

# ON $K_*$ -ULTRAHOMOGENEOUS GRAPHS

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ABSTRACT. Let  $\mathcal{C}$  be any class of graphs. A graph  $G$  is  $\mathcal{C}$ -ultrahomogeneous if every isomorphism between induced subgraphs belonging to  $\mathcal{C}$  extends to an automorphism of  $G$ . We study graphs that are  $K_*$ -ultrahomogeneous, where  $K_*$  is the class of complete graphs. We also explicitly classify the graphs that are  $\sqcup K_*$ -ultrahomogeneous, where  $\sqcup K_*$  is the class of disjoint unions of complete graphs.

## 1. INTRODUCTION

A mathematical structure is called ultrahomogeneous if every isomorphism between substructures can be extended to an automorphism. In the context of graph theory, a graph  $G$  is ultrahomogeneous if every isomorphism between induced subgraphs extends to an automorphism of  $G$ . Gardiner [4] gave an explicit classification of the ultrahomogeneous graphs using the previous work of Sheehan [12].

One can vary this basic definition in many interesting ways. For example,  $G$  is homogeneous if for every pair of isomorphic subgraphs  $H_1$  and  $H_2$ , there exists an isomorphism  $H_1 \rightarrow H_2$  that extends to an automorphism of  $G$ . The difference between homogeneity and ultrahomogeneity is whether or not the isomorphism  $H_1 \rightarrow H_2$  is specified. Ronse [11] showed that a graph is homogeneous if and only if it is ultrahomogeneous. Two different variations are considered in [3] and [9].

Yet other variations arise by considering only certain types of subgraphs. A graph  $G$  is connected-ultrahomogeneous if every isomorphism between connected induced subgraphs extends to an automorphism of  $G$ . Gardiner [5] explicitly classified the connected-ultrahomogeneous graphs.

**Definition 1.1.** Let  $\mathcal{C}$  be any class of graphs closed under isomorphism. A graph  $G$  is  $\mathcal{C}$ -**ultrahomogeneous** if every isomorphism between induced subgraphs belonging to  $\mathcal{C}$  extends to an automorphism of  $G$ .

This definition recovers the original notion of ultrahomogeneity by taking  $\mathcal{C}$  to be the class of all graphs. Similarly, we recover  $\mathcal{C}$ -ultrahomogeneity by taking  $\mathcal{C}$  to be the class of all connected graphs.

If  $H$  is a given fixed graph, we say that another graph  $G$  is  $H$ -ultrahomogeneous if it is  $\{H\}$ -ultrahomogeneous. This is equivalent to requiring that the automorphism group  $\text{Aut}(G)$  of the graph  $G$  acts transitively on the set of induced subgraphs that are isomorphic to  $H$ .

In this paper, we choose a few specific examples of reasonable classes  $\mathcal{C}$  and study the  $\mathcal{C}$ -ultrahomogeneous graphs. Because complete subgraphs are a common subject of study for graph theorists (see, for example, [8] or [10]), we are interested the class

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$K_*$  of all complete graphs. Thus, we are considering graphs  $G$  such that any isomorphism between complete subgraphs of  $G$  extends to an automorphism. Note that the  $K_*$ -ultrahomogeneous graphs include all ultrahomogeneous and connected-ultrahomogeneous graphs, since these are stronger conditions than  $K_*$ -ultrahomogeneity.

Unfortunately, we are able to obtain only partial results about the  $K_*$ -ultrahomogeneous graphs. In order to prove something more substantial, we also consider the  $\sqcup K_*$ -ultrahomogeneous graphs, where  $\sqcup K_*$  is the class of graphs that are disjoint unions of complete graphs. The class  $\sqcup K_*$  consists of all graphs of the form

$$K_{r_1} \sqcup K_{r_2} \sqcup \cdots \sqcup K_{r_n}.$$

In Section 4, we explicitly classify the  $\sqcup K_*$ -ultrahomogeneous graphs. We find that the class of  $\sqcup K_*$ -ultrahomogeneous graphs is only slightly larger than the class of ultrahomogeneous graphs. This enlightens us about the nature of ultrahomogeneity for graphs. Considering all subgraphs turns out to be a highly redundant condition. The seemingly much weaker condition of  $\sqcup K_*$ -ultrahomogeneity turns out to be nearly equivalent.

In this paper, we consider ultrahomogeneity only for finite undirected graphs. However, ultrahomogeneity makes sense in many other combinatorial contexts, such as finite geometries [1] and infinite graph theory [7]. We strongly suspect that it is possible to show that ultrahomogeneity in these situations is also highly redundant.

**1.1. Notation.** For any graph  $G$  and any positive integer  $t$ , let  $tG$  be the graph that consists of  $t$  disjoint copies of  $G$ .

If  $t$  and  $n$  are positive integers, let  $K_{t;n}$  be the complete regular multipartite graph containing  $t$  partite sets, each of which has  $n$  elements. Thus,  $K_{t;n}$  is the complete regular multipartite graph with order  $nt$  and degree  $n(t-1)$ . For example;  $K_{1;n}$  is the graph  $nK_1$ , and  $K_{t;1}$  is the graph  $K_t$ .

## 2. $\mathcal{C}$ -ULTRAHOMOGENEOUS GRAPHS

We present in this section some general results about  $\mathcal{C}$ -ultrahomogeneity that will be useful later.

**Definition 2.1.** If  $\mathcal{C}$  is any class of graphs, then  $\overline{\mathcal{C}}$  is the class of graphs whose complements belong to  $\mathcal{C}$ .

For example, the class  $\overline{K_*}$  contains all graphs that consist entirely of disjoint vertices. Also, the class  $\overline{\sqcup K_*}$  consists of all multipartite complete graphs.

The following theorem will allow us to check whether a graph is  $\mathcal{C}$ -ultrahomogeneous simply by examining its complement, which in some cases is an easier task.

**Theorem 2.2.** *A graph  $G$  is  $\mathcal{C}$ -ultrahomogeneous if and only if its complement  $\overline{G}$  is  $\overline{\mathcal{C}}$ -ultrahomogeneous.*

*Proof.* We prove the backward direction. The proof in the other direction is identical because the complement of the complement of a graph is the original graph.

Suppose that  $\overline{G}$  is  $\overline{\mathcal{C}}$ -ultrahomogeneous. We need to show that  $G$  is  $\mathcal{C}$ -ultrahomogeneous. Let  $\phi : H_1 \rightarrow H_2$  be an isomorphism of subgraphs in  $G$ , where  $H_1$  and  $H_2$  belong to  $\mathcal{C}$ . Then  $\phi$  canonically determines an isomorphism  $\overline{\phi} : \overline{H_1} \rightarrow \overline{H_2}$  in  $\overline{G}$ , where  $\overline{H_1}$  and  $\overline{H_2}$  belong to  $\overline{\mathcal{C}}$ . Since  $\overline{G}$  is  $\overline{\mathcal{C}}$ -ultrahomogeneous, there exists an automorphism  $\overline{\psi}$  of  $\overline{G}$  taking  $\overline{H_1}$  to  $\overline{H_2}$ . Now  $\overline{\psi}$  canonically determines an automorphism  $\psi$  of  $G$  taking  $H_1$  to  $H_2$ , and  $\psi$  extends  $\phi$ .  $\square$

3.  $K_*$ -ULTRAHOMOGENEOUS GRAPHS

In this section, we give some examples of  $K_*$ -ultrahomogeneous graphs. Of course, any ultrahomogeneous graph or connected-ultrahomogeneous graph is  $K_*$ -ultrahomogeneous. These are not the only examples. For example, any vertex-transitive, strongly edge-transitive, triangle-free graph is  $K_*$ -ultrahomogeneous.

**Proposition 3.1.** *Let  $G$  be a  $K_*$ -ultrahomogeneous graph. Then the  $t$ -fold Cartesian product  $G^t$  is  $K_*$ -ultrahomogeneous.*

*Proof.* Suppose that  $H$  is an induced subgraph of  $G^t$  such that  $H$  is isomorphic to  $K_n$  for some  $n$ . Then  $H$  is of the form  $\{v_1\} \times \{v_2\} \times \cdots \times J \times \cdots \times \{v_{t-1}\}$ , where  $J$  is isomorphic to  $K_n$ . By permuting factors, there exists an automorphism of  $G$  taking  $H$  to  $H'$ , where  $H'$  is of the form  $J \times \{v_1\} \times \{v_2\} \times \cdots \times \{v_{t-1}\}$ . Since  $G$  is  $K_1$ -ultrahomogeneous (i.e., vertex-transitive), there exists an automorphism of  $G$  taking  $H'$  to  $H''$ , where  $H''$  is of the form  $J \times \{v\} \times \{v\} \times \cdots \times \{v\}$ , for some fixed vertex  $v$ . Finally, since  $G$  is  $K_*$ -ultrahomogeneous, there exists an automorphism taking  $H''$  to any other subgraph of the form  $L \times \{v\} \times \{v\} \times \cdots \times \{v\}$ , where  $L$  is isomorphic to  $K_n$ .  $\square$

Note that for an ultrahomogeneous graph  $G$ , the graph  $G^t$  is not necessarily ultrahomogeneous. In fact,  $K_n \times K_n$  is  $K_*$ -ultrahomogeneous but not ultrahomogeneous for  $n > 3$ . This shows that there are  $K_*$ -ultrahomogeneous graphs which are not already ultrahomogeneous.

It may seem natural to guess that  $G$  is ultrahomogeneous if it is both  $K_*$ -ultrahomogeneous and  $\overline{K}_*$ -ultrahomogeneous, but this is not true. For example,  $K_n \times K_n$  for  $n > 3$  is  $K_*$ -ultrahomogeneous and  $\overline{K}_*$ -ultrahomogeneous but not ultrahomogeneous, as will be shown below in Lemma 3.3.

**Lemma 3.2.** *The graph  $\overline{K_m \times K_n}$  is  $K_*$ -ultrahomogeneous.*

*Proof.* In the graph  $\overline{K_m \times K_n}$ , two vertices  $(v_1, w_1)$  and  $(v_2, w_2)$  are adjacent if and only if  $v_1 \neq v_2$  and  $w_1 \neq w_2$ . Let  $H$  and  $H'$  be induced subgraphs isomorphic to  $K_t$ , and let  $\{(v_1, w_1), (v_2, w_2), \dots, (v_t, w_t)\}$  and  $\{(v'_1, w'_1), (v'_2, w'_2), \dots, (v'_t, w'_t)\}$  be the sets of vertices of  $H$  and  $H'$  respectively. Let  $\phi$  be the isomorphism that takes  $(v_i, w_i)$  to  $(v'_i, w'_i)$ . Note that the vertices  $v_1, v_2, \dots, v_t$  are distinct. Similarly, the vertices  $w_1, w_2, \dots, w_t$  are distinct; the vertices  $v'_1, v'_2, \dots, v'_t$  are distinct; and the vertices  $w'_1, w'_2, \dots, w'_t$  are distinct.

Choose any permutation of the vertices of  $K_m$  that takes each  $v_i$  to  $v'_i$ , and choose any permutation of the vertices of  $K_n$  that takes each  $w_i$  to  $w'_i$ . These permutations induce an automorphism of  $\overline{K_m \times K_n}$ , and this automorphism extends  $\phi$ .  $\square$

**Lemma 3.3.** *For  $n > 3$ , the graph  $K_n \times K_n$  is  $K_*$ -ultrahomogeneous and  $\overline{K}_*$ -ultrahomogeneous but not ultrahomogeneous.*

*Proof.* The graph  $G$  is  $K_*$ -ultrahomogeneous because of Proposition 3.1. Also, by Lemma 3.2,  $\overline{K_n \times K_n}$  is  $K_*$ -ultrahomogeneous. Therefore, by Theorem 2.2,  $K_n \times K_n$  is  $\overline{K}_*$ -ultrahomogeneous. But the classification of [4] shows that  $K_n \times K_n$  is not ultrahomogeneous.  $\square$

It is reasonable to ask whether all the graphs in the class  $K_*$  are relevant. Is there any subclass  $\mathcal{C}$  of  $K_*$  such that the  $K_*$ -ultrahomogeneous graphs are precisely equal to the  $\mathcal{C}$ -ultrahomogeneous graphs? We do not know the answer, but the following example is a preliminary step.

**Example 3.4.** We construct a graph  $G$  that is  $K_1$ -ultrahomogeneous and  $K_2$ -ultrahomogeneous but *not*  $K_3$ -ultrahomogeneous. The idea to use Cayley graphs to find such an example was brought to our attention through the work of Doyle [2] who employed similar techniques to show that there are vertex-transitive, edge-transitive graphs that are not strongly edge-transitive.

Let  $A$  be the group  $\mathbb{Z}_5 \times \mathbb{Z}_5$ . Let  $S = \{(x, 0), (0, x), (x, x) \mid x \in \mathbb{Z}_5, x \neq 0\}$  be the generating set. It can be shown that the resulting Cayley graph  $G = \text{Cay}(A; S)$  is  $K_1$ -ultrahomogeneous and  $K_2$ -ultrahomogeneous. To see that  $G$  is not  $K_3$ -ultrahomogeneous, just consider the induced subgraphs  $H_1$  whose vertices are  $(0, 0)$ ,  $(1, 0)$ , and  $(1, 1)$ , and  $H_2$  whose vertices are  $(0, 0)$ ,  $(1, 0)$ , and  $(2, 0)$ . Then any isomorphism between these two subgraphs does not extend to an automorphism.

The following proposition tells us that we only need to consider connected  $K_*$ -ultrahomogeneous graphs if we wish to understand all  $K_*$ -ultrahomogeneous graphs.

**Proposition 3.5.** *If  $G$  is  $K_*$ -ultrahomogeneous, then  $G$  is isomorphic to  $tH$ , where  $H$  is a connected,  $K_*$ -ultrahomogeneous graph.*

*Proof.* Suppose  $G$  is  $K_*$ -ultrahomogeneous. The case when  $G$  is connected is trivial, so suppose  $G$  is not connected. Then  $G$  has at least two components. Suppose for sake of contradiction that  $G$  has two components  $H_1$  and  $H_2$  that are not isomorphic. Let  $v_1$  be a vertex of  $H_1$  and  $v_2$  a vertex of  $H_2$ . Since  $G$  is  $K_*$ -ultrahomogeneous, there exists an automorphism taking  $v_1$  to  $v_2$ . This is a contradiction since  $H_1$  is not isomorphic to  $H_2$ .  $\square$

#### 4. $\sqcup K_*$ -ULTRAHOMOGENEOUS GRAPHS

Let  $\mathcal{M}$  be the class of all complete (not necessarily regular) multipartite graphs  $K_{r_1, r_2, \dots, r_n}$ , with  $n \geq 1$  and each  $r_i \geq 1$ . Note that  $\mathcal{M}$  is the complement of the class  $\sqcup K_*$ . In order to understand the  $\sqcup K_*$ -ultrahomogeneous graphs, we will study the  $\mathcal{M}$ -ultrahomogeneous graphs and then apply Theorem 2.2.

Note in particular that  $K_n$  belongs to  $\mathcal{M}$  (by taking  $r_i = 1$ ) and that the disjoint union  $tK_1$  of  $t$  vertices also belongs to  $\mathcal{M}$  (by taking  $n = 1$  and  $r_1 = t$ ). Thus, every  $\mathcal{M}$ -ultrahomogeneous graph is  $K_*$ -ultrahomogeneous and also  $\overline{K}_*$ -ultrahomogeneous.

**Definition 4.1.** For any induced subgraph  $H$  of a graph  $G$ , let  $N_H(G)$  be the induced subgraph of  $G$  consisting of the vertices  $\beta$  of  $G$  such that  $\beta$  is adjacent to every vertex of  $H$  but not in  $H$ . Also, let  $\overline{N}_H(G)$  be the induced subgraph of  $G$  consisting of the vertices  $\beta$  of  $G$  such that  $\beta$  does not belong to either  $H$  or  $N_H(G)$ .

Note that when  $H$  consists just of a single vertex  $\alpha$ , then  $N_H(G)$  (which we also write as  $N_\alpha(G)$ ) is the induced subgraph on the neighbors of  $\alpha$ . Also  $\overline{N}_\alpha(G)$  is the induced subgraph on the vertices that are not adjacent to  $\alpha$ .

We will classify the  $\mathcal{M}$ -ultrahomogeneous graphs by induction on the number of vertices. The key inductive step is given by the following proposition.

**Proposition 4.2.** *If  $G$  is  $\mathcal{M}$ -ultrahomogeneous and  $H$  is an induced subgraph of  $G$  that belongs to  $\mathcal{M}$ , then  $N_H(G)$  is also  $\mathcal{M}$ -ultrahomogeneous.*

*Proof.* Let  $\phi : K \rightarrow K'$  be any isomorphism between induced subgraphs of  $N_H(G)$  such that  $K$  and  $K'$  belong to  $\mathcal{M}$ . We need to show that  $\phi$  extends to an automorphism of  $N_H(G)$ .

Let  $L$  be the induced subgraph of  $G$  consisting of the vertices of  $H$  together with the vertices of  $K$ . Define  $L'$  similarly, using the vertices of  $H$  and of  $K'$ . Then there is an isomorphism  $\tilde{\phi} : L \rightarrow L'$ ; it is the identity on the vertices of  $H$ , and it is  $\phi$  on the vertices of  $K$ .

Note that  $L$  and  $L'$  also belong to  $\mathcal{M}$ . Therefore,  $\tilde{\phi}$  extends to an automorphism  $\psi$  of  $G$  since  $G$  is  $\mathcal{M}$ -ultrahomogeneous. Since  $\psi$  fixes  $H$ , it restricts to an automorphism of  $N_H(G)$ .  $\square$

We now give the explicit classification of  $\mathcal{M}$ -ultrahomogeneous graphs.

**Theorem 4.3.** *A graph is  $\mathcal{M}$ -ultrahomogeneous if and only if it is isomorphic to:*

- (1)  $tK_n$  for  $t \geq 1$  and  $n \geq 1$ ;
- (2)  $K_{t;n}$  for  $t \geq 2$  and  $n \geq 2$ ;
- (3)  $K_n \times K_n$  for  $n \geq 3$ ; or
- (4)  $C_5$ .

We give the main steps in the proof here, but the technical details are recorded in lemmas later in this section.

*Proof.* By inspection, each listed graph is  $\mathcal{M}$ -ultrahomogeneous. For the other implication, suppose that  $G$  is  $\mathcal{M}$ -ultrahomogeneous. If  $G$  is disconnected, then  $G$  belongs to the above list by Lemma 4.5 below.

Now we may assume that  $G$  is connected. The proof is by induction on the number of vertices in  $G$ . Choose any vertex  $\alpha$  of  $G$ . By Proposition 4.2,  $N_\alpha(G)$  is again  $\mathcal{M}$ -ultrahomogeneous, and it has strictly fewer vertices than  $G$ . By the induction assumption,  $N_\alpha(G)$  belongs to the list in the statement of the theorem. Lemmas 4.7, 4.8, and 4.9 below indicate that  $N_\alpha(G)$  must be isomorphic to

- (1)  $2K_n$  with  $n \geq 2$ ; or
- (2)  $tK_1$  with  $t \geq 1$ .
- (3)  $K_{t;n}$  with  $t \geq 2$  and  $n \geq 1$ ;

The situation when  $N_\alpha(G) = K_n$  is included in Case (3). Lemmas 4.10, 4.11, and 4.12 provide an explicit description of  $G$  in cases (1), (2), and (3) respectively.  $\square$

The main point of Theorem 4.3 is to provide a classification of the  $\square K_*$ -ultrahomogeneous graphs. This is stated in the following corollary.

**Corollary 4.4.** *A graph is  $\square K_*$ -ultrahomogeneous if and only if it is isomorphic to:*

- (1)  $tK_n$  for  $t \geq 1$  and  $n \geq 1$ ;
- (2)  $K_{t;n}$  for  $t \geq 2$  and  $n \geq 2$ ;
- (3)  $\overline{K_n \times K_n}$  for  $n \geq 3$ ; or
- (4)  $C_5$ .

*Proof.* By Theorem 2.2, we just need to find the complements of the graphs listed in Theorem 4.3.  $\square$

The only difference between our classification and the classification of ultrahomogeneous graphs [4] is that the graph  $\overline{K_n \times K_n}$  is not ultrahomogeneous when  $n > 3$ .

The rest of this section is dedicated to proving the technical lemmas necessary for the proof of Theorem 4.3.

**Lemma 4.5.** *If  $G$  is disconnected and  $\mathcal{M}$ -ultrahomogeneous, then  $G$  is isomorphic to  $tK_n$  for some  $t \geq 2$  and  $n \geq 1$ .*

*Proof.* Since  $G$  is  $2K_1$ -ultrahomogeneous, every pair of non-adjacent vertices belong to distinct components of  $G$ . This implies that  $G$  is a disjoint union of complete graphs. But  $G$  is also vertex-transitive (i.e.,  $K_1$ -ultrahomogeneous), so each component has the same order.  $\square$

**Remark 4.6.** Many of the proofs that follow use similar techniques, which we introduce here. Some of these ideas were inspired by the methods of [11].

Let  $G$  be a connected  $\mathcal{M}$ -ultrahomogeneous graph. Then  $G$  is vertex-transitive (i.e.,  $K_1$ -ultrahomogeneous), so the graph  $N_\alpha(G)$  is independent (up to isomorphism) of a choice of vertex  $\alpha$  in  $G$ .

Let  $\beta$  be any vertex of  $\overline{N}_\alpha(G)$  (i.e.,  $\beta$  is not adjacent to  $\alpha$ ). Since  $G$  is connected and since  $G$  is  $2K_1$ -ultrahomogeneous, the distance between any two non-adjacent vertices must equal 2. In other words,  $\alpha$  and  $\beta$  have at least one common neighbor. In fact, every pair of non-adjacent vertices has the same number of common neighbors. Let  $m$  be this number; note that it is at least one.

Let  $r$  be the number of vertices in  $\overline{N}_\alpha(G)$  (i.e., the number of vertices not adjacent to  $\alpha$ ). The total number of paths of length 2 from  $\alpha$  to  $\overline{N}_\alpha(G)$  is equal to  $rm$ .

Now let  $d$  be the number of vertices in  $N_\alpha(G)$  (i.e., the degree of  $\alpha$ ) and let  $k$  be the degree of each vertex in  $N_\alpha(G)$  (i.e., the number of copies of  $K_3$  containing any given edge). Then each vertex in  $N_\alpha(G)$  has exactly  $d - k - 1$  neighbors in  $\overline{N}_\alpha(G)$ , so the total number of paths of length 2 from  $\alpha$  to  $\overline{N}_\alpha(G)$  is equal to  $d(d - k - 1)$ . Thus,

$$(4.1) \quad rm = d(d - k - 1).$$

We will use this equation frequently in the following proofs.

The graph  $H = N_{\alpha,\beta}(G)$  does not depend (up to isomorphism) on the choice of  $\beta$  in  $\overline{N}_\alpha(G)$  because  $G$  is  $2K_1$ -ultrahomogeneous. Let  $s$  be the number of vertices in  $\overline{N}_\alpha(G)$  that are adjacent to every vertex of  $H$ . Since  $\beta$  is such a vertex,  $s \geq 1$ . The graph spanned by  $\alpha$  together with  $H$  might belong to  $\mathcal{M}$ ; in practice, this condition will be satisfied in all the cases that we study. Then  $s$  is also independent of the choice of  $\beta$ . If  $p$  is the number of subgraphs of  $N_\alpha(G)$  that are isomorphic to  $H$ , then

$$(4.2) \quad r = sp.$$

Let  $v$  be the number of vertices in  $G$ , and let  $q$  be the number of copies of  $K_n$  in the subgraph  $N_\alpha(G)$ . Then  $\alpha$  is contained in exactly  $q$  copies of  $K_{n+1}$ . Since  $G$  is  $K_1$ -ultrahomogeneous, every vertex is contained in exactly  $q$  copies of  $K_{n+1}$ . Therefore,  $qv$  equals  $n + 1$  times the number of copies of  $K_{n+1}$  in  $G$ , so

$$(4.3) \quad n + 1 \text{ divides } qv.$$

**Lemma 4.7.** *If  $G$  is connected and  $\mathcal{M}$ -ultrahomogeneous and  $\alpha$  is any vertex of  $G$ , then  $N_\alpha(G)$  is not isomorphic to  $C_5$ .*

*Proof.* For sake of contradiction, suppose that  $N_\alpha(G)$  is isomorphic to  $C_5$ . Now  $G$  is regular of degree 5, and  $G$  has  $6 + r$  vertices. Equation 4.1 tells us that  $rm = 10$ , so  $r$  is a divisor of 10. Letting  $n = 2$ , Equation 4.3 tells us that 3 divides  $5(6 + r)$ , so  $r$  is a multiple of 3. But there are no divisors of 10 that are also multiples of 3.  $\square$

**Lemma 4.8.** *If  $G$  is connected and  $\mathcal{M}$ -ultrahomogeneous and  $\alpha$  is any vertex of  $G$ , then  $N_\alpha(G)$  is not isomorphic to  $K_n \times K_n$  with  $n \geq 3$ .*

*Proof.* For sake of contradiction, suppose that  $N_\alpha(G)$  is isomorphic to  $K_n \times K_n$  with  $n \geq 3$ . To fix notation, let  $\{(\delta_i, \delta_j) | i, j = 1, 2, \dots, n\}$  be the vertices of  $N_\alpha(G)$ , where  $(\delta_i, \delta_j)$  and  $(\delta_k, \delta_l)$  are adjacent if and only if  $i = k$  or  $j = l$ .

Let  $\beta$  be any vertex of  $\overline{N}_\alpha(G)$ , and let  $\gamma$  be a vertex in  $N_\alpha(G)$  that is also adjacent to  $\beta$ . We may assume that  $\gamma$  is the vertex  $(\delta_1, \delta_1)$ . Consider  $N_\gamma(G)$ , which is also isomorphic to  $K_n \times K_n$ ; we know that it contains  $\alpha$ ,  $\beta$ ,  $(\delta_i, \delta_1)$ , and  $(\delta_1, \delta_i)$  for every  $i$ . It follows that  $\beta$  is adjacent to at least two more vertices of  $N_\alpha(G)$ , one of the form  $(\delta_1, \delta_j)$  and one of the form  $(\delta_1, \delta_k)$ . Moreover,  $\beta$  is not adjacent to any other vertices of the form  $(\delta_1, \delta_l)$  or  $(\delta_l, \delta_1)$ .

This implies that  $N_{\alpha, \beta}(G)$  is isomorphic to a non-empty disjoint union of cycles of even length. But  $N_{\alpha, \beta}(G)$  must be  $\mathcal{M}$ -ultrahomogeneous by Proposition 4.2, so  $N_{\alpha, \beta}(G)$  is actually isomorphic to  $C_4$  and  $m = 4$ .

Note that the graph spanned by  $\alpha$  together with  $N_{\alpha, \beta}(G)$  is isomorphic to  $K_{1,2,2}$ ; in particular, it belongs to  $\mathcal{M}$ . Therefore, the number  $s$  is independent of the choice of  $\beta$ . There are  $\binom{n}{2}^2$  copies of  $C_4$  in  $K_n \times K_n$ , so Equation 4.2 tell us that

$$r = s \binom{n}{2}^2.$$

On the other hand, Equation 4.1 tells us that

$$4r = n^2(n-1)^2.$$

It follows that  $s = 1$ , which means that for every copy  $C$  of  $C_4$  in  $N_\alpha(G)$ , there exists exactly one vertex  $\beta$  in  $\overline{N}_\alpha(G)$  such that  $\beta$  is adjacent to each vertex of  $C$ .

Now we know that  $G$  contains  $1 + n^2 + \binom{n}{2}^2$  vertices, and  $N_\alpha(G)$  contains  $2n$  copies of  $K_n$ . By Equation 4.3,  $n + 1$  divides  $2n(1 + n^2 + \binom{n}{2}^2)$ . Performing arithmetic modulo  $n + 1$ ,

$$0 \equiv 2n \left( 1 + n^2 + \binom{n}{2}^2 \right) \equiv 2(-1) \left( 1 + 1(-1)^2 + \binom{n}{2}^2 \right) \equiv -4 - 2 \binom{n}{2}^2.$$

Analyzing the cases when  $n$  is even and odd separately, it turns out that  $2 \binom{n}{2}^2$  is always congruent to 2 modulo  $n + 1$ , so  $-4 - 2 \binom{n}{2}^2$  is congruent to  $-6$  modulo  $n + 1$ . Thus 0 is congruent to  $-6$  modulo  $n + 1$ ,  $n + 1$  must divide 6. Since  $n \geq 3$ , the only possibility is  $n = 5$ .

We are left only with the case  $n = 5$ . Since  $n \geq 3$ , the graph  $N_\alpha(G)$  contains three vertices that are pairwise non-adjacent. Since  $G$  is  $3K_1$ -ultrahomogeneous, every set of three pairwise non-adjacent vertices has a common neighbor. If  $\beta_1$  and  $\beta_2$  are vertices of  $\overline{N}_\alpha(G)$  with no common neighbor in  $N_\alpha(G)$ , then  $\beta_1$  and  $\beta_2$  must be adjacent; otherwise,  $\alpha$ ,  $\beta_1$ , and  $\beta_2$  are three pairwise non-adjacent vertices with no common neighbor.

Let  $\beta$  be the vertex of  $\overline{N}_\alpha(G)$  that is adjacent to every vertex of a given copy of  $C_4$  in  $N_\alpha(G)$ . There are 51 of copies of  $C_4$  in  $N_\alpha(G)$  that do not intersect this given copy of  $C_4$ , and there are 51 vertices of  $\overline{N}_\alpha(G)$  that correspond to these 51 copies of  $C_4$ . None of these 51 vertices has a common neighbor with  $\beta$  in  $N_\alpha(G)$ , so these 51 vertices must all be adjacent to  $\beta$  because of the remark in the previous paragraph. This is impossible since  $G$  is 25-regular.  $\square$

**Lemma 4.9.** *Let  $G$  be connected and  $\mathcal{M}$ -ultrahomogeneous, and let  $\alpha$  be any vertex of  $G$ . If  $N_\alpha(G)$  is isomorphic to  $tK_n$ , then  $n = 1$  or  $t \leq 2$ .*

*Proof.* Let  $A_1, A_2, \dots, A_t$  be the components of  $N_\alpha(G)$ , so each  $A_j$  is a complete graph. Let  $\gamma$  be any vertex of  $N_\alpha(G)$ ; then  $N_{\alpha,\gamma}(G)$  is isomorphic to  $K_{n-1}$ . Because  $G$  is  $K_2$ -ultrahomogeneous, every pair of adjacent vertices has exactly  $n-1$  common neighbors.

If  $\gamma_1$  and  $\gamma_2$  both belong to  $A_j$ , then they are adjacent. Their common neighbors consist entirely of  $\alpha$  together with the other vertices of  $A_j$ . In other words, if  $\beta$  is not a neighbor of  $\alpha$ , then  $N_\beta(G) \cap A_j$  contains at most one vertex. This implies that  $1 \leq m \leq t$ . Equation 4.1 gives us the formula

$$(4.4) \quad rm = n^2 t(t-1).$$

The graph spanned by  $\alpha$  together with  $N_{\alpha,\beta}(G)$  is isomorphic to  $K_{1,m}$ , which belongs to  $\mathcal{M}$ . Therefore, the number  $s$  is independent of the choice of  $\beta$ . The number of copies of  $mK_1$  in  $N_\alpha(G)$  is  $\binom{t}{m} n^m$ , so Equation 4.2 gives the formula

$$r = \binom{t}{m} n^m s.$$

Combining the previous two equations, we get the formula

$$(4.5) \quad sn^{m-2} \binom{t-1}{m-1} = t-1.$$

This means that  $t-1$  cannot be strictly smaller than  $\binom{t-1}{m-1}$ , so there are only four possible values for  $m$ : 1, 2,  $t-1$ , and  $t$ . We treat each case separately. In each case, we assume that  $t \geq 3$  and  $n \geq 2$  and reach a contradiction.

First, note that when  $t \geq 3$ , the graph  $N_\alpha(G)$  has three pairwise non-adjacent vertices. Since  $G$  is  $3K_1$ -ultrahomogeneous, every set of three non-adjacent vertices in  $G$  must have a common neighbor.

*Case 1:  $m = 1$ .* Let  $\beta$  be any vertex in  $\overline{N}_\alpha(G)$ , and let  $\gamma$  be its unique neighbor in  $N_\alpha(G)$ . If  $\delta$  is another vertex in  $\overline{N}_\alpha(G)$  that is not adjacent to  $\gamma$ , then  $\beta$  and  $\delta$  must be adjacent; otherwise  $\alpha$ ,  $\beta$ , and  $\delta$  would form a set of three pairwise non-adjacent vertices with no common neighbor. In other words, every vertex of  $\overline{N}_\alpha(G)$  must be adjacent to  $\beta$  or  $\gamma$  (or both). From Equation 4.5, we know that  $s = n(t-1)$ . Thus,  $\gamma$  has  $s = n(t-1)$  neighbors in  $\overline{N}_\alpha(G)$ , so  $\beta$  has at least  $r - n(t-1)$  neighbors in  $\overline{N}_\alpha(G)$ . Now  $nt \geq n^2 t(t-1) - n(t-1)$  since  $G$  is  $nt$ -regular and since  $r = n^2 t(t-1)$ . This inequality can never hold when  $n \geq 2$  and  $t \geq 3$ .

*Case 2:  $m = 2$ .* First of all, Equation 4.5 tells us that  $s = 1$ . Let  $\beta$  be any vertex in  $\overline{N}_\alpha(G)$ , and let  $\gamma_1$  and  $\gamma_2$  be its neighbors in  $N_\alpha(G)$ . Since  $\gamma_1$  and  $\gamma_2$  have exactly  $n-1$  neighbors in  $N_\alpha(G)$  and since  $G$  is  $tn$  regular,  $\gamma_1$  and  $\gamma_2$  have exactly  $n(t-1)$  neighbors in  $\overline{N}_\alpha(G)$ . Thus, there are exactly  $2n(t-1) - 1$  vertices in  $\overline{N}_\alpha(G)$  that are adjacent to  $\gamma_1$  or  $\gamma_2$  (or both). If  $\delta$  is another vertex in  $\overline{N}_\alpha(G)$  that is adjacent to neither  $\gamma_1$  nor  $\gamma_2$ , then  $\beta$  and  $\delta$  must be adjacent; otherwise  $\alpha$ ,  $\beta$ , and  $\delta$  would form a set of three pairwise non-adjacent vertices with no common neighbor.

Therefore,  $\beta$  has at least  $r - 2n(t-1) + 1$  neighbors in  $\overline{N}_\alpha(G)$ . Taking into account the two neighbors of  $\beta$  in  $N_\alpha(G)$ ,  $nt \geq \frac{1}{2}n^2 t(t-1) - 2n(t-1) + 3$  since  $G$  is  $nt$ -regular and since  $r = n^2 t(t-1)/2$ . This inequality can never hold when  $n \geq 2$  and  $t \geq 3$ .

*Case 3:  $m = t-1$ .* Now Equation 4.5 becomes  $sn^{t-3} = 1$ , so  $t \leq 3$ . Thus  $t = 3$  and  $m = 2$ . We have reduced this case to Case 2.

*Case 4:*  $m = t$ . Now Equation 4.5 becomes  $sn^{t-2} = t - 1$ , so  $t - 1 \geq n^{t-2}$ . Since  $n \geq 2$  and  $t \geq 3$ , this can only happen when  $n = 2$ ,  $s = 1$ , and  $t = 3$ . It follows that  $m = 3$  and  $r = 8$ . Exhaustion of cases shows that  $G$  cannot exist.  $\square$

**Lemma 4.10.** *Let  $G$  be an  $\mathcal{M}$ -ultrahomogeneous graph, and let  $\alpha$  be any vertex of  $G$ . If  $N_\alpha(G)$  is isomorphic to  $2K_n$  with  $n \geq 2$ , then  $G$  is isomorphic to  $K_{n+1} \times K_{n+1}$ .*

*Proof.* As in the proof of Lemma 4.9,  $1 \leq m \leq 2$ .

Suppose that  $m = 1$ . Using Equation 4.4,  $r = 2n^2$ . Thus  $G$  has  $1 + 2n + 2n^2$  vertices. Also,  $N_\alpha(G)$  contains exactly 2 copies of  $K_n$ . Equation 4.3 tells us that  $n + 1$  divides  $2 + 4n + 4n^2$ . Performing arithmetic modulo  $n + 1$ ,

$$0 \equiv 2 + 4n + 4n^2 \equiv 2 + 4(-1) + 4(-1)^2 \equiv 2.$$

This can never happen when  $n \geq 2$ .

We have contradicted the assumption that  $m = 1$ , so  $m$  must equal 2. Again using the Equation 4.4,  $r = n^2$ . Thus  $G$  has  $1 + 2n + n^2$  vertices.

Let  $\gamma$  be any neighbor of  $\alpha$ ; then  $N_{\alpha,\gamma}(G)$  is isomorphic to  $K_{n-1}$ . Since  $G$  is  $K_2$ -ultrahomogeneous,  $N_{\gamma_1,\gamma_2}(G)$  is isomorphic to  $K_{n-1}$  for any pair of adjacent vertices  $\gamma_1$  and  $\gamma_2$ .

Let  $A$  and  $B$  be the two copies of  $K_n$  in  $N_\alpha(G)$ , and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be the vertices of  $A$ . Now each vertex  $\alpha_i$  in  $A$  is adjacent to the copy of  $K_n$  consisting of  $\alpha$  together with the other vertices of  $A$ . The vertex  $\alpha_i$  must be adjacent to another copy  $B_i$  of  $K_n$  whose vertices are not in  $A$ ,  $B$ , or equal to  $\alpha$ . The subgraph  $N_{\alpha_i,\alpha_j}(G)$  consists of the vertex  $\alpha$  together with the other vertices of  $A$ . Therefore, none of the  $B_i$  intersect.

The vertex  $\alpha$  together with the vertices of  $A, B, B_1, \dots, B_n$  give us  $1 + 2n + n^2$  vertices, so we have accounted for all the vertices. We still must account for the edges between vertices belonging to  $B, B_1, \dots, B_n$ . In fact, we must account for exactly  $n$  edges incident to each such vertex.

Let  $\beta$  be any vertex of  $B$ . If  $\beta$  had two neighbors  $\gamma_1$  and  $\gamma_2$  in  $B_i$ , then  $\beta$  would belong to  $N_{\gamma_1,\gamma_2}(G)$ . But this is not possible because  $N_{\gamma_1,\gamma_2}(G)$  is the copy of  $K_{n-1}$  consisting of  $\alpha_i$  and the other  $n - 2$  vertices of  $B_i$ . Therefore,  $\beta$  has at exactly one neighbor in each  $B_i$ . From this fact it follows that  $G$  must be isomorphic to  $K_{n+1} \times K_{n+1}$ .  $\square$

**Lemma 4.11.** *Let  $G$  be connected and  $\mathcal{M}$ -ultrahomogeneous, and let  $\alpha$  be any vertex of  $G$ . Suppose that  $N_\alpha(G)$  is isomorphic to  $tK_1$ .*

- (1) *If  $t = 1$ , then  $G$  is isomorphic to  $K_2$ .*
- (2) *If  $t = 2$ , then  $G$  is isomorphic to  $C_4$  or  $C_5$ .*
- (3) *If  $t \geq 3$ , then  $G$  is isomorphic to  $K_{2;t}$ .*

*Proof.* When  $t = 1$ , the graph  $G$  is 1-regular, so it is isomorphic to  $K_2$ .

When  $t = 2$ , the graph  $G$  is connected and 2-regular. This means that  $G$  is isomorphic to a cycle. By inspection, the only  $\mathcal{M}$ -ultrahomogeneous cycles are  $C_4$  and  $C_5$ .

In the rest of the proof, we assume that  $t \geq 3$ . We follow the same outline as the proof of Lemma 4.9. In the case  $m = 1$ , we obtain the inequality  $t \geq (t - 1)^2$ ; this can never hold when  $t \geq 3$ .

In the case  $m = 2$ , we get  $t \geq \frac{1}{2}t(t - 1) - 2(t - 1) + 3$ , which implies that  $t \leq 5$ . Handling the possibilities  $t = 3$ ,  $t = 4$ , and  $t = 5$  separately, exhaustion of cases shows that  $G$  cannot exist.

In the case  $m = t - 1$ , Equation 4.1 tells us that  $r = t$ . Each vertex  $\beta$  in  $\overline{N}_\alpha(G)$  has  $t - 1$  neighbors in  $N_\alpha(G)$ , so it must have exactly one neighbor  $\gamma$  in  $\overline{N}_\alpha(G)$ . Since  $\beta$  and  $\gamma$  both have  $t - 1$  neighbors amongst the  $t$  vertices of  $N_\alpha(G)$ , they must have a common neighbor  $\delta$ . Then  $\beta$ ,  $\gamma$ , and  $\delta$  are the vertices of a copy of  $K_3$ . This is a contradiction since  $G$  contains no copies of  $K_3$ .

Finally, in the case  $m = t$ , the graph  $\overline{N}_\alpha(G)$  consists entirely of vertices that are adjacent to every vertex of  $N_\alpha(G)$ . Equation 4.1 tells us that  $r = t - 1$ . Since  $G$  is  $t$ -regular, there are no more vertices and  $G$  is isomorphic to  $K_{2,t}$ .  $\square$

**Lemma 4.12.** *Let  $G$  be connected and  $\mathcal{M}$ -ultrahomogeneous, and let  $\alpha$  be any vertex of  $G$ . If  $N_\alpha(G)$  is isomorphic to the complete regular multipartite graph  $K_{t;n}$  with  $t \geq 2$  and  $n \geq 1$ , then  $G$  is isomorphic to  $K_{t+1;n}$ .*

*Proof.* Let  $A_1, A_2, \dots, A_t$  be the partite sets of  $N_\alpha(G)$ . Choose any vertex  $\beta$  of  $A_1$ . Since  $N_\beta(G)$  is also isomorphic to  $K_{t;n}$ , there exist  $n - 1$  vertices  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$  of  $G$  such that each  $\alpha_j$  is adjacent to  $\beta$  and to every vertex of  $A_2, \dots, A_t$  but is not adjacent to  $\alpha$ . Also, the vertices  $\alpha_j$  and  $\alpha_k$  are not adjacent for  $j \neq k$ .

It remains only to show that each  $\alpha_j$  is adjacent to every vertex of  $A_1$ . To do this, choose a vertex  $\gamma$  in  $A_2$  and consider  $N_\gamma(G)$ , which is again isomorphic to  $K_{t;n}$ .  $\square$

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