ON QUADRATIC WITT GROUPS OVER POLYNOMIAL RINGS

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Abstract. We study the quadratic Witt Group for a commutative ring R, as defined in [reference Sharlau]. We show that $WQ(R[t]) \cong WQ(R)$ for any principal ideal domain R of characteristic not equal to two. This will be done by constructing a group $Nil_{WQ}(R)$ and the following exact sequence:

$$Nil_{WQ}(R) \longrightarrow WQ(R[t]) \longrightarrow WQ(R) \longrightarrow 0$$

for any commutative ring R. We then show that $Nil_{WQ}R$ is actually zero if R is a principal ideal domain.

1. Introduction

A result of Gerstein and Quebbemann establishes that the Witt ring, $W(K) \cong W(K[t])$ when K is a field with characteristic not equal to 2 [Scharlau p. 212]. In this work, we extend this result to any principal ideal domain with characteristic not equal to 2, by replacing WR with $WQ(R)$, the quadratic Witt group. We will use the terminology of [Scharlau], to which we refer the reader for more expanded definitions.

All rings R in this note will be assumed to be commutative with unity, and we will further assume that 2 is not a zero divisor. Recall that an even form over R is a pair (M, b) where M is a finitely generated free R-module and b is a symmetric bilinear form on M such that $b(x, x) \in 2R$ for all $x \in M$. If $\hat{b}$, the adjoint map, is an isomorphism, we say that (M, b) is nonsingular. An even, nonsingular form (M, b) is hyperbolic if there exists a free summand $W \subset M$ such that $W = W^{\perp}$, see []. Furthermore, an even, nonsingular form m is stably hyperbolic if there exists a hyperbolic form m_0 such that $m \oplus m_0$ is hyperbolic.

Two even, nonsingular forms m and n are Witt equivalent if there exist hyperbolic forms m_0 and n_0 such that $m \oplus m_0 \approx n \oplus n_0$. The quadratic Witt group over R, denoted $WQ(R)$, is the set of Witt equivalence classes of even, nonsingular forms with the operation of orthogonal direct

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The National Science Foundation, Frank Connolly, and Dan Isaksen.

sum. See [Scharlau Ch. 1] for more details. Note that if 2 is invertible in R, then $WQ(R)$ is identical to $W(R)$, the Witt ring over R.

2. A Quadratic Version of Higman Linearization

In this section we modify a K-theoretic technique, developed by Higman, so that it applies to $WQ(R)$. This technique shows that any element of $WQ(R[t])$ can be represented by a form $(M[t], b)$ where the $deg(b) \leq 1$. We note that

$$(M[t], b) = (M[t], \beta_0 + t\beta_1 + \ldots + t^k\beta_k).$$

Here each β_i is a symmetric bilinear form on M over R and $M[t] = R[t] \otimes_R M$. Each bilinear form β on M over R extends uniquely to a bilinear form (still written β) on $M[t]$ over $R[t]$. This extension is uniquely specified by the rule: $\beta(1 \otimes m, 1 \otimes m') = \beta(m, m')\ \forall m, m' \in M$. This technique will yield:

$$[M[t], b] = [M[t], b_0 + tb_1].$$

Lemma 2.1. *Let $M[t], b)$ be an even, nonsingular form over $R[t]$. Then there exists a hyperbolic form $(N[t], g)$ over $R[t]$ such that*

$$(M[t], b) \oplus (N[t], g) \approx ((M \oplus N)[t], b_0 + tb_1)$$

where b_0 and b_1 are R bilinear forms over $M \oplus N$.

Proof. Let A be the matrix $(M[t], b)$ with respect to some basis for M. A is a matrix of polynomials with some degree, k. We can write $A = A_0 + A_1t + A_2t^2 + \ldots + A_kt^k$. The matrix A then specifies the isomorphism class of $(M[t], b)$ and we will write $[A]$ for $[M[t], b]$. Since b is even, we can write $A_k = B + B^t$ for some $B \in M_n(R)$. $[A] = [A']$ where $A' = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$ since a hyperbolic space represents the zero element in $WQ(R)$,

where each block matrix I is $n \times n$. Now let $P = \begin{pmatrix} I & Bt^{k-1} & -tI \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$,

We now have $[A] = [PA'P^t]$, where

$$PA'P^t = \begin{pmatrix} A_0 + \ldots + A_{k-1}t^{k-1} & -tI & Bt^{k-1} \\ -tI & 0 & I \\ B^tt^{k-1} & I & 0 \end{pmatrix}.$$

Continue this process inductively on the degree, each time reducing the degree of the bilinear form by one, while tripling its rank, until the matrix has degree ≤ 1.

□

Suppose $M \times M \xrightarrow{b} R$ is a symmetric bilinear form over R and $M \xrightarrow{\nu} M$ is any R map. We use M, b, ν to construct an $R[t]$ bilinear form on $M[t]$, namely $M[t] \times M[t] \xrightarrow{b^\nu} R[t]$. Specifically, b^ν is the unique $R[t]$ bilinear map such that:

$$b^\nu(x,y) = b(x,y) + tb(\nu x, y) \qquad \forall\ x, y \in M.$$

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Lemma 2.2. *Let $(M[t], b^\nu)$ be an even, nonsingular form. Then there exists a nilpotent endomorphism $M \xrightarrow{\nu} M$ satisfying:*

- $b^\nu(x,y) = b(x,y) + tb(\nu x, y)$
- $b(\nu x, x) \in 2R$
- $b(\nu x, y) = b(x, \nu y) \forall x, y \in M[t]$

Proof. Let A be the matrix of $(M[t], b^\nu)$ with respect to some basis for M. Then by Lemma 2.1, $A = A_0 + A_1 t$. Now we'll see that $A = A_0(I + t\nu)$ where ν is nilpotent and I is the identity. $(I + t\nu)$ is invertible because A and A$_0$ are. Let $B = (I + t\nu)^{-1} = B_0 + B_1 t + \ldots + B_k t^k$. Then

$$I = (I + t\nu)B = B_0 + (\nu B_0 + B_1)t + \ldots + (\nu B_{k-1} + B_k)t^k.$$

So B$_0$=I, B$_1$ $= -\nu$, B$_2$ $= \nu^2$, b$_3$ $= -\nu^3$ and so on. But then $B = B_0 + B_1 t + \ldots + B_k t^k + B_{k+1} t^{k+1}$ where $B_{k+1} = 0$. Since $B_{k+1} = \pm \nu^{k+1} = 0$, ν is nilpotent.

□

3. The Group $Nil_{WQ}(R)$ and the Exact Sequence

In this section we define the abelian group $Nil_{WQ}(R)$ of nil forms and construct the exact sequence

$$Nil_{WQ}(R) \longrightarrow WQ(R[t]) \longrightarrow WQ(R) \longrightarrow 0$$

Definition 3.1. A *nil form* is a triple (M, b, ν) where (M, b) is a hyperbolic form over R, and ν is a nilpotent R-map $M \xrightarrow{\nu} M$ such that:

- $b(\nu x, x) \in 2R$

- $b(\nu x, y) = b(x, \nu y). \forall\ x, y \in M$

If, in addition, there exists a free summand $W \subset M$ such that $W = W^{\perp}$ and $\nu(W) \subset W$ we say the nil form (M, b, ν) is *split.*

Two nil forms $(M, b, \nu), (M', b', \nu')$ are *isometric* if there exists an isomorphism $M \xrightarrow{f} M'$ such that $\forall\ x, y \in M$:

- $b(x, y) = b'(f(x), f(y))$
- $\nu' f = f \nu$.

Let m, n be nil forms over R. We say they are *Witt-equivalent* if there exists split nil forms m_0, n_0 such that $m \oplus m_0 \approx n \oplus n_0$.

Definition 3.2. The set of all Witt equivalence classes of nil forms is:

$$Nil_{WQ}(R)$$

We make this set into an abelian group by defining addition is as follows:

$$[M, b, \nu] + [M', b', \nu'] := [M \oplus M', b \oplus b', \nu \oplus \nu']$$

where $b \oplus b'$ is the orthogonal direct sum of forms [see Scharlau]. This is clearly a well defined operation. $Nil_{WQ}(R)$ is an abelian group under this binary operation. The identity element $[0, 0, 0]$. The inverse of $[M, b, \nu]$ is $[M, -b, \nu]$.

Definition 3.3. We define the map η:$Nil_{WQ}(R) \longrightarrow WQ(R[t])$ as follows:

$$\eta([M, b, \nu]) = [M, b^{\nu}]$$

Theorem 3.4. *The sequence* $Nil_{WQ}(R) \xrightarrow{\eta} WQ(R[t]) \xrightarrow{\epsilon_*} WQ(R) \longrightarrow 0$ *is exact.*

Proof. First we show that the sequence $WQ(R[t]) \xrightarrow{\epsilon_*} WQ(R) \longrightarrow 0$ is exact. The evaluation map $\mathbb{R}[t] \xrightarrow{\epsilon} R$ satisfies $\epsilon \circ i = id_R$, where $R[t] \xrightarrow{i} R$ is the inclusion map. It follows that $\epsilon_* \circ i_* = id_{WQ(R)}$ and that ϵ_* is an epimorphism.

Next, we show the sequence is exact at $WQ(R[t])$. Recall that every element of $WQ(R[t])$ can be written in the form $M[t], b^{\nu}]$ where $[M, b]$ is even and nonsingular and ν is nilpotent (by Lemmas 2.1 and 2.2). The image of η consists of elements of the form $[M[t], b^\nu]$ where $[M, b]$ is hyperbolic. The kernel of ϵ_* is the set of all elements of $WQ(R[t])$ which are sent to stably hyperbolic forms in $WQ(R)$. The elements of $\ker(\epsilon_*)$ must therefore have the form $[M, b^\nu]$ where b is a hyperbolic form over R. These are precisely the elements in the image of η. Thus the sequence is exact at $WQ(R[t])$.

□

4. Analysis of the Group $Nil_{WQ}(R)$

We now shift our analysis from an arbitrary ring with characteristic unequal to two to a principal ideal domain. In this context, we analyze the group $Nil_{WQ}(R)$ and show that it is equal to zero.

Definition 4.1. Let (M, b, ν) be a nil form over R. A free summand N of M is a *sublagrangian* of (M, b, ν) if $N \subset N^\perp$ and $\nu(N) \subset N$.

Construction 4.2 (The Sublagrangian Construction)**.** *Let (M, b, ν) be a nil form over R and N be a sublagrangian of (M, b, ν). Set $M_N = N^\perp\!/ N$. We define an even bilinear form on M_N,*

$$M_N \times M_N \xrightarrow{b_N} R$$

by $b_N([x], [y]) = b(x, y)$ for all $x, y \in N^\perp$. (Here $[x] := \pi(x)$ where $N^\perp \xrightarrow{\pi} N^\perp\!/ N$ is the quotient map.) We also define $\nu_N : M_N \longrightarrow M_N$ by the rule $\nu_N([x]) = [\nu(x)]$ for all $[x] \in N^\perp$. b_N and ν_N are easily seen to be well defined. ν_N has the range stated for it because of 3.1.1. It is well known that b_N is nonsingular.

Note: $\nu(x) \in N^\perp$, and $\hat{b} \circ \nu$ is symmetric.

Theorem 4.3. *If (M, b, ν) is a nil form, and N is a sublagrangian then $[M, b, \nu] = [M_N, b_N, \nu_N]$ in $Nil_{WQ}(R)$.*

We note that since $N^\perp$ is a direct summand, and b_N is nonsingular, $(N^\perp)^\perp = N$.

Proof. We show that $(M \oplus M_N, b \oplus -b_N, \nu \oplus \nu_N)$ is split by $\Delta_N = \{x \oplus [x] : x \in N^{\perp}\}$ Let β be $b \oplus -b_N$. Consider

$$\Delta_N^{\perp} = \{x \oplus [y] \in M \oplus M_N : x \in M, \;\; y \in N^{\perp} \;\;\; and \;\;\; \beta(x \oplus [y], z \oplus [z]) = 0 \;\;\; \forall z \in N^{\perp \cdot}\}$$

$$= \{x \oplus [y] \in M \oplus M_N | x \in M, \;\; y \in N^{\perp}, \;\;\; b(x,z) - b(y,z) = 0 \;\;\; \forall z \in N^{\perp}\}$$

$$= \{x \oplus [y] \in M \oplus M_N | x \in M, \;\; y \in N^{\perp}, \;\;\; x - y \in (N^{\perp})^{\perp}\}$$

$$= \{x \oplus [y] \in M \oplus M_N | x \in M, \;\; y \in N^{\perp}, \;\;\; x \equiv y \;\; mod \;\; N\}$$

$$= \{x \oplus [x] \in M \oplus M_N | x \in N^{\perp}\}$$

$$= \Delta_N$$

Now we show that$(\nu \oplus \nu_N)(\Delta_N) \subset \Delta_N$. For any $x \oplus [x] \in \Delta_N$,

$$(\nu \oplus \nu_N)(x \oplus [x]) = \nu(x) \oplus \nu_N([x]) = \nu(x) \oplus [\nu(x)]$$

Since $\nu(N^{\perp}) \subset N^{\perp}, \nu(x) \oplus [\nu(x)] \in \Delta_N$.

□

Lemma 4.4. *Let A be a finitely generated free R-module, and B a submodule of A. $\overline{B} = \{a \in A | ra \in B$ for some $r \in \mathbb{R} - \{0\}\}$. Then $\overline{B}$ is a direct summand of A.*

Proof. Consider the quotient $A/\overline{B}$. If $[a] \in A/\overline{B}$ and $n[a] = 0$ for some $r \in \mathbb{R} - \{0\}$ then $[a] = 0$. $A/\overline{B}$ is free since it is a finitely generated and torsion free R-module. Thus $0 \longrightarrow \overline{B} \longrightarrow A \longrightarrow A/\overline{B} \longrightarrow 0$ splits. So $\overline{B}$ is a direct summand. □

Theorem 4.5. *Let (M, b, ν) represent a typical element of $Nil_{WQ}(R)$, where $\nu^{k+1} = 0$ for some k, and $\nu^k \neq 0$. Then $N = \overline{\nu^k(M)}$ is a sublagrangian.*

Proof. By Lemma 4.4, N is a direct summand. Now we show that $N \subset N^{\perp}$. For any two elements X and Y in N, where
$rX = \nu^k(x)$ and $sY = \nu^k(y)$ for some $r, s \in R$ and $x, y \in M$,

(4.4.1)
$$rsb(X,Y) = b(rX, sY) = b(\nu^k(x), \nu^k(y)) = b(\nu^{2k}(x), y) = b(0, y) = 0.$$

And therefore, b(X,Y)=0, so $N \subset N^{\perp}$.

$$r\nu(X) = \nu(rX) = r\nu(\nu^k(x)) = r\nu^{k+1}(x) = 0.$$

Therefore $r\nu(X) = 0 \implies \nu(X) = 0$ since M is a free R-module, M is torsion free.

□

Theorem 4.6. $Nil_{WQ}(R) = 0$.

Proof. From theorem 4.5 any element of $[M, b, \nu]$ of $Nil_W Q$, (M, b, ν) contains a sublagrangian. By theorem 4.3
$[M, b, \nu] = [M_N, b_N, \nu_N]$ where $\text{rank}[M, b, \nu] > \text{rank}[M_N, b_N, \nu_N]$. By induction on the rank, we conclude that
$[M, b, \nu] = [0, 0, 0]$.

□

Theorem 4.7. $WQ(R[t]) \cong WQ(R)$.

Proof. By theorem 3.5, the sequence $Nil_{WQ}(R) \xrightarrow{\eta} WQ(R[t]) \xrightarrow{\epsilon_*} WQ(R) \longrightarrow 0$ is exact. But $Nil_{WQ}(R) = 0$, so $0 \longrightarrow WQ(R[t]) \longrightarrow WQ(R) \longrightarrow 0$
is exact.

□

5. References