On the Reliability of System Identification: Applications of Bootstrap Theory

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ABSTRACT: The estimation of structural damping from systems with unknown input can be particularly challenging but is vital to furthering the understanding energy dissipation in wind-induced motion. The Random Decrement Technique and Spectral Analysis are commonly invoked system identification schemes for this problem. Although there is well-established theory regarding the bias and variance errors associated with these two approaches, the theoretical error formulae give no indication of errors inherent to the estimated dynamic properties. This study utilizes bootstrapped replicates of the random decrement segments and raw power spectra to assess the quality of the system identification by providing surrogate estimates of damping and natural frequency to generate useful statistics and confidence intervals.

1 INTRODUCTION

With the advancement of modern structures to new heights, the issues of serviceability and occupant comfort have come to the forefront in design. A simple exercise in random vibrations reiterates the importance of damping in maintaining acceptable levels of response, particularly in the case of the accelerations, which are used to assess occupant comfort. Despite its critical role in the performance of structures, damping continues to be an enigma in design. Through the examination of existing structures, efforts towards international databases have been undertaken; however, there is considerable scatter in the data, partly attributed to the amplitude dependence of damping, but more so due to errors in the identification of this parameter. Most of the full-scale data relies on ambient excitations, providing the analyst with no measured input for system identification. As a result, system identification must be conducted using "unknown input" schemes or those that make some general assumption about the nature of the input. This restriction is often problematic and leads to a host of possible errors.

The intent of this work is to provide practical tools to assess the quality of the system identification performed. Although no method can identify precisely how much an estimate deviates from the true system characteristics, the ability to provide statistical information on the obtained dynamic properties provides some simple reliability measure.

1.1 *Traditional Approaches to System Identification and Inherent Difficulties*

As mentioned previously, the identification of dynamic system properties solely from response data is being considered in this study. With the general assumption that the driving process is a random, stationary, zero-mean, Gaussian, white noise process, two approaches are most frequency employed: Spectral Analysis (SA) and the Random Decrement Technique (RDT).

1.1.1 *Spectral Analysis: Inherent Errors*

The estimation dynamic properties has commonly been accomplished through the use of the Spectral Density (PSD) of the process, generated by a family of N_s raw spectra resulting from the Fast Fourier Transform (FFT) of blocks of the response data.

As basic principles of signal processing suggest, the quality of the PSD is limited by the degree of bias and variance error present. The bias error relates directly to the frequency resolution of the spectra. Concurrently, the variance error must also be minimized, as given by:

$$\mathrm{var}\left[\hat{S}_{xx}(f)\right] \approx \frac{S^2{}_{xx}(f)}{N_s}. \tag{1}$$

where $\hat{S}xx(f)$ =estimated PSD; $Sxx(f)$ = true PSD.

1.1.2 *Random Decrement Technique: Inherent Errors*

The Random Decrement Technique has become one of the most popular approaches used for system identification from wind-excited responses because of its ability to overcome the strict requirements for lengthy stationary data imposed by the traditional spectral approach. The decrement is generated by capturing a prescribed length of the time history upon the satisfaction of a threshold condition (Cole 1973). This triggering condition, in its strictest sense, will specify both amplitude and slope criteria. The segments meeting these conditions are averaged to remove the random component of the response, assumed to be zero mean, leaving a signature, $D_{x_o}(\tau)$, proportional to $R_X(\tau)$, the autocorrelation function of the system (Vandiver et al. 1982). The method by which segments are captured varies in the literature; however, strict adherence to the threshold condition may only be achieved by capturing peaks of a specified amplitude (Tamura & Suganuma 1996).

The RDS will be unbiased with variance that can be expressed by (Vandiver et al. 1982):

$$\mathrm{var}[D_{x_o}(\tau)] = E[D^2{}_{x_o}(\tau)] - E[D_{x_o}(\tau)]^2 = R_x(0)/N_r[1 - R_x^2(\tau)/R_x^2(0)] \tag{2}$$

where N_r=the number of segments averaged in the estimate. Representing the system as a linear oscillator excited by Gaussian, zero mean, white noise, the autocorrelation function will have the same form as the free vibration of a system with critical damping ratio of ξ and natural frequency of f_n.

From Equations 1 and 2, it is evident that the variance of both the PSD and Random Decrement Signature (RDS) can be determined theoretically; however, they are dependent on the true PSD and autocorrelation, the very quantities that are hopefully being obtained by this process. Further, any one of the theoretical assumptions may be violated, leading to inaccurate estimates of the variance. Thus, the notion of bootstrapping will be exercised as an alternative means by which to estimate their true variance (Efron & Tibshirani 1993).

1.2 *Errors in System Identification: An Exercise*

The first example presented in this paper reflects the blind process that is undertaken when system identification is performed by SA or RDT. In accordance with common practice, the minimum spectral resolution was determined to limit the normalized spectral bias error to –2% for a 0.2 Hz SDOF oscillator with 1% damping. The required number of FFT points would be 16,384. Assuming the signal to be sampled at 10 Hz, that is roughly one half hour of data. To minimize normalized random error to 10%, approximately 50 hours of data would be necessary to generate 100 sufficiently resolved raw spectra. Response data was generated by passing stationary, band-limited Gaussian white noise with zero mean and unit PSD through the system. Though often not obtainable in practice, approximately 50 hours of stationary data is considered for illustrative purposes.

In only the most general sense, the user understands that averaging 100 raw spectra should provide acceptable results, but the level of error in the frequency and damping identified from the PSD is still not known. To illustrate the variability possible, the same system is simulated 50 times. In each case, the same amount of data is used so that the normalized bias and random errors on paper are the same, and the system is then identified by the simplest means available: the Half Power Bandwidth (HPBW). For comparison, the same simulated response is also analyzed by RDT. As illustrated by Equation 2, the variance in the RDS increases with each cycle of oscillation, thus the most reliable system identification was achieved by a least squares fit of only the first two cycles.

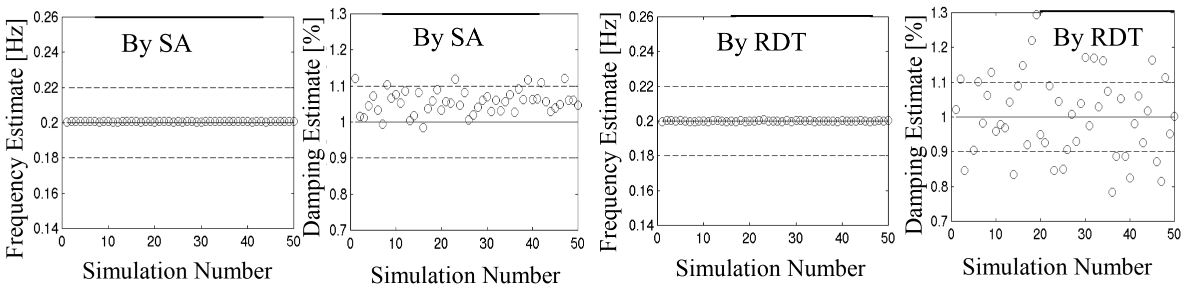

Figure 1. Identified dynamic properties of simulated data: identification by SA and by RDT (circles). Solid line indicates actual value and dashed lines represent 10% error bounds.

The amplitude selected for the triggering condition was set as the mean peak value to maximize the number of segments averaged in RDT. Even armed with a favorable amount of data, the variability inherent in the random process can cause considerable scatter in the identified parameters, as shown by Figure 1. Table 1 summarizes the statistics of the simulations, including the mean μ, standard deviation σ, and coefficient of variation (*CoV*) defined as σ/μ .

From Figure 1 and Table 1, it is not surprising to confirm that the frequency is relatively simple to identify with accuracy. The damping, on the other hand, is much more difficult. As expected, SA produces a biased estimate of the damping, as the smoothing of the spectrum results in an underestimation of the spectral peak and thus an overestimation of the damping. Despite the reasonable variance in the estimates, the bias may be problematic. What should be noted is that, due to the random nature of the process, identical amounts of data subjected to the same analysis technique can produce an estimate of damping which has no error or as much as 10% error. Though 10% error may be reasonable, this situation also illustrates a very favorable and perhaps unrealistic amount of data. Under these ideal conditions, the damping estimated by independent simulations has inherent variability, which is no doubt enhanced under less ideal conditions.

At the same time, RDT approach produces a relatively unbiased estimate of damping. Unfortunately, there is a considerable amount of scatter in RDT result, leading to a *CoV* that is an order of magnitude greater than SA result. In this case, the identification of the system from any one of these simulations can produce estimates of damping that are near exact or in error by up to 20%. If one has the luxury of repeating an experiment 50 times under identical conditions, acceptable results are obtainable in the average. However, as this is not possible, how can one access the accuracy of identified parameters from one data set?

Table 1. Statistics of Monte Carlo Simulations.

	$\mu[f_n]$ (Hz)	$\sigma[f_n]$ (Hz)	Bias $[f_n]$ (Hz)	Cov $[f_n]$	$\mu[\xi]$ (%)	$\sigma[\xi]$ (%)	bias$[\xi]$ (%)	CoV$[\xi]$
SA	0.2007	0.00026	0.0007	0.13%	1.0556	0.0329	0.0556	3.12%
RDT	0.2001	0.000256	0.0001	0.13%	0.9955	0.1262	-0.0045	12.67%

2 BOOTSTAPPING SCHEMES: APPLICATIONS TO SYSTEM IDENTIFICATION

If the distribution of a random variable were known, then theory or simulation may be invoked to calculate various statistics. However, in most practical applications, this is not possible, but the bootstrap may be used to make the best of what information is available. The bootstrap approach is a computer-based method for assigning accuracy to statistical estimates based on independent data points or samples, which, in its simplest form, is non-parametric, requiring no assumptions about the distribution of the parameters (Efron & Tibshirani 1993).

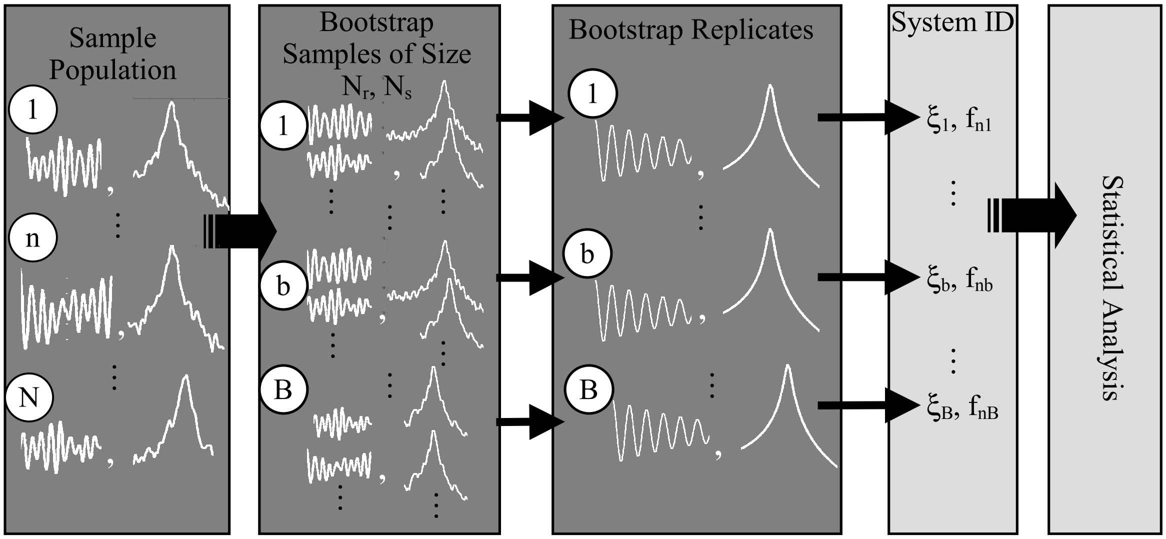

Figure 2. Proposed bootstrapping scheme for system identification.

When the sample population is large enough, the Central Limit Theorem (CLT) can usually be invoked to assert that the estimator is approximately distributed as a Gaussian random variable. Luckily, the underlying assumptions of bootstrap analysis are valid in cases where there are only limited samples available, say 10 or 15 samples, though both will approach one another in situations where the population is large. The bootstrap is also capable of capturing non-Gaussian statistics such as skewness that could not be captured relying on CLT.

It takes a number of hours of measured response data to obtain a reasonable estimate of the system's frequency and damping. By the nature of the random process driving the structure, the estimated damping and frequency are not deterministic, but are also random due to the inherent variability in the PSD and RDS from which they are drawn. From this single estimate, there can be no true inference of the potential variance. Bootstrapping is used in this case to get B more estimates of the system's dynamic properties, according to Figure 2. As first implemented in Vandermeulen et al. (2000) and Kijewski & Kareem (2000), the N_s raw spectra and the N_r time-history segments that satisfy RDT trigger condition form the sample population. From this population, N_s or N_r samples are drawn with replacement to form one bootstrap sample. This is repeated B times. Each bootstrap sample is then averaged to form a smooth PSD estimate or RDS, termed a bootstrap replicate. From these B replicates, the system's dynamic properties may be determined and then used

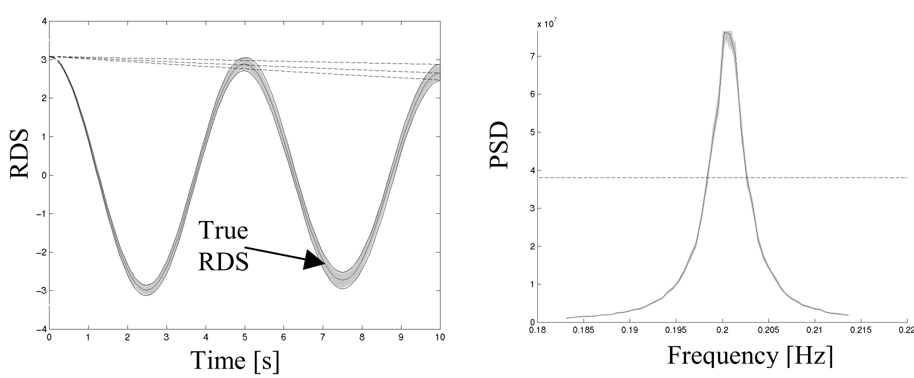

Figure 3. Variance envelopes for RDT (left) and PSD. Grey lines indicate variance envelope; black line indicates traditional RDT and PSD estimate; dotted lines indicate RDS decay and HPBW.

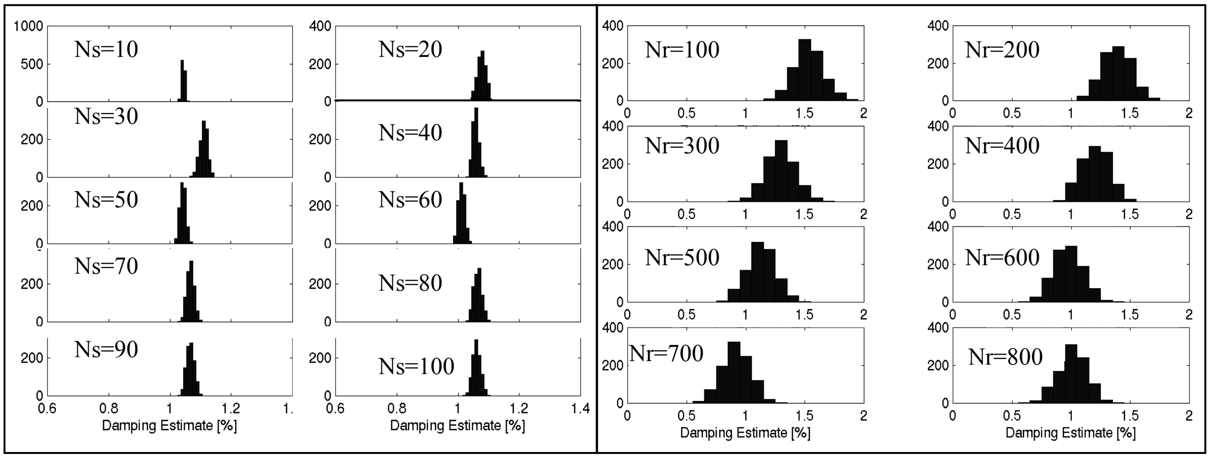

Figure 4. Histogram of bootstrap simulations on PSD (left) and RDS for data set 1.

to make statistical inferences on the reliability of the estimate. The bootstrap replicates can also be plotted atop one another to create variance envelopes, shown in gray by Figure 3. Significant deviations from the estimated PSD and RDS, shown in black, highlight the areas of highest variance.

It was shown in Kijewski & Kareem (2000) that, under ideal theoretical conditions, the Bootstrap Approach provides an estimate of the standard error consistent with theoretical predictions, verifying its ability to estimate the standard error of the RDS. Performing the same calculation at each frequency can generate a similar estimate of the standard deviation of the PSD. It is hoped that the introduction of such a scheme will provide practitioners with a simple means by which to estimate the variance of their power spectra and random decrement signatures and provide a measure of the reliability when theoretical assumptions are not entirely met.

3 STATISTICAL ANALYSIS VIA BOOTSTRAPPING SCHEME

The bootstrapping scheme outlined in Figure 2 was undertaken on some of the records simulated in the previous Monte Carlo analysis to illustrate the information that can be gained, particularly when limited information is available. Increments of the total time history were analyzed so that the influence of the number of raw spectra (N_s) and the number of segments in RDT (N_r) could be determined. As the frequency is estimated with near certainty every time, it shall not be discussed here for brevity. The tables that follow contain the bootstrap estimate of the damping, $\xi boot$, defined as the mean of the damping identified from the bootstrap replicates in Figure 2. The plug-in estimate of damping, $\xi plugin$, was determined by the traditional approach without any resampling. The bias in the estimate is then defined as the difference, $\xi boot$- $\xi plugin$. The standard deviation of the bootstrap estimate σ and the *CoV* can readily be calculated by standard formulae. Additionally, by virtue of the bootstrapping scheme, histograms depicting the distribution of the damping estimate from a given time history can be obtained. These are useful tools for identifying the underlying distribution of damping estimates and its associated characteristics. An example of the histogram of 1000 bootstrap replicates of the damping estimates from a single time history is given in Figure 4. Keep in mind that the simulated system had 1% damping.

Additionally, confidence intervals are defined for the damping estimate. By traditional analysis, only a single estimate of damping is available from each simulated time history. However, through the bootstrapping scheme, this estimate is enhanced by a family of associated estimates, which can give valuable insight into the reliability of a given damping estimate. In the most elementary formulation, a level of confidence can be selected, and then the replicates that correspond to this level can be identified from the distribution of bootstrap replicates. By virtue of the non-parametric nature of this approach, there is no need to make any assumptions about the normality of the damping estimates. This is especially useful in cases where the PSD or RDS is generated from a limited number of samples. Under these conditions, it cannot be assumed with certainty that the CLT applies, therefore casting doubt on any confidence intervals generated by standard approaches. 95% confidence intervals, as determined by the bootstrap data and by an assumed normal distribution, are provided in the final columns of Table 2 for comparison.

Table 2. Statistics of bootstrap replications of damping as percent of critical: data set 1 by SA.

Ns	ξboot	ξplugin	bias[ξ]	σ[ξ]	CoV	95% boot	95% normal
10	1.044	1.044	-0.0001	0.0049	0.47%	(1.036, 1.052)	(1.034, 1.054)
20	1.077	1.080	-0.0031	0.0143	1.33%	(1.053, 1.101)	(1.049, 1.106)
30	1.110	1.110	-0.0005	0.0130	1.17%	(1.088, 1.130)	(1.084, 1.136)
40	1.058	1.061	-0.0033	0.0105	0.99%	(1.041, 1.077)	(1.037, 1.079)
50	1.043	1.044	-0.0015	0.0095	0.91%	(1.027, 1.059)	(1.024, 1.062)
60	1.012	1.012	0.0002	0.0104	1.03%	(0.996, 1.029)	(0.991, 1.033)
70	1.067	1.075	-0.0081	0.0121	1.13%	(1.048, 1.088)	(1.043, 1.091)
80	1.064	1.065	-0.0009	0.0134	1.26%	(1.043, 1.086)	(1.038, 1.091)
90	1.067	1.069	-0.0024	0.0130	1.22%	(1.043, 1.089)	(1.041, 1.093)
100	1.061	1.065	-0.0045	0.0132	1.24%	(1.039, 1.083)	(1.035, 1.087)

3.1 *Discussion of resampled results for SA*

From the histograms of data set 1 in Figure 4, it is evident that in cases of limited data there are shifts in the damping estimate distribution about various mean values. It is only for $N_s>60$ (approximately 30 hours of data) that the behavior stabilizes and the damping estimates distribute about a relatively constant mean, indicating that the variance is sufficiently minimal. The bootstrap affirms that the damping estimates do not reflect significant spread, as evidenced by the moderate *CoV* in Table 2. This reiterates the findings of the Monte Carlo Simulations. However, the bootstrap, when considering all the data ($N_s=100$), was not capable of reproducing the same *CoV* as the Monte Carlo results, though capturing the order of magnitude.

It is interesting to note that, even with modest amounts of data (N_s=10, approximately 5 hours of data), a relatively good estimate of the damping is obtainable, while the addition of further recorded data leads to poorer estimates. This may seem counterintuitive, as the variance of the PSD has been shown to reduce with the number of raw spectra being averaged. However, the inherent randomness of the process must be considered. It just so happens that, in this case, the first 5 hours of data was able to provide a reasonable estimate of damping. Likewise the variance is also dependent on the magnitude of the PSD itself, which also fluctuates in each case considered. These random fluctuations can and should be expected when limited data is used, as the variance is high (Vandermeulen et al. 2000). In the limit, a more stable and reliable PSD will result as variance decreases. By evidence of Figure 4 and Table 2, it would indicate that this is achieved for $N_s>60$.

As evident from Figure 1, there is a discernable amount of bias in the spectral estimate, shifting the mean damping values approximately 5%. The inherent bias in the estimate cannot be overcome by the bootstrapping approach, as also noted in Vandermeulen et al. (2000). Spectra with an outright bias cannot be enhanced by this approach, as they are not truly representative of the deterministic system to be identified, but rather a biased representation of that system. Understandably, for this approach to work, it must be assumed that the sample population is representative of the actual process. The bootstrap cannot repair sampled data, but can merely make inferences about its various statistics. Thus, the confidence intervals and all relevant statistical distributions will be clustered about this biased estimate, as illustrated by the histograms shown in Figure 4. In this case, even placing 95% confidence intervals on the estimate will not capture the true damping value.

Note that the 95% confidence intervals given by the last two sets of columns in Table 2 show some consistency with those predicted from a normal distribution.

Table 3. Statistics of bootstrap replications of damping as percent of critical: data set 1 by RDT.

Nr	ξboot	ξplugin	bias[ξ]	σ[ξ]	CoV	95% boot	95% normal
100	1.533	1.532	0.0004	0.1239	8.08%	(1.334, 1.747)	(1.285, 1.781)
200	1.386	1.381	0.0054	0.1223	8.82%	(1.181, 1.591)	(1.142, 1.631)
300	1.296	1.296	0.0002	0.1259	9.72%	(1.089, 1.500)	(1.044, 1.548)
400	1.203	1.188	0.0159	0.1174	9.76%	(1.010, 1.397)	(0.969, 1.438)
500	1.132	1.126	0.0056	0.1231	10.88%	(0.921, 1.333)	(0.886, 1.378)
600	0.974	0.976	-0.0013	0.1243	12.76%	(0.772, 1.093)	(0.726, 1.223)
700	0.917	0.917	0.0005	0.1220	13.30%	(0.717, 1.203)	(0.673, 1.161)
800	1.005	1.000	0.0048	0.1218	12.13%	(0.808, 1.201)	(0.761, 1.248)

3.2 *Discussion of resampled results for RDT*

As shown in Figure 4, the distribution of RDT damping estimates on the same data manifest considerably more scatter than SA estimates, consistent with the findings of the simulations in Section 1.2. Once again, the distributions shift as more samples are considered. Beyond N_r=500 (approximately 25 hours of data), their behavior tends to stabilize about a consistent mean value, similar to what was found in SA approach. Even in this stable range, there is still some fluctuation in the estimate, consistent with the findings of Kijewski & Kareem (2000).

However, unlike SA results, the behavior of RDT estimate clearly indicates that limited amounts of data offer little hope of an accurate result, overestimating the damping far more significantly than SA. Rather, the results steadily improve with the number of samples being considered: $N_r>500$, the damping estimates are within 10%. However, when considering the full amount of data available ($N_s=100$, $N_r=800$), the two approaches are comparable, with RDT producing an estimate that is nearly exact, as illustrated by the Monte Carlo simulation, though with considerably larger *CoV*. When considering the full amount of data ($N_r=800$), the bootstrap analysis of a single observation from this population was able to capture the standard deviation and *CoV* of the Monte Carlo Simulation quite accurately. This was not possible in SA.

In this instance, the confidence intervals predicted by bootstrapping RDT results will encase the predicted result for $N_r>400$. Comparisons with the confidence bands predicted by a normal distribution are reasonably in agreement with the bootstrapped result. However, it should be reiterated that, in the case of the bootstrap approach, there was no need to invoke CLT.

Data set 1 was one of the situations where RDT performed rather well; however, as Figure 1 illustrates, there were also outlier simulations in which the performance was not as ideal. To investigate the true merits of a bootstrap analysis of the damping estimates, the statistics of data set 2 are presented in Table 4. In this case, even considering the full amount of data ($N_r=800$), the results are still in error by approximately 15%. Once again, there is sizeable variation. The performance is especially poor for $N_r=100$, where negative damping was actually detected. By visual inspection of the RDS with so few segments averaged, one will note an unstable behavior and increase in the decay curve. Still, the *CoV* of the system is again captured relatively accurately from $N_r>400$. Further, for $N_r>400$, the 95% confidence intervals, though widely spaced, do capture the true result. Traditional damping estimates lack any type of interval estimate, thus such tools can offer insight where previously there was none.

In defense of RDT, its performance under idealized conditions, in comparison to SA, is apparently lacking. However, its merits have been documented in full-scale applications where the response is no longer considered stationary (Jeary 1992). In addition, to the credit of RDT, it is also capable of detecting amplitude-dependent damping (Tamura and Suganuma 1996). The same cannot be said for the PSD.

Table 4. Statistics of bootstrap replications of damping as percent of critical: data set 2 by RDT.

Nr	ξboot	ξplugin	bias[ξ]	σ[ξ]	CoV	95% boot	95% normal
100	-0.052	-0.053	0.0015	0.0977	-189.3%	(-0.222, 0.102)	(-0.247, 0.144)
200	0.532	0.532	0.0001	0.1101	20.71%	(0.353, 0.716)	(0.311, 0.752)
300	0.775	0.774	0.0012	0.1160	14.97%	(0.595, 0.969)	(0.543, 1.007)
400	0.851	0.852	-0.0005	0.1176	13.82%	(0.656, 1.048)	(0.616, 1.086)
500	0.868	0.871	-0.0031	0.1157	13.33%	(0.683, 1.057)	(0.637, 1.099)
600	0.808	0.812	-0.0034	0.1137	14.07%	(0.621, 1.000)	(0.581, 1.036)
700	0.813	0.805	0.0081	0.1129	13.89%	(0.633, 0.996)	(0.587, 1.039)
800	0.847	0.850	-0.0034	0.1206	14.25%	(0.649, 1.051)	(0.605, 1.088)

4 CONCLUSIONS

Two common approaches for the estimation of structural damping from systems with unknown input were evaluated in this study: the Random Decrement Technique and traditional Spectral Analysis. Despite extensive theory, there is no indication of errors inherent to the dynamic properties estimated. This study utilized bootstrapped replicates of the random decrement segments and raw power spectra to assess the quality of the resulting system identification by providing surrogate estimates of damping and natural frequency to generate useful statistics and confidence intervals. Though the bias inherent in spectral estimates prohibits confidence intervals from capturing the true damping estimate, its variance was less marked. On the other hand, the broad RDT confidence bands can capture the true damping estimate when sufficient segments are considered. In both cases, the bootstrap statistics can provide new insights into the reliability of system identification.

5 ACKNOWLEDGEMENTS

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