REU PROJECTS

LIVIU I. NICOLAESCU

1. Introduction to geometric probability

Suppose R is a "reasonable" compact region in the plane sitting inside the a disk D. For example, R could be the finite union of compact convex sets K_0, K_1, K_2 depicted in Figure 1. In the nineteenth century J.J. Sylvester asked the following question?

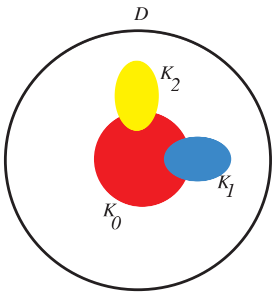

Figure 1. *A polyconvex set*

What is the probability that a line that intersects the disk D will also intersect a given reasonable region R contained inside the circle inside the circle D?

Lets denote by $\mu_1(R|D)$ this yet unknown probability , where we use the subscript 1 because we are intersecting R with one-dimensional objects, 1. Note a few properties of this quantity.

$\mathbf{P_1}$ μ_1 satisfies the inclusion exclusion principle meaning that if R_1, R_2 are two regions in D then

$$\mu_1(R_1 \cup R_2|CD) = \mu_1(R_1|D) + \mu_1(R_2|D) - \mu_1(R_1 \cap R_2|D).$$

In plain English this means that the "number of lines" intersecting R_1 or R_2 is equal to the "number of lines" intersecting R_1 plus he "number of lines" intersecting R_2 minus the he "number of lines" intersecting both R_1 and R_2.

$\mathbf{P_2}$ If we slightly rotate R inside C the quantity $\mu_1(R|C)$ does not change. Also, $\mu_1(R)$ should not change too much if we slightly deform R.

$\mathbf{P_3}$ Suppose D is the disk of radius one centered at the origin and R is a disk of radius r also centered at the origin. Then

$$\mu_1(R|D) = r.$$

To visualize this note that a line is determined by two numbers: the distance ρ away from the origin, and the angle θ it makes with the horizontal axis. The set of lines intersecting D can thus be identified with the rectangle $0 \leq \rho \leq$, $0 \leq \theta \leq \pi$ which has area π while the lines intersecting R can be identified with the rectangle $0 \leq \rho \leq r$, $0 \leq \theta \leq \pi$ which has area π. The ratio of the two areas is r.

For our region we have

$$\mu_1(R) = \mu(K_0) + \mu(K_1) + \mu(K_2) - \mu(K_0 \cap K_1) - \mu(K_0 \cap K_2).$$

Date: February, 2006.

Observe that all the regions K_i and $K_i \cap K_j$ are convex and thus is suffices to understand $\mu_1(R)$ only for convex regions.

Do we know a quantity associated to plane regions which satisfies the inclusion-exclusion principle? One example comes to mind immediately. Fix a real number c and set

$$\mu(R) = c \cdot \text{Area}\,(R).$$

There is a problem with this choice. Area is measured in square feet (sqft) while $\mathbf{P_4}$ shows that μ_1 is measured in feet (ft).

The goal of this project is to find all the functions μ which associate to any reasonable region in $\mathbb{R}^2$ or $\mathbb{R}^3$ a real number $\mu(R)$ which are continuous (change very little when R is changed slightly, are invariant under isometries of the plane ($\mu(R)$ does not change if we rotate R arbitrarily) and satisfy the inclusion-exclusion principle. The reasonable regions will be finite unions of bounded convex sets, while the functions μ with the sought for properties are called *valuations*. It turns out that for regions in $\mathbb{R}^2$ there are three very important valuations μ_0 (also known as the *Euler characteristic*), μ_2 (also know as *area*) and the valuation μ_1 which appears in Sylvester problem. Any other valuation μ can be written as a linear combination

$$\mu = \mathcal{T}_0\mu_0 + t_1\mu_1 + t_2\mu_2.$$

where t_0, t_1, t_2 are fixed real constants. Similarly there are only four interesting valuations for regions in $\mathbb{R}^3$. In this REU we will begin an investigation of these objects and some of their surprising properties. As guide we will follow the excellent booklet [**KR**].

2. Geodesics, curvature and Gauss-Bonnet

By early nineteenth century many mathematicians have started to suspect that *Euclid's parallel postulate* which is equivalent to the fact that the sum of angles of a triangle is 180° may not be universally true. Gauss in particular got interested in this and was shocked to find out that the sum of angles of a triangles formed by three far away stars is less than 180°.

Already several interesting question arise. For example, what is the meaning of triangle? To confuse you some more, have a look at Figure 2. On top we have two triangular regions on the sphere. Which of the two would merit the label of *"genuine" triangle*? Similarly, at the bottom of Figure 2 we have two triangular shapes in the plane. Which of the two would merit the label of *"genuine" triangle*?

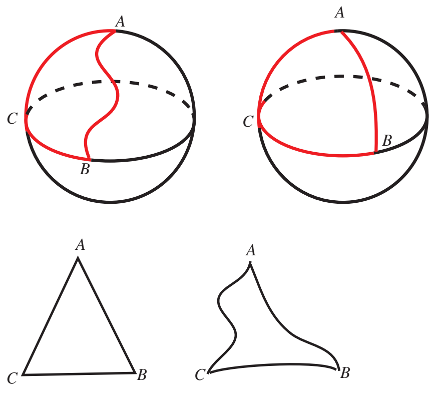

Figure 2. *Find the genuine triangles in this picture.*

In the standard Euclidean plane a the edges of a triangle must be straight line segments. Clearly no straight line segment fits on the surface of a sphere. To avoid this problem we can replace the idea of straight line with the idea of shortest path along the sphere connecting two given points. Such a curve is called a *geodesic* and a geodesic triangle is a region bounded by three geodesics.

This still does not solve the problem, it only gives it a new name. How do we measure the length of a curve on a surface? Do geodesics exists? If so, how do we determine them?

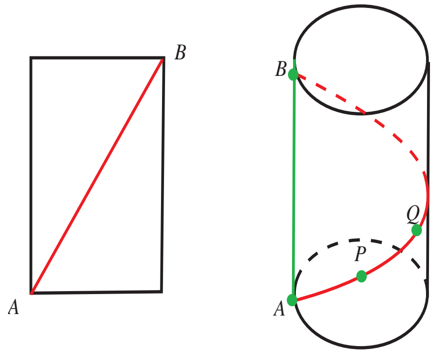

FIGURE 3. *Geodesics on the cylinder*

To add more to the "mystery" consider the following experiment depicted in Figure 3. Consider a rectangular sheet of paper, draw a straight line segment connecting a pair of opposite corners and then wrap it around a circular cylinder by gluing a pair of opposite edges. The straight line segment now looks like a helix and since we have not "stretched" the sheet of paper, a curve on the flat sheet is transformed to a curve on the cylinder of the same length. Notice however that the red helicoidal path on the cylinder is not the shortest path between A and B. The shortest path is the vertical green segment. However, the shortest path between the points P and Q is still the helicoidal red arc connecting them. This experiment should indicate that the "shortest path definition" should be taken with a grain of salt: the helix is a geodesic yet it may not be the shortest path between any of two points on it.

We can attempt to do a similar thing with a sphere. Try to wrap a piece of flat surface along a sphere, without distorting the lengths of curves: think of wrapping a piece of flat cloth around a sphere. It will now work and we will notice the appearance of folds. What goes wrong with the sphere? what makes it different from the cylinder?

In this REU we will gradually answer these questions and display their connection with the global appearance of a surface. As sources we will use the classics, [**HC**, **S**]. For a preview of some of the things we will cover see web preprint [**N**].

REFERENCES

[**HC**] D. Hilbert, S. Cohn-Vossen: *Geometry and the Imagination*, AMS Chelsea, 1999.

[**KR**] D.A. Klein: *Introduction to Geometric Probability*, Cambridge University Press, 1999.

[**N**] L.I. Nicolaescu: *The many faces of the Gauss-Bonnet theorem*, `http://www.nd.edu/~lnicolae/GradStudSemFall2003.pdf`

[**S**] D.J. Struik: *Lectures on Classical Differential Geometry*, Dover,1988.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556-4618.
E-mail address: `nicolaescu.1@nd.edu`
URL: `http://www.nd.edu/~lnicolae/`