HYPERHAMILTONICITY OF THE CARTESIAN PRODUCT OF TWO DIRECTED CYCLES

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ABSTRACT. Let $\mathbb{Z}_a \times \mathbb{Z}_b$ be the product of two directed cycles, let $\mathbb{Z}_c$ be a subgroup of $\mathbb{Z}_a$, and let $\mathbb{Z}_d$ be a subgroup of $\mathbb{Z}_b$. Also, let $A = \frac{a}{c}$ and $B = \frac{b}{d}$. We say that $\mathbb{Z}_a \times \mathbb{Z}_b$ is $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian if there is a spanning connected subgraph of $\mathbb{Z}_a \times \mathbb{Z}_b$ that has degree $(2, 2)$ at the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ and degree $(1, 1)$ everywhere else. We show that the graph $\mathbb{Z}_a \times \mathbb{Z}_b$ is $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian if and only if there exist positive integers m and n such that $Am + Bn = AB + 1$, $\gcd(m, n) = 1$ or 2, and when $\gcd(m, n) = 2$, then $\gcd(dm, cn) = 2$.

1. INTRODUCTION

Curran and Witte [3] proved using the theory of torus knots that the Cartesian product $\mathbb{Z}_a \times \mathbb{Z}_b$ of two directed cycles is hamiltonian if and only if there exists a pair of relatively prime positive integers m and n such that $am + bn = ab$. Gallian and Witte [4] defined a digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ to be hyperhamiltonian if there is a spanning connected subgraph of $\mathbb{Z}_a \times \mathbb{Z}_b$ which passes through one vertex exactly twice and all others exactly once. They showed that the digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ is hyperhamiltonian if and only if there exist positive integers m and n such that $am + bn = ab + 1$ and $\gcd(m, n) = 1$ or 2.

Note that passing through one vertex from $\mathbb{Z}_a \times \mathbb{Z}_b$ twice as in [4] is equivalent to passing through the vertices of the subgroup $\mathbb{Z}_1 \times \mathbb{Z}_1$ twice. In this paper, we define the graph $\mathbb{Z}_a \times \mathbb{Z}_b$ to be $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian if there is a spanning connected subgraph in which the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ have two in-edges and two out-edges and all other vertices have one in-edge and one out-edge. Here, $\mathbb{Z}_c$ is a subgroup of $\mathbb{Z}_a$ and $\mathbb{Z}_d$ is a subgroup of $\mathbb{Z}_b$. Hence, our result is a natural generalization of Gallian and Witte's result.

The methods used in this paper are similar to the ones used in [1]. That paper generalized results for hamiltonicity of vertex-deleted digraphs [5] to subgroup-deleted digraphs.

2. BACKGROUND

We recall some definitions and results that will be useful. We refer to [2] for the basic language of digraphs, but we remind the reader of two useful definitions. First, a vertex of a digraph has degree (r, s) if it has r in-edges and s out-edges. Second, a digraph is connected if there is a directed path from any vertex to any other vertex.

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Definition 2.1. Let G be a digraph, and let V be a set of vertices of G. Then G is V-hyperhamiltonian if there is a connected spanning subgraph that has degree $(2,2)$ at the vertices of V and degree $(1,1)$ at all other vertices. Such a subgraph is called a V-hyperhamiltonian circuit of G.

The idea is that a digraph G is V-hyperhamiltonian if there exists a closed directed walk that passes through each vertex of V exactly twice and passes through the other vertices exactly once. See Figure 1 for a picture of the digraph $\mathbb{Z}_9 \times \mathbb{Z}_4$ with a $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-hyperhamiltonian circuit.

In this paper, we only consider the case when G is the digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ and V consists of the vertices belonging to the subgroup $\mathbb{Z}_c \times \mathbb{Z}_d$.

Definition 2.2. Let H be a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit on $\mathbb{Z}_a \times \mathbb{Z}_b$. Then a vertex (x,y) **travels by** $(1,0)$ if H contains the directed edge from (x,y) to $(x+1,y)$. Similarly, a vertex (x,y) **travels by** $(0,1)$ if H contains the directed edge from (x,y) to $(x,y+1)$.

Note that in a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit, a vertex (x,y) that does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ travels by either $(1,0)$ or by $(0,1)$. A vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$ travels by both $(1,0)$ and $(0,1)$.

Definition 2.3. Let $g = \gcd(a,b)$. For any integer p, let $\langle p \rangle$ be the subset of $\mathbb{Z}_a \times \mathbb{Z}_b$ consisting of pairs (x,y) such that $x+y = p$ modulo g.

The subset $\langle 0 \rangle$ is the subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ generated by $(1,-1)$, and $\langle p \rangle$ is the coset $(p,0) + \langle 0 \rangle$. The subgroup $\langle 0 \rangle$ has index g, so there are exactly g distinct cosets.

The following lemma shows why such cosets are useful.

Lemma 2.4. *Let H be a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of $\mathbb{Z}_a \times \mathbb{Z}_b$.*

(1) *If $(x+1,y-1)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ and $(x+1,y-1)$ travels by $(1,0)$, then (x,y) also travels by $(1,0)$.*
(2) *If $(x+1,y)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ and $(x+1,y-1)$ travels by $(0,1)$, then (x,y) also travels by $(0,1)$.*

Proof. In part (1), the vertex $(x+1,y)$ must have at least one in-edge. By assumption, this in-edge does not come from $(x+1,y-1)$, so it must come from (x,y).

In part (2), the vertex $(x+1,y)$ can have at most one in-edge. By assumption, it has an in-edge from $(x+1,y-1)$, so it cannot have an in-edge from (x,y). Thus (x,y) does not travel by $(1,0)$, so it must travel by $(0,1)$. □

We recall the following result from [4].

Theorem 2.5 (Gallian-Witte)**.** *The digraph $\mathbb{Z}_a \times \mathbb{Z}_b$ is hyperhamiltonian if and only if there exist positive integers m and n such that $am+bn = ab+1$ and $\gcd(m,n) = 1$ or 2.*

In Theorem 2.5, the numbers m and n have useful geometric interpretations. If we embed $\mathbb{Z}_a \times \mathbb{Z}_b$ in the torus in the obvious way, then a hyperhamiltonian circuit consists of two embedded directed loops that meet at a single vertex, and the total knot class of these two loops is equal to (m,n). See [6] for more details on knot classes.

Using intersection numbers, Gallian and Witte [4] showed that when $\gcd(m,n) = 1$, these two directed loops have knot classes (m_1,n_1) and (m_2,n_2), where $m_1 n_2 -$

m_2n_1 equals 1 or -1. When $\gcd(m,n) = 2$, the two directed loops both have knot class $(\frac{m}{2}, \frac{n}{2})$.

3. The Main Theorem

We now come to our main result.

Theorem 3.1. *Let $A = \frac{a}{c}$ and $B = \frac{b}{d}$. Then $\mathbb{Z}_a \times \mathbb{Z}_b$ is $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian if and only if there exist positive integers m and n such that*

(1) $Am + Bn = AB + 1$,
(2) $\gcd(m,n) = 1$ *or* 2, *and*
(3) *when* $\gcd(m,n) = 2$, *then* $\gcd(dm, cn) = 2$.

When $A = a$, $B = b$, then $c = d = 1$. In this case, the conditions in Theorem 3.1 reduce to the conditions in Theorem 2.5.

Example 3.2. We give two examples illustrating the theorem. First, consider the digraph $\mathbb{Z}_9 \times \mathbb{Z}_4$, as shown in Figure 1. This digraph has a $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-hyperhamiltonian circuit. The values $m = 1$ and $n = 2$ satisfy the three conditions of the theorem.

However, the digraph $\mathbb{Z}_{10} \times \mathbb{Z}_6$ does not have a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-hyperhamiltonian circuit. When $m = n = 2$, the first two conditions are satisfied, but the third condition is not.

In order to prove this theorem, we need the following facts about $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian graphs.

Lemma 3.3. *If $\mathbb{Z}_a \times \mathbb{Z}_b$ is $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian, then $\gcd(A, B) = 1$ and every coset contains at least one vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$.*

Proof. Suppose for contradiction that $\langle -1 \rangle$ contains no vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. Since $(0,0)$ belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$, the vertex $(-1,0)$ travels by $(1,0)$. A repeated application of part (1) of Lemma 2.4 implies that every vertex of $\langle -1 \rangle$ also travels by $(1,0)$. This is a contradiction because the vertex $(0,-1)$ travels by $(0,1)$. This means that $\langle -1 \rangle$ contains at least one vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. Let $(\alpha a, \beta B)$ be such a vertex, where α and β are integers.

From the definition of a coset we find that $\alpha A + \beta B = -1 \bmod g$. There exists an integer γ such that $\alpha A + \beta B = -1 + g\gamma$. Now $\gcd(A, B)$ divides g, αA, and βB so it must also divide -1. This shows that $\gcd(A, B) = 1$.

Now consider an arbitrary coset $\langle p \rangle$. Multiplying the equation of the previous paragraph by $-p$, we get $(-\alpha p)A + (-\beta p)B = p \bmod g$. From the definition of a coset, we know $((-\alpha p)A, (-\beta p)B)$ is in $\langle p \rangle$. □

Corollary 3.4. *If $\mathbb{Z}_a \times \mathbb{Z}_b$ has a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit, then it is unique.*

Proof. Let (x, y) be an arbitrary vertex. Since every coset contains a vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$ by Lemma 3.3, (x, y) can be written (uniquely) in the form $(x_0 - k, y_0 + k)$, where (x_0, y_0) belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$, k is a non-negative integer, and $(x_0 - j, y_0 + j)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$ for $0 < j \leq k$. By induction on k, we show that there is no choice in the directions in which (x, y) must travel.

First, if $k = 0$, then (x, y) belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$. Thus, it must travel both by $(1,0)$ and by $(0,1)$. Now assume that $k > 0$, so (x, y) travels either by $(1,0)$ or by $(0,1)$ but not by both. If $(x+1, y)$ belongs to $\mathbb{Z}_c \times \mathbb{Z}_d$, then (x, y) must travel by $(1,0)$. Hence we may assume that $(x+1, y)$ does not belong to $\mathbb{Z}_c \times \mathbb{Z}_d$; this allows us to apply Lemma 2.4.

If $k = 1$, then part (2) of Lemma 2.4 tells us that (x, y) must travel by $(0, 1)$. Now suppose for sake of induction $k \geq 2$ and that we know the direction in which $(x_0 - (k-1), y_0 + (k-1))$ travels. Parts (1) and (2) of Lemma 2.4 tell us that $(x_0 - k, y_0 + k)$ must travel in the same direction as $(x_0 - (k-1), y_0 + (k-1))$. □

The following corollary tells us that $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuits on $\mathbb{Z}_a \times \mathbb{Z}_b$ are suitably periodic.

Corollary 3.5. *Suppose $\mathbb{Z}_a \times \mathbb{Z}_b$ has a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit. Let α and β be integers. Then (x, y) and $(x + \alpha A, y + \beta B)$ travel in the same direction.*

Proof. Let H be a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of $\mathbb{Z}_a \times \mathbb{Z}_b$. Then $\phi(x, y) = (x + \alpha A, y + \beta B)$ is an automorphism of $\mathbb{Z}_a \times \mathbb{Z}_b$ that preserves $\mathbb{Z}_c \times \mathbb{Z}_d$. So $\phi(H)$ is a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of $\mathbb{Z}_a \times \mathbb{Z}_b$ as well. But the $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit is unique (by Corollary 3.3), so $\phi(H) = H$. Thus, any vertex (x, y) travels in the same direction as the vertex $\phi(x, y)$. □

Note that the proofs of Lemma 3.3, Corollary 3.4, and Corollary 3.5 are similar to the proofs found in [1].

Lemma 3.6. *Suppose that H is a spanning subgraph of $\mathbb{Z}_a \times \mathbb{Z}_b$ such that the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ have degree $(2, 2)$ in H and all other vertices have degree $(1, 1)$ in H and such that every directed loop in H contains a vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. Then H is connected if and only if for any two vertices v and w in $\mathbb{Z}_c \times \mathbb{Z}_d$, there is a directed path in H from v to w.*

Proof. One direction follows from the definition of connectedness. For the other direction, suppose that for any two vertices v_1 and v_2 in $\mathbb{Z}_c \times \mathbb{Z}_d$, there is a directed path in H from v_1 to v_2. We need to show is that there is a directed path from w_1 to w_2 for any two vertices w_1 and w_2 of $\mathbb{Z}_a \times \mathbb{Z}_b$. Under our assumptions, there is a directed path from any vertex w_1 to some vertex v_1 in $\mathbb{Z}_c \times \mathbb{Z}_d$. Similarly, there is a directed path from some vertex v_2 in $\mathbb{Z}_c \times \mathbb{Z}_d$ to the vertex w_2. Finally, there is a directed path from v_1 to v_2 by assumption. Therefore, there is a path from w_1 to w_2. □

Now we are ready to prove Theorem 3.1.

Proof. First suppose that H is a $(\mathbb{Z}_c \times \mathbb{Z}_d)$-hyperhamiltonian circuit of $\mathbb{Z}_a \times \mathbb{Z}_b$. We define a function f from $\mathbb{Z}_a \times \mathbb{Z}_b$ to $\mathbb{Z}_A \times \mathbb{Z}_B$ by $f(x, y) = (x \mod A, y \mod B)$. Define a subgraph H' of $\mathbb{Z}_A \times \mathbb{Z}_B$ by requiring the vertex (x, y) in $\mathbb{Z}_A \times \mathbb{Z}_B$ to travel in the same direction as the vertices $(x + \alpha A, y + \beta B)$ in $\mathbb{Z}_a \times \mathbb{Z}_b$ where α and β are integers. We can define a subgraph in this way because Corollary 3.4 states that all vertices of the form $(x + \alpha A, y + \beta B)$ travel in the same direction. Note that H' is just $f(H)$.

Let (x_1, y_1) and (x_2, y_2) be any two vertices in $\mathbb{Z}_a \times \mathbb{Z}_b$. There is a directed path in H from (x_1, y_1) to (x_2, y_2), so there is a directed path in $f(H)$ from $f(x_1, y_1)$ to $f(x_2, y_2)$. This means that H' is connected, so $\mathbb{Z}_A \times \mathbb{Z}_B$ is $(\mathbb{Z}_1 \times \mathbb{Z}_1)$-hyperhamiltonian. By Theorem 2.5, conditions (1) and (2) are satisfied.

In order to show condition (3), assume that $\gcd(m, n) = 2$. The two directed loops of H' both have knot class $(\frac{m}{2}, \frac{n}{2})$, so there is a directed path from any vertex $(\alpha A, \beta B)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$ to the vertex $(\alpha A + (\frac{m}{2})A, \beta B + (\frac{n}{2})B)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$. Since H is connected, there is a directed path in H from any vertex in $\mathbb{Z}_c \times \mathbb{Z}_d$ to any other vertex in $\mathbb{Z}_c \times \mathbb{Z}_d$. This means that $(\frac{m}{2}, \frac{n}{2})$ generates $\mathbb{Z}_c \times \mathbb{Z}_d$. In particular,

$\mathbb{Z}_c \times \mathbb{Z}_d$ must be cyclic, so $\gcd(c, d) = 1$. This allows us to define an isomorphism $\phi : \mathbb{Z}_c \times \mathbb{Z}_d \to \mathbb{Z}_{cd}$ by the formula $\phi(x, y) = dx + cy$. Since $\phi(\frac{m}{2}, \frac{n}{2})$ must be a generator of $\mathbb{Z}_{cd}$, $\gcd(cd, d(\frac{m}{2}) + c(\frac{n}{2})) = 1$. With this equation and because c divides $c(\frac{n}{2})$, we see that $\gcd(d(\frac{m}{2}), c) = 1$. Similarly, $\gcd(d, c(\frac{n}{2})) = 1$. By assumption $\gcd(\frac{m}{2}, \frac{n}{2}) = 1$. Using these equations, we see that $\gcd(d(\frac{m}{2}), c(\frac{n}{2})) = 1$ or $\gcd(dm, cn) = 2$. This finishes one direction of the theorem.

Now suppose conditions (1), (2), and (3). The first two conditions imply that $\mathbb{Z}_A \times \mathbb{Z}_B$ has a $(\mathbb{Z}_1 \times \mathbb{Z}_1)$- hyperhamiltonian circuit H' with total knot class (m, n). We construct a spanning subgraph H of $\mathbb{Z}_a \times \mathbb{Z}_b$ by requiring each vertex (x, y) in $\mathbb{Z}_a \times \mathbb{Z}_b$ to travel in the same direction as the vertex $(x \mod A, y \mod B)$ in $\mathbb{Z}_A \times \mathbb{Z}_B$. With this construction, the vertices of $\mathbb{Z}_c \times \mathbb{Z}_d$ have degree $(2, 2)$ in H, and all other vertices have degree $(1, 1)$. All that is left to show is that H is connected. Since every directed loop in H' contains the vertex $(0, 0)$, every directed loop in H contains a vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$. By Lemma 3.5, we need only show that H contains a directed path from any vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$ to any other vertex of $\mathbb{Z}_c \times \mathbb{Z}_d$.

When $\gcd(m, n) = 1$, let (m_1, n_1) and (m_2, n_2) be the knot classes of the two directed loops in H'. There are directed paths from any vertex $(\alpha A, \beta B)$ of $\mathbb{Z}_c \times \mathbb{Z}_d$ to the vertices $(\alpha A + m_1 A, \beta B + n_1 B)$ and $(\alpha A + m_2 A, \beta B + n_2 B)$, so we want to show that (m_1, n_1) and (m_2, n_2) generate the group $\mathbb{Z}_c \times \mathbb{Z}_d$. This is equivalent to showing that for any element (x, y) of $\mathbb{Z}_c \times \mathbb{Z}_d$, we have two integers e and f satisfying the equation

$$(x, y) = e(m_1, n_1) + f(m_2, n_2).$$

We can write this as the matrix equation

$$\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The determinant of the matrix on the left is 1 or -1, because of the remarks after Theorem 2.5. The inverse of this matrix is again an integer matrix, which gives us equations for e and f. This finishes the case when $\gcd(m, n) = 1$.

When $\gcd(m, n) = 2$, we have condition (3) which states that $\gcd(dm, cn) = 2$. It follows that $\gcd(d(\frac{m}{2}), c(\frac{n}{2})) = 1$. The two directed loops of H' both have knot class $(\frac{m}{2}, \frac{n}{2})$, so we want to show that $(\frac{m}{2}, \frac{n}{2})$ generates $\mathbb{Z}_c \times \mathbb{Z}_d$. Since c and d are relatively prime, $\mathbb{Z}_c \times \mathbb{Z}_d$ is cyclic; we again use the isomorphism ϕ from above.

Now $\phi(\frac{m}{2}, \frac{n}{2}) = d(\frac{m}{2}) + c(\frac{n}{2})$. The element $(\frac{m}{2}, \frac{n}{2})$ is a generator of $\mathbb{Z}_c \times \mathbb{Z}_d$ if and only if $\gcd(cd, d(\frac{m}{2}) + c(\frac{n}{2})) = 1$. This last equation follows from the facts that $\gcd(c, d) = 1$, $\gcd(c, \frac{m}{2}) = 1$, and $\gcd(d, \frac{n}{2}) = 1$. □

4. Questions

Most questions about hamiltonian circuits on digraphs have analogies about hyperhamiltonian circuits. We end with a few specific examples. The first question extends our problem to larger dimensions.

Question 4.1. *When does the graph $\mathbb{Z}_{a_1} \times \mathbb{Z}_{a_2} \times \cdots \times \mathbb{Z}_{a_n}$ have a $(\mathbb{Z}_{c_1} \times \mathbb{Z}_{c_2} \times \cdots \times \mathbb{Z}_{c_n})$-hyperhamiltonian circuit?*

Not every subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$ is of the form $\mathbb{Z}_c \times \mathbb{Z}_d$. This leads to our next question.

Question 4.2. *Let A be any subgroup of $\mathbb{Z}_a \times \mathbb{Z}_b$. When does $\mathbb{Z}_a \times \mathbb{Z}_b$ have an A-hyperhamiltonian circuit?*

Instead of just considering the vertices of one subgroup, it is also possible to consider the vertices belonging to more than one coset of a subgroup.

Question 4.3. *Choose a positive number r. If V is a disjoint union of r cosets of $\mathbb{Z}_c \times \mathbb{Z}_d$ in $\mathbb{Z}_a \times \mathbb{Z}_b$, when is $\mathbb{Z}_a \times \mathbb{Z}_b$ V-hyperhamiltonian?*

Gallian and Witte [4] determined when the digraph $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_1 \times \mathbb{Z}_1)$ is hyperhamiltonian.

Question 4.4. *If V is a coset of $\mathbb{Z}_c \times \mathbb{Z}_d$ in $\mathbb{Z}_a \times \mathbb{Z}_b$, when does the digraph $(\mathbb{Z}_a \times \mathbb{Z}_b) - (\mathbb{Z}_c \times \mathbb{Z}_d)$ have a V-hyperhamiltonian circuit?*

References

[1] V. Barone, M. Mauntel, and M. Miller, Hamiltonicity of the cartesian product of two directed cycles minus a subgroup, submitted.
[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, New York (1996).
[3] S. J. Curran and D. Witte, Hamilton paths in cartesian products of directed cycles. *Annals Discrete Math.* 27 (1985) 35-74.
[4] J. Gallian and D. Witte, When the cartesian product of two directed cycles is hyperhamiltonian. *J. Graph Theory.* 11 (1987) 21-24.
[5] L. E. Penn and D. Witte, When the cartesian product of two directed cycles is hypohamiltonian. *J. Graph Theory.* 7 (1983) 441-443.
[6] D. Rolfsen, *Knots and Links*, Publish or Perish, Berkeley, CA (1976).

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