Equivalent Statistical Cubicization for System and Forcing Nonlinearities

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Abstract: This technical note outlines the treatment of a single-degree-of-freedom (SDOF) system containing statistically symmetric nonlinearities in both its stiffness characteristics and its excitation. Via equivalent statistical cubicization, the nonlinearities that are not in polynomial form are cast as polynomials containing first- and third-order terms. This allows the use of a Volterra series approach that yields a system of two differential equations for the first- and third-order components of the response. Transforming this system into the frequency domain produces transfer functions from which power spectral density and statistics up to the fourth order may be obtained for the system response. Finally, using the statistics within the framework of a moment-based Hermite transformation model yields an estimate of the non-Gaussian probability-density function (PDF) for the system response.

Introduction

In previous work by the writers, it has been shown that the method of equivalent statistical cubicization, in tandem with the Volterra theory, can be an effective tool for developing the power spectral densities, response statistics, and probability density functions (PDFs) for systems containing statistically symmetric nonlinearities in their excitation or in their system characteristics (Kareem and Zhao 1993, 1994; Tognarelli et al. 1995, 1997). Here, it will be shown that this method works well when a statistically symmetric nonlinearity is present both in the excitation and in the system itself. To the writers' knowledge, this is the first treatment of such a pair of nonlinearities using the present techniques. Equivalent statistical linearization and quadratization techniques fail to capture important response features introduced by statistically symmetric nonlinearities. Indeed, in such cases the method of equivalent statistical quadratization reverts to the method of equivalent statistical linearization, which can reflect neither non-Gaussian characteristics of the response nor the modified frequency character of response introduced by the presence of statistically symmetric nonlinearities. Conversely, equivalent statistical cubicization has been observed to overcome such shortcomings via the statistically symmetric nature of the cubic approximating polynomial that it introduces. Thus far, however, the technique has been applied only for the case in which a single nonlinearity appears in any given system.

Analysis

Consider an offshore system modeled by the following single-degree-of-freedom (SDOF) equation of motion

$$M\ddot{x} + C\dot{x} + K(x + \varepsilon x^3) = K_m\dot{u} + K_d|u - \dot{x}|(u - \dot{x}) \tag{1}$$

where x = system displacement response; u and $\dot{u}$ = water-particle velocity and acceleration, respectively; and K_m and K_d = Morison inertia and drag force coefficients, respectively. Such an equation of motion could be used to model a catenary-moored platform exposed to viscous hydrodynamic loads. The water-particle kinematics are modeled as Gaussian processes based on a Pierson-Moskowitz (P-M) elevation spectrum. This system contains statistically symmetric nonlinearities (e.g., Tognarelli et al., in press, 1997) in the forms of both Morison drag force without current and Duffing stiffness. The response process is treated as the sum of the outputs of first- and third-order Volterra systems

$$x = x_1 + \frac{1}{6}x_3 = \int_{-\infty}^{\infty} h_{x_1}(\tau_1)f(t - \tau_1)\,d\tau_1 + \frac{1}{6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot h_{x_3}(\tau_1, \tau_2, \tau_3)f(t - \tau_1)f(t - \tau_2)f(t - \tau_3)\,d\tau_1\,d\tau_2\,d\tau_3 \tag{2}$$

where $h_{x_1}(\tau_1)$ and $h_{x_3}(\tau_1, \tau_2, \tau_3)$ = first- and third-order Volterra kernels of the system response. Nonlinear terms in the governing equations may then be expanded in a Taylor series containing terms that are nonlinear in terms of the Gaussian, first-order response only, and linear in terms of both the first- and the third-order response as follows:

$$f_N(x, \dot{x}) = f_N(x_1, \dot{x}_1) + \frac{\partial f_N}{\partial x}(x_1, \dot{x}_1)\frac{x_3}{6} + \frac{\partial f_N}{\partial \dot{x}}(x_1, \dot{x}_1)\frac{\dot{x}_3}{6} + O(x_3^2, \dot{x}_3^2) \tag{3}$$

For the particular nonlinearities in (1), this gives

$$(u - \dot{x})|u - \dot{x}| \approx (u - \dot{x}_1)|u - \dot{x}_1| - 2|u - \dot{x}_1|\frac{\dot{x}_3}{6}$$

$$\approx (u - \dot{x}_1)|u - \dot{x}_1| - 2E\{|u - \dot{x}_1|\}\frac{\dot{x}_3}{6} \tag{4}$$

$$x^3 \approx x_1^3 + 3x_1^2\frac{x_3}{6} + \frac{1}{2}(6x_1)\left(\frac{x_3}{6}\right)^2$$

$$\approx x_1^3 + 3E\{x_1^2\}\frac{x_3}{6} + \frac{1}{12}E\{x_3^2\}x_1 \tag{5}$$

where $E\{\cdot\}$ denotes the expectation operator. Expectations are introduced in (4) and (5) to avoid the presence of higher-order mixed terms involving both the first- and third-order Volterra system responses, primarily to simplify the analysis. The simulation results, presented later, seem to support the use of this simplifying assumption.

The equivalent statistical cubicization process itself involves the casting of the nonlinear term $(u - \dot{x}_1)|u - \dot{x}_1|$ in (4) as a polynomial containing first- and third-order terms in x_1 and $\dot{x}_1$. [Note, the nonlinear initial term on the right side of (5) is already in polynomial form.] Incorporating the relative fluid-structure velocity to address the drag nonlinearity, this procedure gives

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$$(u - \dot{x}_1)|u - \dot{x}_1| \approx \alpha_1(u - \dot{x}_1) + \frac{\alpha_3}{6}(u - \dot{x}_1)^3 \tag{6}$$

Defining the error

$$\varepsilon = (u - \dot{x}_1)|u - \dot{x}_1| - \alpha_1(u - \dot{x}_1) - \frac{\alpha_3}{6}(u - \dot{x}_1)^3 \tag{7}$$

mean-square error minimization by setting $\partial E\{\varepsilon^2\}/\partial\alpha_i = 0$, $i = 1, 3$ yields the coefficient values

$$\alpha_1 = \sqrt{\frac{2}{\pi}}\,\sigma_\nu; \quad \alpha_3 = \sqrt{\frac{8}{\pi}}\,\frac{1}{\sigma_\nu} \tag{8}$$

where $\nu = u - \dot{x}_1$; and $\sigma_\nu = E\{\nu^2\}$. The fact that the random processes involved in the cubicization procedure are Gaussian eases the determination of the coefficients significantly. In an alternative approach to equivalent statistical quadratization, which was very successful for fatigue analyses of systems containing statistically asymmetric nonlinearities, Spanos and Donley (1991) used a Gram-Charlier series to model the probability density of the system response since, in their analysis, the nonlinearity was not first expanded as a Taylor series. This added computational complexity to the determination of the polynomial coefficients and, in some cases, yielded negative tails in the response PDF.

Finally, substituting the Volterra series representation given in (2) for x and the polynomial forms (5) and (6) for the nonlinear terms in the original SDOF system, the governing equation, (1), may be re-expressed as a Volterra system of two equations

$$\ddot{x}_1 + \left(2\xi\omega_N + \frac{K_d\alpha_1}{M}\right)\dot{x}_1 + \omega_N^2\left(1 + \frac{\varepsilon\beta_1}{12}\right)x_1 = \frac{K_m}{M}\dot{u} + \frac{K_d\alpha_1}{M}u \tag{9}$$

$$\ddot{x}_3 + \left(2\xi\omega_N + \frac{2K_d\alpha_1}{M}\right)\dot{x}_3 + \omega_N^2(1 + 3\varepsilon\beta_3)x_3 = \frac{K_d\alpha_3}{M}(u - \dot{x}_1)^3 - 6\omega_N^2\varepsilon x_1^3 \tag{10}$$

where $\beta_1 = \sigma_{x_3}^2$; $\beta_3 = \sigma_{x_1}^2$; and it is observed that $E\{|u - \dot{x}_1|\} = \alpha_1$. Note that for Gaussian u and $\dot{u}$, the first-order response is indeed Gaussian, as suggested in the preceding. This system gives first- and third-order transfer functions [e.g., Schetzen (1980)]

$$H_x^{(1)}(\omega) = \frac{(i\omega K_m + K_d\alpha_1)/M}{\left[\omega_N^2\left(1 + \frac{\varepsilon\beta_1}{12}\right) - \omega^2\right] + i\omega\left(2\xi\omega_N + \frac{K_d\alpha_1}{M}\right)} \tag{11}$$

and using the shorthand $H_x^{(1)}(\omega_1) = H_x^{(1)}(1)$ and $H_\nu(\omega_1) = H_\nu(1)$

$$H_x^{(3)}(\omega_1, \omega_2, \omega_3) = \frac{\left[\frac{K_d\alpha_3}{M}H_\nu(1)H_\nu(2)H_\nu(3) - 6\omega_N^2\varepsilon H_x^{(1)}(1)H_x^{(1)}(2)H_x^{(1)}(3)\right]}{[\omega_N^2(1 + 3\varepsilon\beta_3) - \omega_T^2] + i\omega_T\left(2\xi\omega_N + \frac{2K_d\alpha_1}{M}\right)} \tag{12}$$

where $\omega_T = \omega_1 + \omega_2 + \omega_3$; and $H_\nu(\omega) = 1 - i\omega H_x^{(1)}(\omega)$. Using these, the two-sided response power spectral density $D_{xx}(\omega)$ as well as statistics in the form of response cumulants k_i may be obtained from the transfer functions and the power spectral density of the input process $D(\omega)$, via the following frequency domain integrals:

$$\begin{aligned} k_2 = &\int_{-\infty}^{\infty}|H_x^{(1)}(1)|^2 D(1) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_x^{(1)}(1)H_x^{(3)}(-1, 2, -2)\,D(1)\,D(2) \\ &+ \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_x^{(3)}(1, 2, -2)H_x^{(3)}(-1, 3, -3)\,D(1)\,D(2)\,D(3) \\ &+ \frac{1}{6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|H_x^{(3)}(1, 2, 3)|^2 D(1)\,D(2)\,D(3) \end{aligned} \tag{13}$$

$$\begin{aligned} k_4 = 4 &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_x^{(1)}(1)H_x^{(1)}(2)H_x^{(1)}(3)H_x^{(3)}(-1, -2, -3)\,D(1)\,D(2)\,D(3) \\ &+ 3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_x^{(1)}(1)H_x^{(1)}(2)H_x^{(3)}(-1, 3, -3)H_x^{(3)}(-2, 4, -4) \\ &\cdot D(1)\,D(2)\,D(3)\,D(4) + 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_x^{(1)}(1)H_x^{(1)}(2) \\ &\cdot H_x^{(3)}(-1, -2, 3)H_x^{(3)}(-3, 4, -4)D(1)\,D(2)\,D(3)\,D(4) \\ &+ 3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_x^{(1)}(1)H_x^{(1)}(2)H_x^{(3)}(-1, 3, 4)H_x^{(3)}(-2, -3, -4) \\ &\cdot D(1)\,D(2)\,D(3)\,D(4) + O(x_3^3) \end{aligned} \tag{14}$$

$$\begin{aligned} D_{xx}(\omega) = &\left|H_x^{(1)}(\omega) + \frac{1}{2}\int_{-\infty}^{\infty}H_x^{(3)}(\omega, \theta, -\theta)\,D(\theta)\,d\theta\right|^2 D(\omega) \\ &+ \frac{1}{6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|H_x^{(3)}(\theta, \tau, \omega - \theta - \tau)|^2 D(\theta)\,D(\tau)\,D(\omega - \theta - \tau)\,d\theta\,d\tau \end{aligned} \tag{15}$$

where for the sake of brevity we have allowed $H_x^{(1)}(1) = H_x^{(1)}(\omega_1)$, $H_x^{(3)}(1, 2, 3) = H_x^{(3)}(\omega_1, \omega_2, \omega_3)$, and $D(1) = D(\omega_1)\,d\omega_1$ in the cumulant expressions. The response of the system is also statistically symmetric; therefore, the odd-order cumulants vanish.

Having the higher-order cumulants, the PDF of the response may be estimated by using an appropriate non-Gaussian model. In this study, the moment-based Hermite transformation model (Grigoriu 1984; Winterstein 1985) has been chosen. This model does not yield negative tails for the range of response and response statistics considered herein. In the moment-based Hermite framework for the softening system that we will consider, the non-Gaussian response, $x(t)$ is cast as a nonlinear function of $u(t)$, which is Gaussian, as follows (the argument t is omitted for brevity):

$$\frac{x - \mu_d}{\sigma_x} = \alpha[u + h_3(u^2 - 1) + h_4(u^3 - 3u)] \tag{16}$$

where μ_x = mean displacement; σ_x = its standard deviation; and $\alpha = (1 + 2h_3^2 + 6h_4^2)^{-1/2}$. A form of this transformation is also available for hardening systems (Winterstein 1985). Winterstein's approximate expressions relating the response cumulants to the model coefficients for a softening system are

$$h_3 = \gamma_3/(4 + 2\sqrt{1 + 1.5\gamma_4}) \tag{17}$$

$$h_4 = (\sqrt{1 + 1.5\gamma_4} - 1)/18 \tag{18}$$

where $\gamma_3 = k_3/k_2^{3/2}$; and $\gamma_4 = k_4/k_2^2$. Using these as initial guesses, a pair of coupled nonlinear equations are iteratively solved to improve the representation such that it reflects, exactly, the third- and fourth-order cumulants of the process that have been predicted via the equivalent statistical cubicization technique. These equations are

$$\gamma_3 = \alpha^3(8h_3^3 + 108h_3h_4^2 + 36h_3h_4 + 6h_3) \tag{19}$$

$$\begin{aligned} \gamma_4 = \alpha^4(&60h_3^4 + 3{,}348h_4^4 + 2{,}232h_3^2h_4^2 + 60h_3^2 + 252h_4^2 \\ &+ 1{,}296h_4^3 + 576h_3^2h_4 + 24h_4 + 3) - 3 \end{aligned} \tag{20}$$

NUMERICAL EXAMPLE

Consider the SDOF system in (1) for which $M = 7.1286 \times 10^7$ kg, $C = 4.4791 \times 10^5$ N·s/m (5%), $K = 2.8143 \times 10^5$ N/m, $\omega_N = 0.06283$ rad/s, $\varepsilon = 0.2$ m^{-2}, $K_m = 4.0 \times 10^7$ kg, and $K_d = 1.5 \times 10^6$ N·s^2/m^2. The P-M wave-elevation spectrum considered is characterized by significant wave height $H_s = 12$ m and peak wave frequency $\omega_P = 0.3628$ rad/s. For this system Fig. 1 indicates a comparison of the power spectral densities and higher-order statistics obtained from both equivalent statistical cubicization and Monte Carlo simulation. The simulation involves first creating realizations of the Gaussian water-particle velocity and acceleration processes based on the aforementioned P-M spectrum using random-amplitude/random-phase techniques, then numerically integrating the equation of motion via a fourth/fifth-order Runge-Kutta scheme with a time step of 0.33 s. The statistics of the present simulation are based on an ensemble of 100 realizations of the response. Each realization contains 16,384 points, where the first several periods have been truncated to eliminate transients. Larger numbers of realizations do not change the statistical character of the simulated response. By definition, the odd-order statistics of the response for purely statistically symmetric nonlinearities vanish and are not indicated in the figure. Notably, a spectral peak in the low-frequency range is captured via this technique, and the response statistics closely match those from Monte Carlo simulation. The higher-frequency spectral peak reflects the linear dependence that both the inertia and drag force inputs have on the wave elevation. The lower-frequency peak arises from spreading of the energy bands in the input wave-elevation process introduced by the cubic dependency of both the Morison drag force and the Duffing stiffness term. This spreading leads to excitation of the system at its natural frequency, which is slightly higher than ω_N because of the additional Duffing stiffness.

A comparison of the moment-based Hermite PDF versus that obtained via Monte Carlo simulation is given in Fig. 2. The moment information for the Hermite transformation model is that which has been obtained via the cubicization technique. A Gaussian distribution characterized by the standard deviation obtained from the cubicization technique is also shown. Note that while the moment-based Hermite transformation model slightly overestimates the response PDF in the

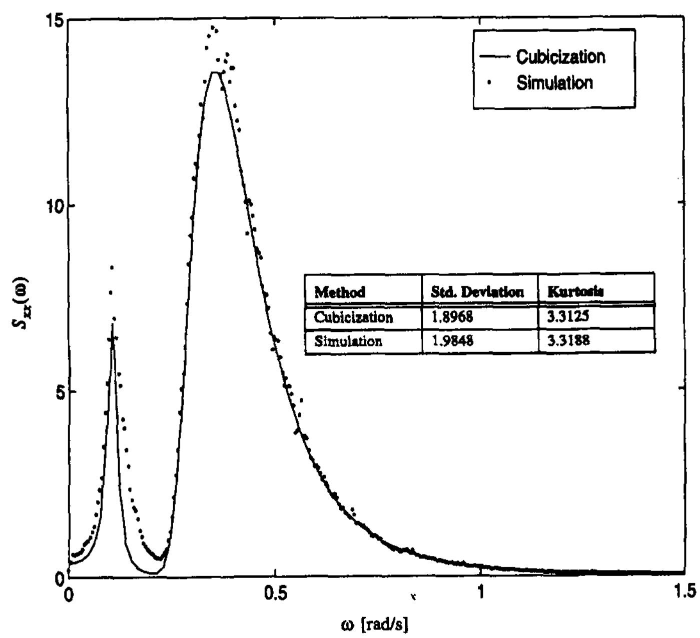

FIG. 1. Comparison of Techniques for Prediction of Power Spectral Density and Higher-Order Statistics of Response

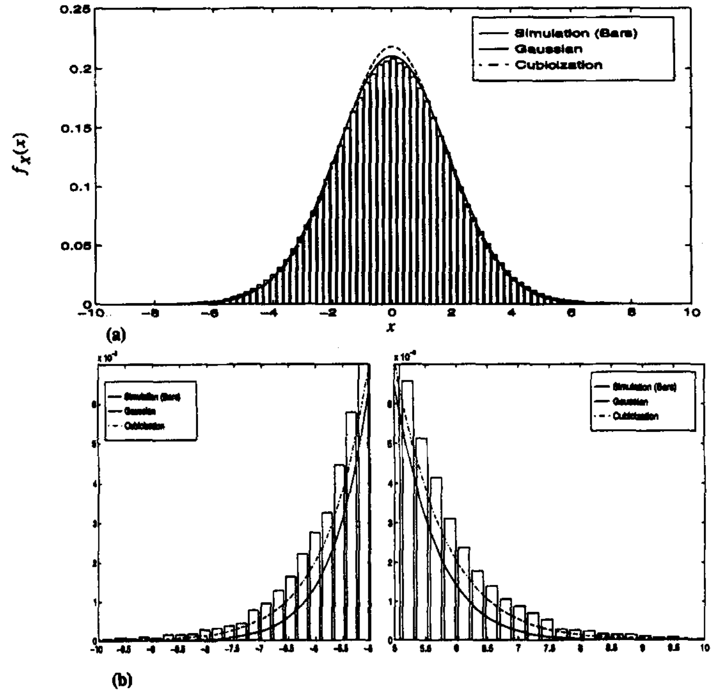

FIG. 2. Comparison of Moment-Based Hermite PDF versus that Obtained by Monte Carlo Simulation: (a) Response PDFs Obtained from Cubicization and Simulation; (b) Details of Tail Regions of Distributions

area of the mean, it accurately reflects the departure of the PDF from Gaussianity in the tail regions. This is very important for analyses in which the extremes of the response are the primary focus.

It has been observed that the present technique is more effective when the Duffing stiffness parameter is relatively small and the system damping is relatively large. Some terms containing higher powers of x_3, the magnitude of which is damping dependent, have been neglected in the preceding development of the fourth cumulant for computational convenience. In a numerical parameter study, it was observed that the relative importance of the third-order response with respect to the first-order response is diminished by increased damping. Since the overall damping depends not only on the system characteristics themselves but also on an additional contribution from the Morison drag term, it is important to consider both of these when making analyses. Essentially, the additional damping increases as the significance of the Morison drag term increases. This can be seen by examining the additional damping terms in the denominators of (11) and (12). As such, often more favorable results are observed for larger nonlinear contributions due to Morison drag force. This is in contrast to the effects resulting from increased Duffing stiffness.

CONCLUSION

It has been shown that an equivalent statistical cubicization technique can be effective for analyses of systems containing nonlinearities not only in their excitation, but also in their inherent characteristics. Within the context of a framework developed previously by the writers, an SDOF example containing statistically symmetric nonlinearities exemplified by Duffing stiffness and Morison drag force without current has been presented. Results in the form of response power spectral densities, higher-order statistics, and PDFs have been compared favorably with those from Monte Carlo simulation.

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