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On the reliability of a class of system identification techniques: insights from bootstrap theory

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Abstract

The estimation of structural damping from systems with unknown input can be particularly challenging but is vital to furthering the understanding of energy dissipation in structural motions. The random decrement technique and spectral analysis are commonly invoked system identification schemes for this problem. Although there is well-established theory regarding the bias and variance errors of power spectra and random decrement signatures, the theoretical error formulae give only approximate indications of errors inherent to the estimated dynamic properties. This study addresses this through a bootstrap approach to assess the quality of system identification by providing surrogate estimates of damping and natural frequency to generate useful statistics and confidence intervals. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

With the advancement of modern structures to new heights, the issues of serviceability and occupant comfort have come to the forefront in design, making damping, now more than ever, a critical design parameter. Despite its significant role, damping continues to be an enigma in design [1,2]. As damping does not relate to physical properties of the structure in a direct way like mass and stiffness, it cannot be estimated with much certainty in design. While the examination of existing structures has resulted in international databases of actual damping values, there is considerable scatter in the data, partly attributed to the amplitude dependence of damping, but more so due to errors in the identification of this parameter. This is complicated by the fact that much of the full-scale data results from ambient excitations, providing the analyst with no measured

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input for system identification (SI). As a result, system identification must be conducted using "unknown input" system identification schemes or those that only make some general assumptions about the nature of the input. This restriction is often problematic and leads to a host of possible errors.

In light of the variability of the identified parameters and without the advantage of repeated measurements, the typical dynamic analyst is forced to make the best of only limited observations, with no indication of the accuracy of the resulting parameters. Although no method can identify precisely how much an estimate deviates from the true system characteristics, statistical information on the obtained dynamic properties provides a valuable reliability measure. As many commonly used techniques lack such information, the following study seeks to address the issue by developing tools to supplement traditional SI schemes. The proposed bootstrap approach resamples the limited data, treating those observed samples as representative of the entire population. While this assumption may be problematic if the data is a poor indicator of the true system, as shown in subsequent examples, the bootstrapping scheme provides the potential to extract information on the reliability of identified system properties through a computationally simple scheme.

1.1. Traditional approaches to system identification

In unknown input SI, the general assumption is that the driving random process meets certain restrictions of stationarity and Gaussianity, as well as being zero-mean, white noise. The satisfaction of these conditions then permits the use of two approaches: spectral analysis (SA) and the random decrement technique (RDT). Though there are some other system identification techniques in use, these two approaches are among the most widely used, with the spectral approach being the more traditional of the two.

1.1.1. Spectral analysis and inherent errors

The estimation of dynamic properties has commonly been accomplished through the use of the Power Spectral Density (PSD) of an assumed stationary, Gaussian process. As discussed, for example, in Bendat and Piersol [3], the PSD estimate can be simply generated by segmenting the measured response time history into blocks of sufficient length T to provide adequate spectral resolution Δf, defined as the difference between adjacent discrete frequencies, given by:

$$f_k = \frac{k}{T} = \frac{k}{N\Delta t} k = 0, 1, \ldots, N-1 \tag{1}$$

where N is the actual number of data values in the block of length T, sampled at the time step Δt. Commonly, N is selected to the nearest power of 2 to permit use of the Fast Fourier Transform (FFT), as was done in this study. The FFT is applied to each block of data to produce a raw spectrum. The squared magnitudes of these Ns raw spectra are then averaged, under standard ergodic assumptions, to obtain the PSD estimate. Note that additional windowing measures can be taken to suppress side lobe leakage [3]; however, since such windowing can increase the effective bandwidth of the spectra and the level of variance, this added step was not considered in this study.

The normalized bias error of the resulting power spectrum in the vicinity of the resonance frequency f_r reflects the importance of the frequency resolution achieved in this process [3]:

$$\varepsilon_b\left[\hat{S}_{xx}(f_r)\right] \approx -\frac{1}{3}\left(\frac{\Delta f}{B_r}\right)^2 \tag{2}$$

where $\hat{S}xx(f)$ is the estimated PSD and B_r is the half power bandwidth, approximated by [3]:

$$B_r = 2\xi f_n \tag{3}$$

where ξ is the critical damping ratio and f_n is the natural frequency of the system. Note the introduction of the half power bandwidth (HPBW) in Eq. (3) is an approximation valid only for lightly damped systems, i.e. $\xi < 0.1$, for which the resonant frequency may be taken as the natural frequency of the system [3]. Conveniently, this relationship permits a very simple and direct means of system identification from the response PSD, given that the input spectrum is constant. This assumption is typically acceptable when the ambient excitations of the system can be approximated as white noise, at least in the vicinity of the spectral peak. Fig. 1 illustrates that the half power bandwidth may be determined by identifying, via interpolation between the discrete frequencies in Eq. (1), the two frequencies f_2 and f_1 that correspond to half the maximum amplitude of the PSD. Assuming symmetry of the spectral peak, the HPBW is then defined as the difference between these two frequencies: $B_r = f_2 - f_1$, with the frequency corresponding to the spectral peak taken as the natural frequency of the system. This permits the system damping to be identified readily from Eq. (3).

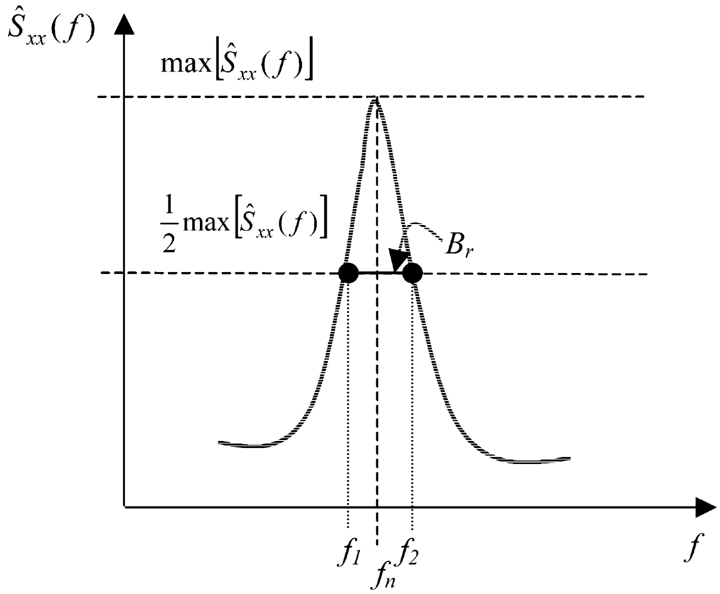

Fig. 1. Schematic of half power bandwidth method of spectral system identification.

The quality of system identification using the PSD is not only dependent upon the minimization of normalized bias errors in Eq. (2), but also depends heavily upon limiting the normalized variance of the spectrum, as given by [3]:

$$\mathrm{var}\left[\hat{S}_{xx}(f)\right] \approx \frac{1}{N_s}. \tag{4}$$

Note that this expression assumes that the raw spectra were determined from independent, disjoint blocks of data of length T without the inclusion of windowing.

Eqs. (2) and (4) are used commonly as guidelines in the system identification process to determine the amount of data necessary to simultaneously minimize bias and variance errors and have become the benchmark by which the reliability of system identification using the PSD is assessed. Unfortunately, while estimates of the bias and variance of the PSD are attractive, they do not directly provide an indication of the bias and variance of the damping estimate obtained by the aforementioned procedure, in part motivating this investigation.

1.1.2. Random decrement technique and inherent errors

The random decrement technique is a time-domain approach that has recently become popular for system identification of wind-excited responses. Assuming that the process is zero-mean, Gaussian and stationary, time history segments of a prescribed length are captured, upon the satisfaction of a threshold condition [4,5]. The triggering condition, in its strictest sense, will specify both amplitude x_o and slope $\dot{x}_o$ criteria, though applications of the RDT in the literature have adopted varying trigger conditions. Note that the duration of segments captured is subjective, but typically is on the order of a few cycles of oscillation.

The captured segments are averaged to remove the random component of the response, assumed to be zero mean, as shown schematically in Fig. 2. This conceptualization reflects a common perception of the RDT, in which the total response is represented by the superposition of the forced vibration response with the homogeneous component or free vibration decay from given initial conditions. As the random component averages to zero, a Random Decrement Signature (RDS) $D_{x_o}(\tau)$ is obtained. This signature, proportional to $R_X(\tau)$, the autocorrelation function of the system, was shown by Vandiver et al. [5] to reduce to

$$D_{x_o}(\tau) \equiv E\left[x(t_2)\middle|x(t_1) = x_o \text{ and } \dot{x}(t_1) = \dot{x}_o\right] = x_o R_x(\tau)/R_x(0) \tag{5}$$

where $\tau = t_2 - t_1$. Though the RDT essentially provides an estimate of the autocorrelation function, it is able to produce this estimate without the same strict requirements for lengthy stationary data and permits the identification of amplitude-dependent damping, as discussed by Jeary [6].

In the case of single degree of freedom (SDOF) linear oscillators excited by Gaussian, zero-mean, white noise, the autocorrelation function normalized by a constant C takes the following form:

$$R_x(\tau) = C\mathrm{e}^{-\xi\omega\tau}\cos(\omega_D\tau), \tag{6}$$

analogous to the free vibration response of an oscillator with critical damping ratio of ξ and frequency of $\omega = 2\pi f_n$, substituting the natural frequency ω for the damped natural frequency ω_D

for this lightly damped system. As long as the white noise assumption remains valid (implications of this are discussed in [1,7]), the analogs between Eqs. (5) and (6) may be exploited for system identification, via least squares minimization to obtain best-fit estimates of damping ξ and natural frequency f_n, letting $C=x_o/R_x(0)$. Though this approach was used in this study, logarithmic decrement or other identification techniques may also be used to determine the damping of the system. Though this simplified approach is designated only for SDOF systems, the RDT can be used to analyze multi-degree of freedom (MDOF) systems by the approach described herein with the incorporation of bandpass filtering [8,9] or by introducing the recently developed vector random decrement technique [10]. However, as this study is concerned with establishing the reliability of RDT estimates of system parameters, it is sufficient for demonstrative purposes to consider only the SDOF formulation.

The resulting RDS will be unbiased with variance that can be expressed by [5]:

$$\mathrm{var}\left[D_{x_o}(\tau)\right] = E\left[D_{x_o}^2(\tau)\right] - E\left[D_{x_o}(\tau)\right]^2 = R_x(0)/N_r\left[1 - R_x^2(\tau)/R_x^2(0)\right] \tag{7}$$

where N_r = the number of segments averaged in the estimate. The presence of noise was ignored in this idealized derivation, as was the potential correlation between the captured segments. To

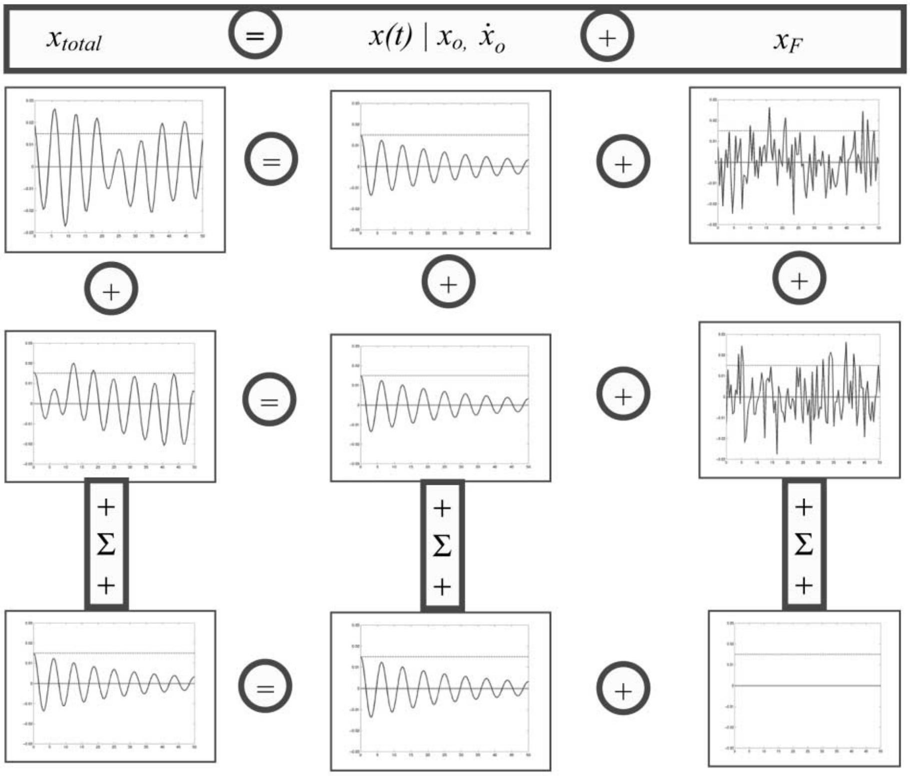

Fig. 2. Conceptualization of the random decrement technique.

satisfy these theoretical constraints, captured segments were not allowed to overlap in this study, though the ramifications of this condition are explored in Kijewski and Kareem [11].

As illustrated by Eq. (7), the variance in the RDS increases with each cycle of oscillation. Thus to achieve reliable results, the system identification advocated in this study considers only the first two cycles of the RDS. Note also that although the amplitude level selected for the triggering condition has no direct effect on the variance, it will dictate the number of segments averaged. Clearly, this number of segments should be increased as much as possible to minimize variance. Relaxation of the triggering condition to merely specify an amplitude level and sign to the slope will generate more segments for averaging; however, this does not provide precise initial conditions for the captured segments. This theoretical requirement may only be achieved by defining a specific value for both the amplitude and slope of the trigger, e.g. by capturing only peaks of specified amplitude. This strategy was proposed by Tamura and Suganuma [8] to permit more precise amplitude-dependent damping estimation and shall be used in this study as a strict triggering condition. Unfortunately, this condition will require more measured data, thus any peak within a certain percentage (e.g. 3%) of the mean peak value is chosen as the amplitude trigger x_o in order to generate more eligible samples for this linear system.

1.2. Variability of estimated dynamic properties

From Eqs. (4) and (7), it is evident that the approximate variance of both the PSD and RDS can be determined theoretically; however, the potential errors of the power spectrum and decrement signature are not truly the reliability measure being sought, as estimates of dynamic parameters are the end result of this process. Seybert [12] recognized this concern and derived approximate expressions for the bias and random errors in the damping estimation using the Half Power Bandwidth of power spectra. Even though these depend on the very quantity being estimated, they have proven insightful and indicate that the normalized bias in the damping estimate is approximately 1.5 times the normalized bias of the power spectrum. This derivation was made realizing that the bias of the PSD itself in the vicinity of the HPBW points will directly affect the estimates of damping. Seybert [12] also derived an approximate expression for the normalized variance of the damping estimate, again confirming that the damping variance is proportional to the number of raw spectra considered. However, the expression has limited utility in this case, as the measure of the coherence function between the input and output processes at the HPBW points is required. Another option is empirical expressions for the coefficient of variation, such as those determined by Montpellier [13]. Though useful, these expressions are again approximate and idealized.

Typically, without the advantage of repeated experiments or measured input, the dynamic analyst is forced to make the best of only limited observations to make a reliability estimate. However, non-parametric resampling schemes can be useful in such cases. These approaches amount to treating the observed samples as if they are representative of the larger population, then resampling this limited data to approximate variance [14]. Using this computationally simple scheme, the analyst can effectively "extend" the data used to obtain a single estimate of damping in order to approximate the variance of that estimate. Such approaches further allow the assignment of confidence intervals without necessitating knowledge of the parameter distribution. This practical tool is introduced here to quantify random errors of frequency and damping estimates garnered from these two common system identification approaches.

It should be noted that other system identification techniques have also considered the issue of statistical reliability of the identification process. For example, Autoregressive Moving Average (ARMA) schemes (e.g. [15,16]) or simplified Autoregressive (AR) models utilizing the Maximum Entropy Method (e.g. [17]) implicitly provide performance measures in terms of prediction errors, which indirectly reflect the quality of the AR or ARMA identification. More recently, a Bayesian spectral density approach was introduced by Katafygiotis and Yuen [18] to update the PDF of modal parameters for ambiently-excited data. The resulting PDF obtained in the minimization scheme was found to be well approximated by a Gaussian distribution, whose mean is indicative of the optimal parameter estimates and covariance yields a direct measure of uncertainty. Approaches like these provide a variance estimate of the damping parameter as a by-product of the minimization operation or "goodness of fit" to an assumed model. The associated errors then indicate how well the estimated damping parameter value fits the available data, providing one form of reliability measure.

On the other hand, the intent of this work is not to propose an altogether new SI technique with error estimates to quantify a "goodness of fit." Rather, this study provides a practical computational tool to generate the statistical reliability measures for frequency and damping parameters estimated using traditional approaches like SA and RDT, for which no statistical confidence measures are implicitly provided. It is the inherent randomness in the data that influences the quality of the PSD and RDS and thereby the damping estimates garnered from them. As these schemes must average out the effects of randomness, their performance becomes dependent upon sufficient amounts of available data. The intent of this work is to mimic this random characteristic to provide a supplementary tool to gauge the potential variance in the resulting damping estimate from a given spectral or random decrement analysis.

1.3. Errors in system identification: an exercise

System identification by any approach represents the process through which optimal values of system parameters are obtained, though this is complicated without the benefit of measured input. In particular, when using SA or RDT, while some understanding of uncertainty is available, specific measures of variance in frequency or damping estimates are in general not known. To illustrate this, frequency and damping are estimated from the response of ambiently-excited SDOF oscillators sampled at 10 Hz. In accordance with common practice, the minimum spectral resolution is determined by limiting the normalized spectral bias error in Eq. (2) to less than −2% and determining the required number of FFT points (N) conservatively to the nearest power of two. To minimize normalized variance errors to 10%, enough data is generated to yield 100 sufficiently resolved raw spectra. These time histories are generated by passing stationary, Gaussian, zero mean white noise through the SDOF system. This random input was simulated from a target band-limited spectrum, with unit magnitude from 0 Hz to the cutoff frequency f_c [19]. The resulting simulation parameters are documented in Table 1 for three cases.

Though often not obtainable in practice, significant amounts of stationary data are considered for illustrative purposes. The first case represents a lightly damped 1 Hz oscillator. The latter two cases involve more narrowband systems, requiring significantly more data to achieve the same spectral bias and variance errors. In Case 3, though the same amount of data is generated, the length of segments captured in the RDT is shortened from 60 to 30 s in order to increase the

Table 1
Properties of simulated cases

Case	f_c (Hz)	f_n (Hz)	ξ	N	Ns	Nr[a]	Length of data (Hr)
1	5	1.0	0.01	4096	100	~600–700	11.4
2	1	0.2	0.01	16384	100	~700–800	45.5
3	1	0.2	0.01	16384	100	~1100–1200	45.5

[a] Actual Nr will vary in each run dependent upon the number of peaks forming non-overlapping segments.

number of segments being averaged, a critical factor in RDT performance. Individual time histories generated by this approach were retained for the subsequent resampling analysis presented in this paper.

Armed with this data, the normalized bias is assumed to be less than −2% and the normalized variance to be 10%, but the exact error in the frequency and damping identified from the PSD is still not known. To illustrate the variability possible, the systems in Table 1 are each simulated 50 times. In each case, the same amount of data is used so that the normalized bias and random errors on paper are the same, and the system is then identified by both the spectral analysis and random decrement technique discussed respectively in Sections 1.1.1 and 1.1.2.

Even in the presence of a favorable amount of data, the variability inherent in this random process can cause considerable scatter in the identified parameters, as shown by Table 2 summarizing the statistics of the simulations, including the mean μ, standard deviation σ and coefficient of variation (*Co*V) defined as σ/μ. From Table 2 and Fig. 3, which graphically displays the data for Case 2, it is not surprising to confirm that the frequency is relatively simple to identify with accuracy. The damping, on the other hand, is much more difficult. As expected, SA produces a biased overestimation of the damping present, as the smoothing of the spectrum results in an underestimation of the spectral peak and thus an overestimation of the damping. The fact that the SA bias is consistent should not be surprising since the goal was to maintain the same level of bias. Though the normalized bias in the power spectrum is less than −2%, the normalized bias in the damping estimated from it is several times that amount. Recall that Seybert's [5] expression

Table 2
Statistics of Monte Carlo simulations

	$\mu[f_n]$ (Hz)	$\sigma[f_n]$	Bias $[f_n]$ (Hz)	CoV $[f_n]$ (%)	$\mu[\xi]$	$\sigma[\xi]$	Bias$[\xi]$	CoV$[\xi]$ (%)
Case 1								
SA	1.0023	0.0012	0.0023	0.12	0.01041	0.00029	0.00041	2.76
RDT	1.0003	0.0014	0.0003	0.14	0.00921	0.00124	−0.00079	13.50
Case 2								
SA	0.2007	0.00026	0.0007	0.13	0.01056	0.00033	0.00056	3.12
RDT	0.2001	0.000256	0.0001	0.13	0.00995	0.00126	−0.00005	12.67
Case 3								
SA	0.2006	0.00027	0.00060	0.13	0.01052	0.00032	0.00052	3.01
RDT	0.2000	0.00025	−0.00002	0.12	0.01011	0.00109	0.00011	10.79

underestimated this over prediction as only 1.5 times the normalized PSD bias. As expected, the bias does become a bigger problem in the narrowband system. For Case 3, this narrowband system was resimulated, with the same amount of data as in Case 2. As a result, the marginal changes in the statistics associated with the SA are only the result of the new random excitation generated to simulate the system. The variance is largely unchanged with standard error changing only by 4% between the two cases. The *CoV* for the SA in all cases is between 2.75 and 3.15%. It should be relatively constant for all cases as the length of data was selected to maintain the same bias and random errors in the PSD. Note that the variance estimates in Table 2 are still about one half the estimate provided in [13].

Though both approaches have near identical performance with respect to frequency identification, there is, unfortunately, a considerable amount of scatter in the RDT damping result, leading to a *CoV* that is an order of magnitude greater than SA result. This *CoV* is nearly half that observed by Montpellier [13], as expected due to the large amount of data considered. The level of bias in the RDT result is more significant in the broader band system, perhaps due to the limited number of segments being averaged. However, when large amounts of segments are available, the

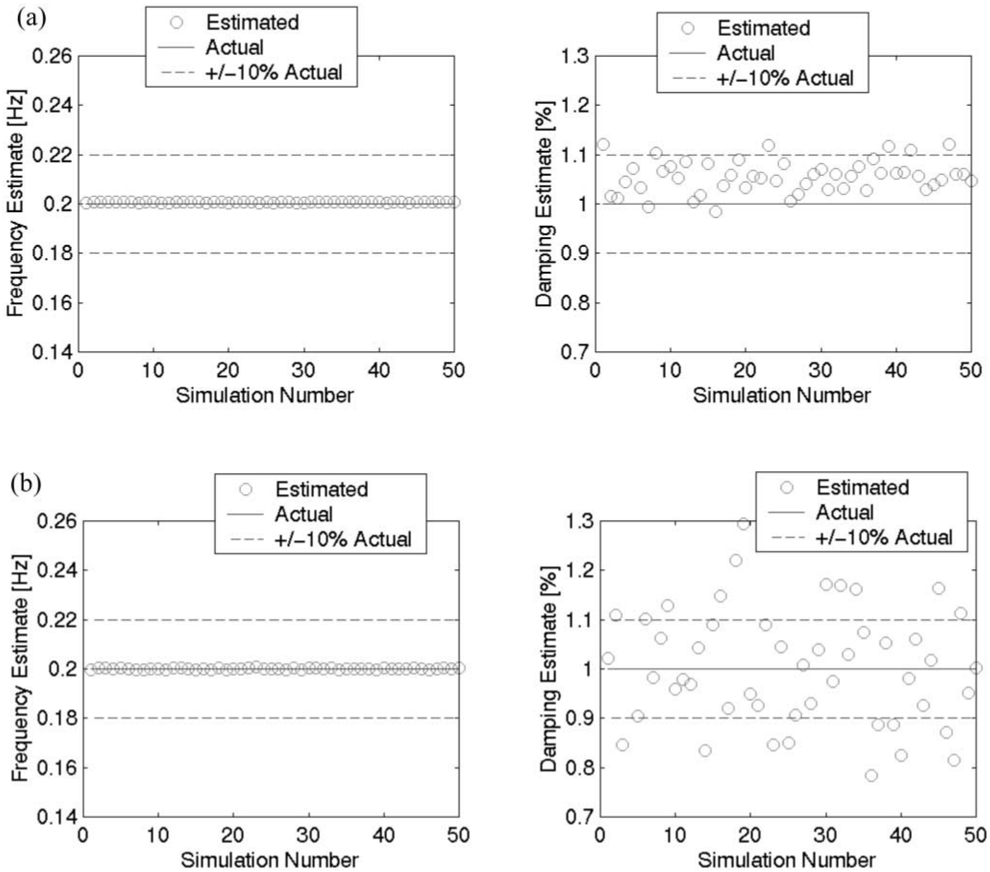

Fig. 3. Identified dynamic properties of simulated data: identification of Case 2 by (a) spectral analysis and (b) random decrement technique.

estimate of damping is nearly unbiased, as expected. This is advantageous, since narrowbanded systems are increasingly difficult to identify by spectral analysis. The larger number of RDS averaged in Case 3 helps to further reduce the deviation of damping estimates by RDT. Increasing *Nr* by a few hundred samples has led to a nearly a 15% decrease in the standard deviation. This tends to be the limiting factor in the RDT, as the number of segments being averaged has direct bearing on the RDS variance.

It is important to note that due to the random nature of the process identical amounts of data subjected to the same SA can produce an estimate of damping which has no error or as much as 10% error. Though 10% error may be reasonable, this situation also illustrates a very favorable and perhaps unrealistic amount of data. Under these ideal conditions, the damping estimated by independent simulations has inherent variability, which is no doubt enhanced under less ideal conditions. On the other hand, the RDT identification of the system from any one of these simulations can produce estimates of damping that are near exact or in error by up to 20–30%. Though Table 2 illustrates that if one has the luxury of repeating an experiment 50 times under identical conditions, acceptable results are obtainable in the average, it is beneficial to be able to assess the accuracy of identified parameters from a given time history. The accuracy of any identification by one of these two approaches is contingent upon the level of randomness in the measured data and the degree to which it has been eliminated in the averaging process.

2. Bootstrapping schemes in system identification

If the distribution of a random variable were known, then theory or simulation may be invoked to calculate various statistics. However, in most practical applications, this is not possible, but the bootstrap may be used to make the best of what information is available. The bootstrap approach is a computer-based method for assigning accuracy to statistical estimates based on independent data points or samples, which, in its simplest form, is non-parametric, requiring no assumptions about the distribution of the parameters [14]. The approach is widely documented in the literature and has a variety of practical applications elaborated in textbooks [20,21]. Even in cases where the statistic is too complicated for theoretical estimates of random errors, the bootstrap can be invoked to make some inference.

When the sample population is large enough, the Central Limit Theorem (CLT) can usually be invoked to assert that the estimator is approximately distributed as a Gaussian random variable. Luckily, the underlying assumptions of bootstrap analysis are valid in cases where there are only limited samples available, say 10 or 15 samples, though both will approach one another in situations where the population is large. The bootstrap is also capable of capturing non-Gaussian statistics such as skewness that could not be captured relying on CLT [20].

This process of statistical inference involves estimating some statistic $\hat{\theta} = s(x)$ of an unknown Probability Density Function (PDF) F of a population based on an observed random sample $x = [x_1, x_2, \ldots x_n]$ drawn from it. Let the sampled data compose a population with empirical distribution function $\hat{F}$. By randomly sampling with replacement from the observed values, a new sample or bootstrap sample $x^* = \left[x_1^*, x_2^*, \ldots x_n^*\right]$ can be generated, which is not the actual data but a randomized or resampled version of it. The resampled data can then be used to estimate the statistic of interest $\hat{\theta}^* = s(x^*)$ to produce a bootstrap replication. The rationale for this metho-

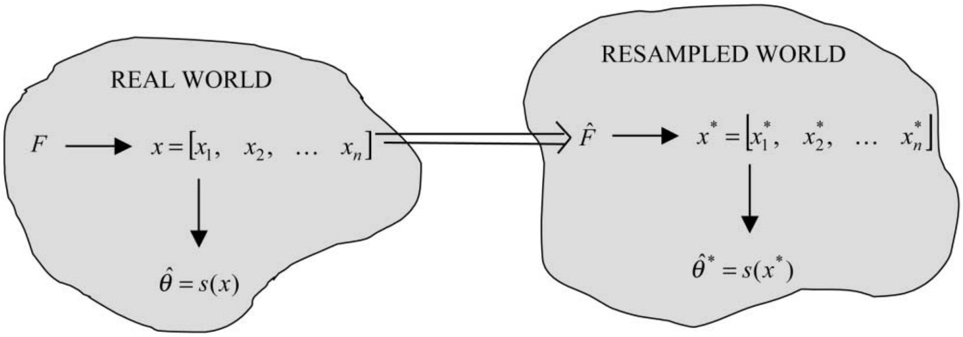

Fig. 4. Schematic diagram of generalized bootstrap concept (adapted from [20]).

dology is based on simple analogies depicted in Fig. 4. F gives x by random sampling, so $\hat{F}$ gives x^* by random sampling; $\hat{\theta}$ is obtained from x via the function $s(x)$, so $\hat{\theta}^*$ is obtained from x^* in the same way [20]. As the resamples are generated as a function of the sample distribution, which is itself a randomized sample of the actual population, the distribution of the resamples varies randomly in the same way that the sample population does, paralleling Monte Carlo simulation, as discussed in [20]. Note that the function $s(x)$ can be virtually any statistic or parameter of the system. This resampling process is repeated B times to form B replications. Based on these B replications, the variance or standard error of the population can be determined. The resampling scheme proposed here merely mimics the randomness of the process itself, more closely approximating the process as more and more data is considered, i.e. as $F \rightarrow \hat{F}$.

For the present application, the measured response of an oscillator can be assumed to be one such random sample, as the driving process is randomly varying. By the nature of the random process driving the system, the estimated damping and frequency are not deterministic, but are also random due to the inherent variability in the PSD and RDS from which they are drawn. Therefore, the lengthy amounts of data analyzed can be segmented into assumed independent random samples—the segmenting discussed previously to produce RDT segments satisfying a trigger or the segmenting in the generation of the PSD. Each segment, itself a random sample, can be subjected to the same bootstrapping methodology, as it is assumed to be randomly drawn. It takes a number of hours of measured response data to obtain a single, reasonable estimate of the system's damping by RDT or SA, yielding a large amount of data for which the empirical distribution of the sampled population is indeed approaching the true PDF.

As the generalities of bootstrapping have been discussed, the actual bootstrapping scheme for this application is now proposed in Fig. 5. As first implemented in Vandermeulen et al. [22] and Kijewski and Kareem [11], the N_s raw spectra and the N_r time-history segments that satisfy the RDT trigger condition form the sample population. From this population, Ns or Nr samples are drawn with replacement to form one bootstrap sample. Each bootstrap sample is then averaged to form a smooth PSD estimate or stable RDS and form a bootstrap replicate. This is repeated B times to form B replicates. These bootstrap replicates can be plotted atop one another to create variance envelopes, shown by the thick outline in Fig. 6, enveloping the plug-in estimates of the random decrement signature and power spectral density, i.e. those that would be obtained from the procedures in Sections 1.1.1 and 1.1.2 without resampling. Significant deviations from the estimated PSD and RDS (shown in black) highlight the areas of highest variance.[1] Each of these B replicates is then used to determine the system's dynamic properties and make statistical inferences on the reliability of the estimate via calculations of the standard errors of these values. The ideal estimate of the standard error of these replicate estimates is obtained by [20].

$$\sigma = \left\{ \sum_{b=1}^{B} \left[\hat{\theta}^*(b) - \hat{\theta}^*(\cdot) \right]^2 / (B-1) \right\}^{1/2} \tag{8}$$

where

$$\hat{\theta}^*(\cdot) = \sum_{b=1}^{B} \hat{\theta}^*(b)/B. \tag{9}$$

Eq. (9) can be viewed as the bootstrap mean, with the accuracy of Eq. (8) enhanced as $B \to \infty$. Discussions on the practical values of B are provided in [20].

Understandably, for this approach to work, it must be assumed that the sample population, represented by the measured time history, is representative of the actual process. If the data is drawn from periods of extremes, it may not be indicative of typical behavior. So it is vital that the samples drawn are sufficient to make inferences about the total process behavior. This is a critical consideration. In cases were data are of poor quality, accurate estimates and information cannot be provided by any approach, even the bootstrap. The bootstrap cannot repair sampled data, but can merely make inferences about its various statistics. Even with this in mind, it is hoped that the introduction of such a scheme will provide practitioners with a simple means by which to estimate the variance of the frequency and damping estimated from power spectra and random decrement signatures, which currently lack such reliability measures. Before doing so, it should be noted that

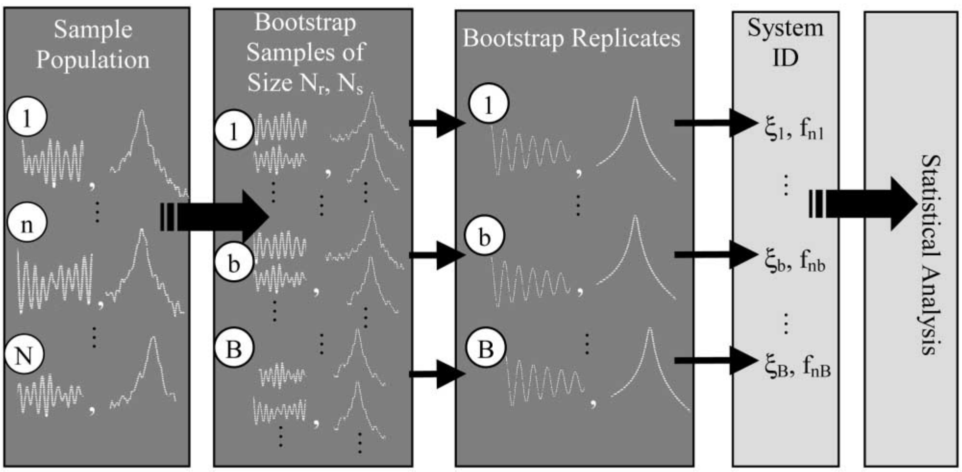

Fig. 5. Proposed bootstrapping scheme for system identification.

[1] It was shown in Kijewski & Kareem [11] that the Bootstrap Approach provides an estimate of the standard error consistent with theoretical predictions.

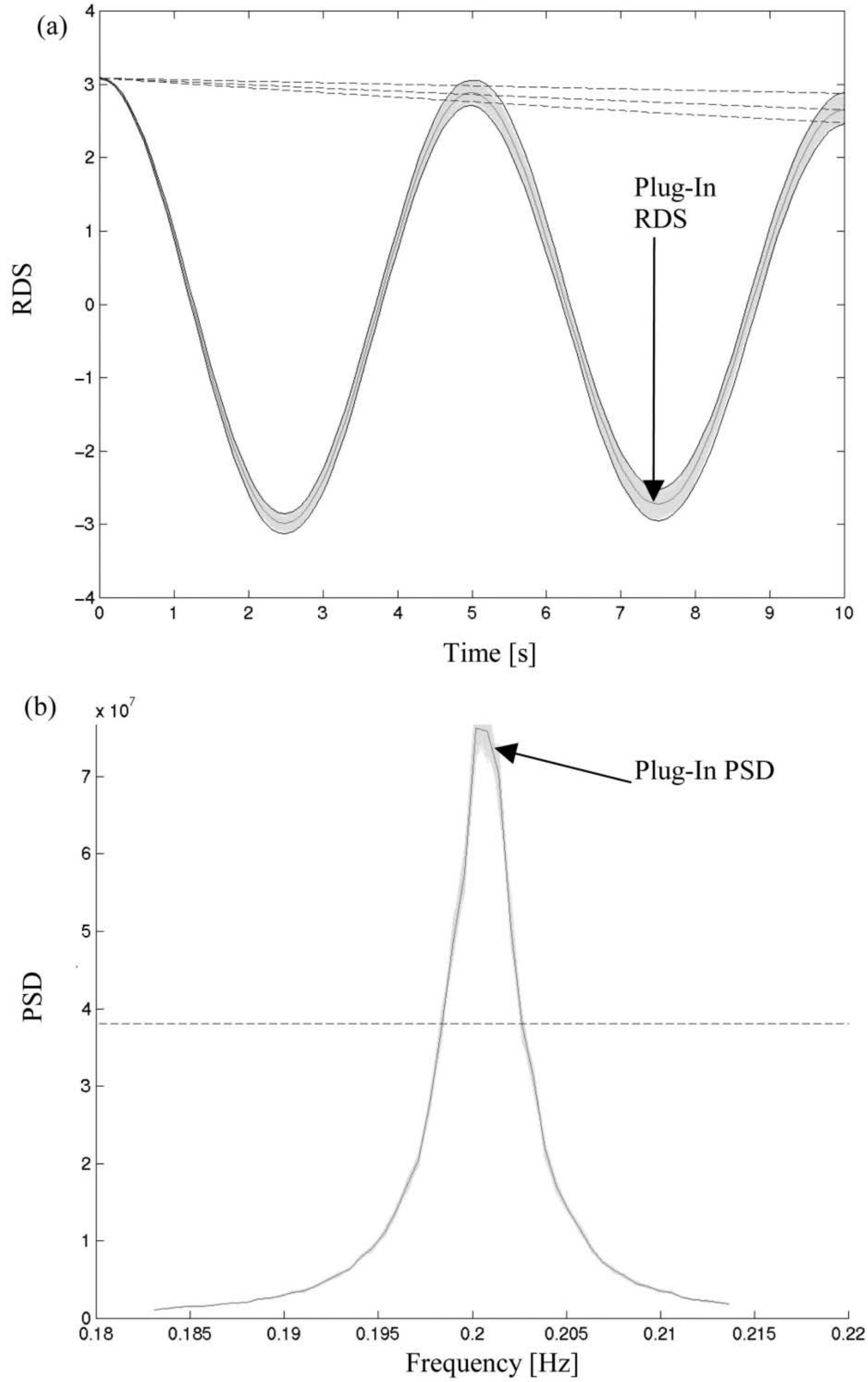

Fig. 6. Variance envelopes for (a) RDT and (b) PSD. Grey lines indicate variance envelope; black line indicates traditional RDT and PSD estimate; dotted lines indicate RDS decay and HPBW.

this concept has also been applied for identification of confidence intervals for instantaneous frequency estimates, which also previously lacked the interval estimates [23].

3. Statistical analysis via bootstrapping scheme

For each case in Table 1, the bootstrapping scheme outlined in Fig. 5 is applied to selected time histories simulated by Monte Carlo in Section 1.3. To illustrate the information that can be gained, particularly when limited data is available, increments of the total time history are analysed so that the influence of the number of raw spectra (N_s) and the number of segments in RDT (N_r) can be determined. As the frequency is estimated with near certainty every time, it shall not be discussed here for brevity. The tables that follow contain the bootstrap estimate of the damping, ξ_{boot}, defined as the mean of the damping identified from $B=1000$ bootstrap replicates [see Eq. (9)]. The traditional estimate of damping, without any resampling, is commonly termed the plug-in estimate, ξ_{plugin}. The standard deviation of the bootstrap replicates σ, as defined by Eq. (8), and the *Co*V, defined as the ratio of the standard error to the bootstrap mean in Eq. (9), are also provided. The bootstrap bias in the estimate is then defined as the difference, $\xi_{\text{boot}} - \xi_{\text{plugin}}$. This measure is important, as a small bootstrap bias, relative to the standard error, confirms that the bootstrap is a good estimate of the parameter, e.g. bias/σ < 0.25. As this measure becomes larger, the bootstrap estimates may no longer be accurate and require bias correction, which is discussed further in Efron and Tibshirani [20]. Additionally, by virtue of the boot-strapping scheme, histograms depicting the distribution of the damping estimate from a given time history can be obtained. These are useful tools for identifying the underlying distribution of damping estimates and its associated characteristics. An example of such a histogram for the damping estimates from a single time history (Case 2) is given in Fig. 7. While viewing these results, please be reminded that the simulated system had a critical damping ratio of 0.01.

Additionally, confidence intervals are defined for the damping estimate. By traditional analysis, only a single estimate of damping is available from each simulated time history. However, through the bootstrapping scheme, this estimate is enhanced by a family of associated estimates, which can give valuable insight into the reliability of a given damping estimate. In the most elementary formulation, a level of confidence can be selected, and then the replicates that correspond to this level can be identified from the resampled distribution [20]. By virtue of the non-parametric nature of this approach, there is no need to make any assumptions about the normality of the damping estimates. This is especially useful in cases where the PSD or RDS is generated from a limited number of samples. Assuming the *CoV* of the bootstrapped replicates is small, e.g. less than 20% and the bootstrap biases are negligible, a Gaussian PSD may be assumed to describe the distribution of the bootstrapped damping parameter. The 95% confidence bands are then approximated to lie within 2 standard deviations of the bootstrap mean [20]. These bands are provided in the last column of the following tables for comparison.

3.1. Discussion of resampled results for spectral analysis

As shown in Table 3, the bootstrap investigations conducted on a single record from the Monte Carlo simulations in Section 1.3 predict significantly smaller values of standard errors than

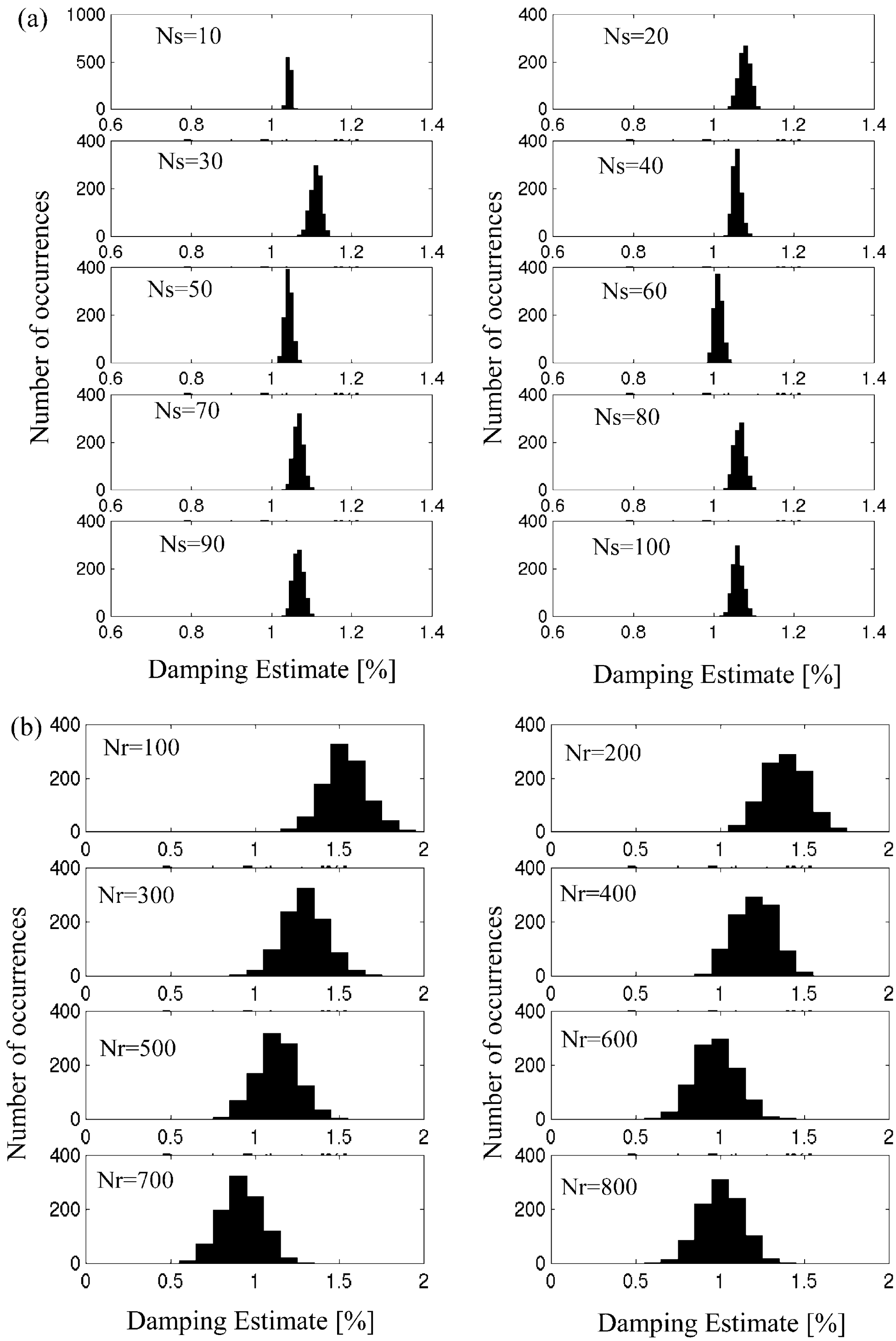

Fig. 7. Histogram of bootstrap simulations on (a) PSD and (b) RDS for Case 2.

anticipated. However, the relative changes are consistent with Monte Carlo predictions, when examining the full data set ($N_s = 100$). Case 1 has smallest variance, thus one would expect the Case 1 bootstraps to reflect that smaller variance in comparison to Cases 2 and 3. Likewise, Case 3 produced a slightly smaller value of standard error than Case 2, a trend reflected in the bootstrap errors. From the sample histogram in Fig. 7, it is evident that in cases of limited data there are shifts in the damping estimate distribution about various mean values. It is only for $N_s > 50$ that the behavior stabilizes and the damping estimates distribute about a relatively constant mean, indicating that the variance is sufficiently minimal.

It is interesting to note that, even with modest amounts of data (e.g. $N_s = 10$), a relatively good estimate of the damping is obtainable, while the addition of further recorded data may lead to

Table 3
Statistics of bootstrap replications of critical damping ratio: estimates by SA

*N*s	ξ_{boot}	ξ_{plugin}	Bias[ξ]	$\sigma[\xi]$	CoV (%)	95% Boot	95% Normal
Case 1							
10	0.010299	0.01030	−0.000011	0.000084	0.82	(0.01016, 0.01044)	(0.01013, 0.01047)
20	0.010478	0.010481	−0.000029	0.000082	0.78	(0.01034, 0.01060)	(0.01031, 0.01064)
30	0.010571	0.010564	0.000064	0.000081	0.77	(0.01044, 0.01071)	(0.01041, 0.01073)
40	0.010167	0.010165	0.000016	0.000079	0.78	(0.01004, 0.01030)	(0.01001, 0.01032)
50	0.010099	0.010096	0.000032	0.000078	0.77	(0.00997, 0.01023)	(0.00994, 0.01025)
60	0.009940	0.009940	0.000041	0.000080	0.80	(0.00981, 0.01007)	(0.00978, 0.01010)
70	0.010127	0.010131	−0.000037	0.000082	0.81	(0.00999, 0.01025)	(0.00996, 0.01029)
80	0.010075	0.010079	−0.000044	0.000079	0.78	(0.00995, 0.01020)	(0.00993, 0.01023)
90	0.010043	0.010041	0.000019	0.000076	0.76	(0.00991, 0.01017)	(0.00989, 0.01019)
100	0.010029	0.010027	0.000013	0.000083	0.83	(0.00989, 0.01016)	(0.00986, 0.01019)
Case 2							
10	0.01044	0.01044	−0.000001	0.000049	0.47	(0.01036, 0.01052)	(0.01034, 0.01054)
20	0.01077	0.01080	−0.000031	0.000143	1.33	(0.01053, 0.01101)	(0.01049, 0.01106)
30	0.01110	0.01110	−0.000005	0.000130	1.17	(0.01088, 0.01130)	(0.01084, 0.01136)
40	0.01058	0.01061	−0.000033	0.000105	0.99	(0.01041, 0.01077)	(0.01037, 0.01079)
50	0.01043	0.01044	−0.000015	0.000095	0.91	(0.01027, 0.01059)	(0.01024, 0.01062)
60	0.01012	0.01012	0.000002	0.000104	1.03	(0.00996, 0.01029)	(0.00991, 0.01033)
70	0.01067	0.01075	−0.000081	0.000121	1.13	(0.01048, 0.01088)	(0.01043, 0.01091)
80	0.01064	0.01065	−0.000009	0.000134	1.26	(0.01043, 0.01086)	(0.01038, 0.01091)
90	0.01067	0.01069	−0.000024	0.000130	1.22	(0.01043, 0.01089)	(0.01041, 0.01093)
100	0.01061	0.01065	−0.000045	0.000132	1.24	(0.01039, 0.01083)	(0.01035, 0.01087)
Case 3							
10	0.010637	0.010637	−0.000001	0.000065	0.61	(0.01053, 0.01075)	(0.01051, 0.01077)
20	0.010104	0.010104	0.000000	0.000117	1.16	(0.00992, 0.01030)	(0.00987, 0.01034)
30	0.010187	0.010193	−0.000007	0.000113	1.11	(0.01001, 0.01038)	(0.00996, 0.01041)
40	0.010453	0.010464	−0.000010	0.000107	1.02	(0.01026, 0.01062)	(0.01024, 0.01067)
50	0.010516	0.010522	−0.000006	0.000103	0.98	(0.01032, 0.01068)	(0.01031, 0.01072)
60	0.010346	0.010348	−0.000002	0.000122	1.18	(0.01015, 0.01052)	(0.01010, 0.01059)
70	0.010089	0.010094	−0.000005	0.000109	1.08	(0.00991, 0.01026)	(0.00987, 0.01031)
80	0.009837	0.009840	−0.000003	0.000123	1.25	(0.00963, 0.01005)	(0.00959, 0.01008)
90	0.009922	0.009917	0.000005	0.000121	1.22	(0.00973, 0.01012)	(0.00968, 0.01016)
100	0.009993	0.009991	0.000002	0.000116	1.16	(0.00980, 0.01020)	(0.00976, 0.01022)

poorer estimates. This may seem counterintuitive, as the variance of the PSD has been shown to reduce with the number of raw spectra being averaged. However, the inherent randomness of the process must be considered. The first few hours of data may not have significant variance in comparison to later components of the overall time history, luckily leading to a reasonable estimate of damping. Likewise the variance is also dependent on the magnitude of the PSD itself, which also fluctuates in each case considered. These random fluctuations can and should be expected when limited data is used, as the variance is high [22]. It is only in the limit that a more stable and reliable PSD results—one without marked fluctuations as more data is considered in the average. As evidenced by Fig. 7 and Table 3, it would indicate that this is achieved for $N_s > 60$ for the narrowband system, but achieved much sooner, $N_s > 40$ for Case 1.

As evident from Fig. 3, there is a discernable amount of bias in the spectral estimate of the narrowband system, shifting the mean critical damping ratios to approximately 0.0105. The inherent bias in the estimate cannot be overcome by the bootstrapping approach, as also noted in Vandermeulen et al. [22]. Spectra with an outright bias cannot be enhanced by this approach, as they are not truly representative of the system to be identified, but rather a biased representation of that system. As the bootstrap cannot repair sampled data, but can merely make inferences about its various statistics, the confidence intervals and all relevant statistical distributions will be clustered about this biased mean, as illustrated by the histograms shown in Fig. 7. In this case, even placing 95% confidence intervals on the estimate will not capture the true damping value. Note that Seybert's [12] derivation can provide a normalized bias estimate for ξ. Knowing that the SA consistently overestimates damping, the rough normalize bias formula may be offered as a correction to the estimates and used to refine the confidence bands.

3.2. Discussion of resampled results for random decrement technique

As shown in Fig. 7, the distribution of RDT damping estimates on the same data manifest considerably more scatter than SA estimates, consistent with the findings of the Monte Carlo simulation. Once again, the distributions shift as more samples are considered. As shown by Table 4, as N_r increases, their behavior tends to stabilize about a consistent mean value, similar to what was found in SA approach. Even in this stable range, there is still some fluctuation in the estimate, consistent with the findings of Kijewski and Kareem [11].

However, unlike SA results, the behavior of RDT estimate clearly indicates that limited amounts of data offer little hope of an accurate result. Rather, the results steadily improve with the number of samples being considered: for $N_r > 500$, the damping estimates are consistently within 10% of the actual value. The level of variance in the RDT estimate, as shown by Monte Carlo simulation, is markedly greater than the SA. This large standard deviation is accurately represented by the bootstrap analysis, in part due to the fact that the RDT is an unbiased estimator. This was not possible in the biased SA where the predicted variance was much less than the observed Monte Carlo result. As the RDT tends to stabilize after a significant number of averages, the changes in standard deviation are quite small when additional samples are added. Between Cases 1 and 2, the Monte Carlo standard deviation changes only marginally, by about 2%, explaining why the bootstrapped estimates for these two cases show only slight difference. In the same way, the standard deviation takes on consistently smaller values in Case 3 than in Case 2, reflecting the reduction in variance due to the inclusion of additional segments in the average.

Table 4
Statistics of bootstrap replications of critical damping ratio: estimates by RDT

N_r	ξ_{boot}	ξ_{plugin}	Bias[ξ]	$\sigma[\xi]$	CoV (%)	95% Boot	95% Normal
Case 1							
100	0.010636	0.010586	0.000051	0.001105	10.39	(0.00876, 0.01244)	(0.00843, 0.01285)
200	0.012293	0.012320	−0.000023	0.001144	9.31	(0.01041, 0.01417)	(0.01001, 0.01458)
300	0.012927	0.012932	−0.000006	0.001207	9.34	(0.01088, 0.01495)	(0.01051, 0.01534)
400	0.011308	0.011340	−0.000032	0.001225	10.83	(0.00934, 0.01339)	(0.00886, 0.01376)
500	0.011149	0.011163	−0.000013	0.001264	11.34	(0.00901, 0.01323)	(0.00862, 0.01368)
600	0.010915	0.010971	−0.000056	0.001237	11.33	(0.00881, 0.01295)	(0.00844, 0.01339)
Case 2							
100	0.01533	0.01532	0.000004	0.001239	8.08	(0.01334, 0.01747)	(0.01285, 0.01781)
200	0.01386	0.01381	0.000054	0.001223	8.82	(0.01181, 0.01591)	(0.01142, 0.01631)
300	0.01296	0.01296	0.000002	0.001259	9.72	(0.01089, 0.01500)	(0.01044, 0.01548)
400	0.01203	0.01188	0.000159	0.001174	9.76	(0.01010, 0.01397)	(0.00969, 0.01438)
500	0.01132	0.01126	0.000056	0.001231	10.88	(0.00921, 0.01333)	(0.00886, 0.01378)
600	0.00974	0.00976	−0.000013	0.001243	12.76	(0.00772, 0.01093)	(0.00726, 0.01223)
700	0.00917	0.00917	0.000005	0.001220	13.30	(0.00717, 0.01203)	(0.00673, 0.01161)
800	0.01005	0.01000	0.000048	0.001218	12.13	(0.00808, 0.01201)	(0.00761, 0.01248)
Case 3							
100	0.004597	0.004578	0.000019	0.001132	24.61	(0.00281, 0.00644)	(0.00233, 0.00686)
200	0.006960	0.007009	−0.000049	0.001151	16.53	(0.00510, 0.00889)	(0.00466, 0.00926)
300	0.008277	0.008304	−0.000027	0.001175	14.20	(0.00635, 0.01014)	(0.00593, 0.01063)
400	0.008907	0.008922	−0.000015	0.001118	12.55	(0.00709, 0.01070)	(0.00667, 0.01114)
500	0.008850	0.008850	0.000000	0.001128	12.74	(0.00701, 0.01067)	(0.00659, 0.01110)
600	0.009955	0.009860	0.000095	0.001152	11.57	(0.00812, 0.01196)	(0.00765, 0.01226)
700	0.010079	0.010108	−0.000029	0.001189	11.80	(0.00810, 0.01204)	(0.00770, 0.01246)
800	0.009909	0.009971	−0.000062	0.001204	12.15	(0.00808, 0.01198)	(0.00750, 0.01232)
900	0.010256	0.010229	0.000027	0.001162	11.33	(0.00834, 0.01219)	(0.00793, 0.01258)
1000	0.010301	0.010358	−0.000057	0.001186	11.52	(0.00835, 0.01236)	(0.00793, 0.01267)

Unlike the biased SA approach, the broad confidence intervals predicted by bootstrapping RDT results will encase the predicted result at minimum for $N_r > 400$. It is important to emphasize that traditional RDT damping estimates lack any type of interval estimate. Thus such tools can offer insight where previously there was none.

In defense of RDT, its performance under idealized conditions, in comparison to SA, is apparently lacking. Recall that the trigger condition used in this study seeks only peaks. This is a strict criterion that limits the amount of segments captured for averaging. As Case 3 illustrates, by shortening the length of segments captured, more unique non-overlapping segments were identified, reducing the random errors, as shown in Table 2. A similar relaxation of the trigger condition would also enhance performance, increasing N_r, the critical factor in limiting the variance of the RDS. Further, the merits of RDT have been documented in full-scale applications where the response is no longer strictly stationary [6]. In addition, to the credit of RDT, it is also capable of detecting amplitude-dependent damping [6,8]. The same cannot be said for the PSD.

Interestingly, a cursory examination of the ratio of bias to standard deviation for both RDT and SA data confirms that SA generally produces relatively large biases in the bootstrap data, whose ratios to the standard error exceed 0.25. On the other hand, RDT produces very small biases in the bootstrap analysis. Large biases often indicate that the bootstrap analysis of the data may not be accurate, as it was shown that the bootstrap approach could not capture the variance predicted by Monte Carlo for SA. Bias corrected data may be resampled, but is should be noted that this is risky practice and is thus not advocated in this study as it often leads to an increase in standard error. More details may be found in Efron and Tibshirani [20].

4. Conclusions

Two common approaches for the estimation of structural damping from systems without measured input were evaluated in this study: the random decrement technique and traditional spectral analysis. The product of this evaluation is a practical computational tool to generate the statistical reliability measures for damping parameters estimated using these two traditional approaches, for which no statistical confidence measures are implicitly provided. This study utilized bootstrapped replicates of the random decrement segments and raw power spectra to assess the quality of the resulting system identification by providing surrogate estimates of damping and natural frequency to generate useful statistics. Statistical information such as the standard deviation and confidence intervals were previously unavailable for a single estimate of frequency and damping by these techniques; however, they can be estimated using the bootstrap approach. Understandably, randomness inherent to the process leads to randomness in identified parameters, even when identical amounts of data are analyzed by the same procedure. With sufficient amounts of data, the power spectrum will produce a biased damping estimate with much smaller variance than the unbiased Random Decrement estimate; however, the amount of data required to obtain stable behavior may be prohibitive. Statistical errors in the damping estimates were provided using a bootstrapping scheme, which approximated the standard deviation of the RDT estimate consistent with Monte Carlo simulations, using a single time history. The bootstrapped standard deviation of the damping estimate derived from spectral analysis was considerably less than its Monte Carlo counterpart, attributed to the fact that it was tainted by the bias inherent in the sample population. This bias makes it impossible for the confidence intervals to be used to capture the true estimate, though its damping estimate is not as variable. To enhance the viability of bootstrapping schemes in such situations, bias correction may be considered. On the other hand, the broad confidence bands of the RDT can be used to capture the true damping estimate when sufficient samples are available. In total, the proposed technique and particularly its variance estimate can provide insight previously lacking in spectral analysis and the random decrement technique.

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