimerà et al. '02, Zhao et al., '05;
Danila et al., '06; Sreenivasan et al. '06

- packets are generated with ***identical rate*** φ at each node
- identical processing capabilities for each node
  (e.g., can send one packet per unit time)
- throughput is limited by the most congested node

$$b_i = \frac{C_i}{2}\overline{R}$$

$$\varphi N b_i < 1, \quad \forall i \quad \Rightarrow \quad \varphi_c = \frac{1}{N b_{max}}$$

Special case of weighted RWs:

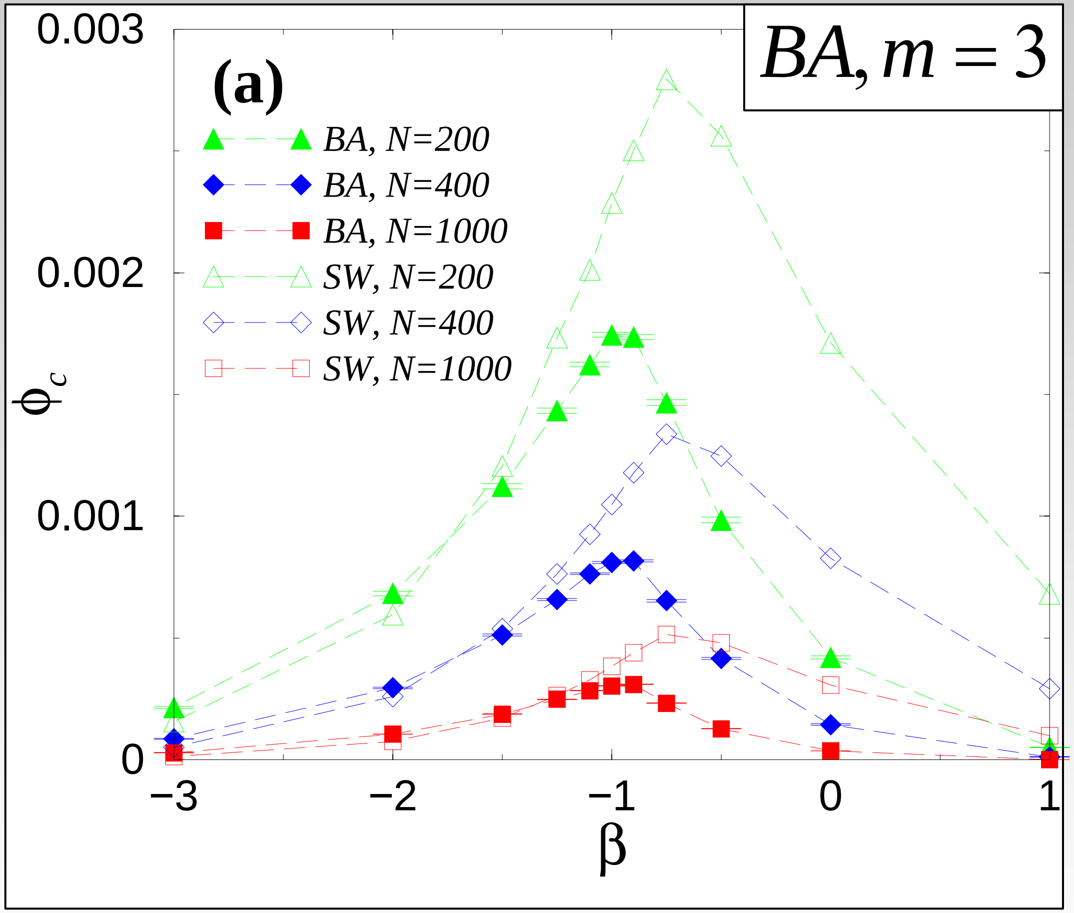


$RW:$

$prob\{i \to j\}:$

$$P_{ij} = \frac{C_{ij}}{C_i} = \frac{A_{ij}(k_i k_j)^\beta}{\sum_l A_{il}(k_i k_l)^\beta} = \frac{A_{ij} k_j^\beta}{\sum_l A_{il} k_l^\beta} \propto A_{ij} k_j^\beta$$

network throughput:



$BA, m = 3$

there exists a β^* where φ_c is maximum

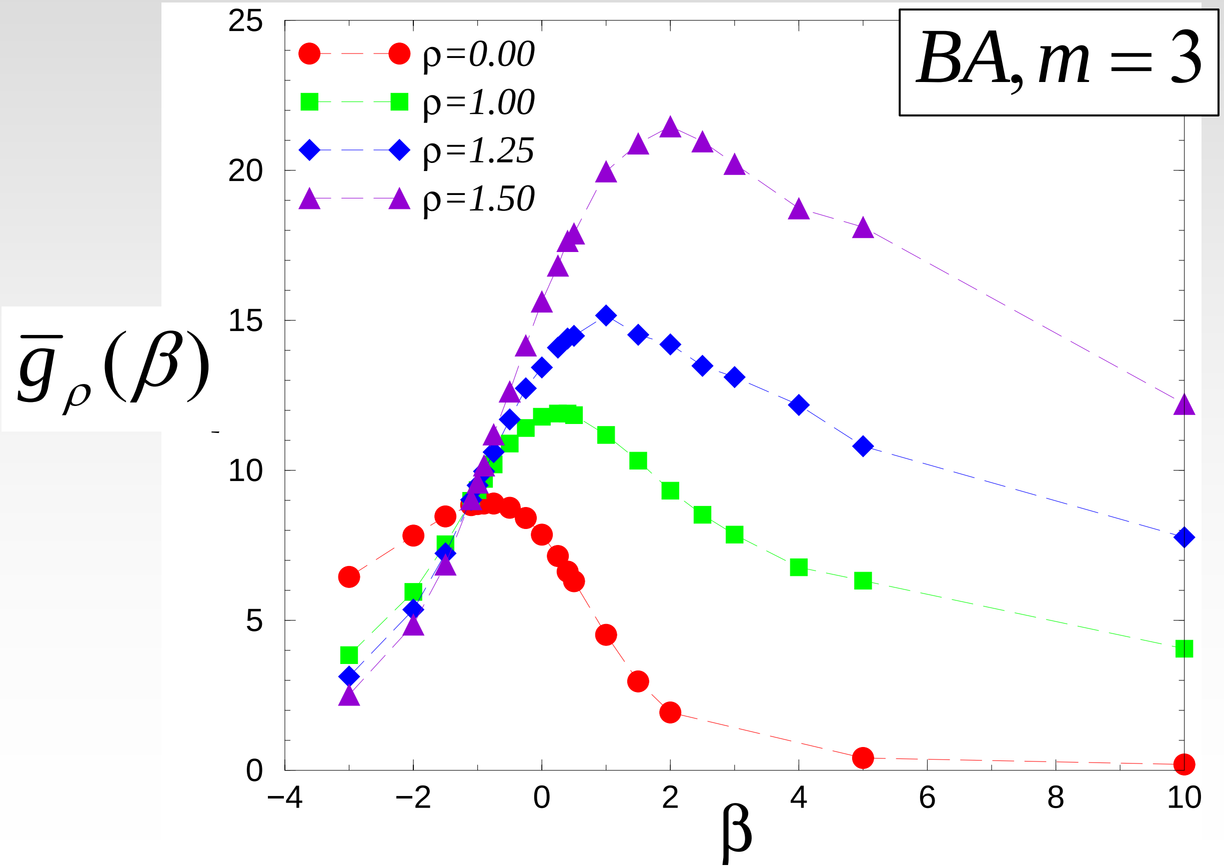
$$\beta^* \approx -1, \quad 1 << m << N$$

$$\overline{b} = \frac{1}{2N}\sum_i C_i \overline{R} = \min$$

$$\overline{\tau} = N\overline{b} = \min$$

"Re-weighting" the average two-point conductance with weighted "in/out flux"

$$\bar{g}_\rho(\beta) = \frac{1}{\mathcal{N}} \sum_{s,t} k_s^\rho k_t^\rho g_{s,t}(\beta) = \frac{1}{\mathcal{N}} \sum_{s,t} k_s^\rho k_t^\rho \frac{1}{R_{st}(\beta)}$$



$$\mathcal{N} = \sum_{s,t} k_s^\rho k_t^\rho$$

(normalization)

$$g_{st} \sim \frac{(k_s k_t)^{1+\beta}}{k_s^{1+\beta} + k_t^{1+\beta}}$$

Summary

- problems in noisy synchronization and transport/flow are intimately related
- performance can be optimized by appropriately distributing the weights in complex network

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G.K., *Phys. Rev. E* (2007); arXiv:cond-mat/0609098.

Guclu et al., Chaos (2007); arXiv:cond-mat/0701301.

G.K. et al., *Phys. Lett. A* **350**, 324 (2006); arXiv:cond-mat/0508056.

B. Kozma, M.B. Hastings, and G.K, *Phys. Rev. Lett.* **95**, 018701 (2005).

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