Math 40480 PROJECT, Due Monday, April 13

Choice of topic due by Wednesday, April 1

Your project should be a well-written explanation of the topic and should convince me that you have a good understanding of it. It should not be something copied from a book. It must include at least one proof (either details or a careful sketch of the key steps). You will probably want to include some examples. You may want to include some of the history of the topic, the topic's relation to other things, and solutions of some relevant exercises. Include references. Here are some possible topics. Topics which are harder or require more background are indicated with a (*) or (**) (a lot harder). If you want to do a topic not on this list, please discuss it with me ahead of time.

1. Learn something about complex dynamical systems. Possibilities include:
   (a) What happens when you iterate a Möbius transformation (a.k.a. linear fractional transformation)? ([Be], [E])
   (b) What happens when you iterate $z \to z^2 + c$? ([Be], [E])
   (c) What is the Mandelbrot set? What is a Julia set? (*) ([Ga], [Be], [E])
2. How many ways can you prove the Fundamental Theorem of Algebra? Learn some proofs that aren't in the textbook. To what extent does complex analysis enter into each of the proofs? ([FR])
3. What does complex analysis, in particular, the Riemann ζ function, have to do with the Prime Number Theorem? (**) ([BN], [SS])
4. What is a Riemann surface? (*) ([Fo], [Gr], [MH], [S]) Possibilities include:
   (a) What is the Riemann surface of an algebraic function?
   (b) How are Riemann surfaces useful in thinking about functions like $\sqrt{z}$ and $\log z$?
5. The proof of the Riemann Mapping Theorem. This requires some understanding of uniform convergence. (*) ([SS])
6. What is an elliptic function? (*) ([SS])
7. What is an elliptic curve? How are elliptic curves related to the proof of Fermat's last theorem? Why was Wiles' 1993 announcement that he proved the theorem front page news in the *New York Times*? (**) ([M])
8. How can you find the inverse Laplace transform of a function (other than finding it in the transform column of a table of Laplace transforms)? This will require learning something about the Fourier transform as well. ([Fi])
9. What is the Poincaré metric in the disc? What does it have to do with some famous Escher prints such as *Circle Limit 1*? ([K1], [D], [Hah])

10. What is the Poincaré metric in the disc? What does it have to do with the Schwarz Lemma? What else can you get out of this geometric approach to the Schwarz Lemma? ([K2])

11. Where are the zeroes of polynomials? ([He], [Fi])

12. What are some differences between functions of one complex variable and functions of several complex variables? (Several = more than one.) Find out about Hartogs' extension phenomenon and learn how to prove it in a simple case. ([R])

13. Why stop at one? What happens if you allow more square roots of -1? What are the quaternions? the octonions? ([CG],[CS],[KS])

14. What is happening to a power series on the circle of convergence? Learn about Abel's Theorem. ([Bo])

15. Learn more about Möbius transformations (a.k.a. linear fractional transformations). Find out how fixed points are used to classify them, what the cross ratio is and why it is important, ... ([Hah], [N])

16. How can complex numbers be used to give simple proofs of such results in Euclidean geometry as the fact that the medians of a triangle all meet in a point and the nine point circle theorem? ([Hah])

17. What is the Weierstrass–Enneper representation of a minimal surface? Learn what a minimal surface is and about the connection between minimal surfaces and complex analysis. This requires some knowledge of differential geometry. (*) ([Har], [Op], [E])

18. You know lots of solutions of the Cauchy–Riemann equations. How can you solve the inhomogeneous Cauchy–Riemann equations $\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = g$? (*) ([NN])

19. How is the Mercator projection related to stereographic projection? What does conformal mapping have to do with map making? ([Fe], [Os], [Pi])

20. What is the icosahedral group, what is $\mathbf{P}^1$ and why is it advantageous to think of the icosahedral group as acting on $\mathbf{P}^1$? ([JS])

21. You know all about the zeroes of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. There aren't any. What can you say about the zeroes of the Nth partial sum, the polynomial $\sum_{n=0}^{N} \frac{z^n}{n!}$? ([Z])

22. What are the Ham Sandwich Theorem and the Jordan Curve Theorem and how can you use complex variables to prove them? ([Br1], [Br2])

23. You know a non-constant entire function is unbounded. There are entire functions that tend to zero on every line. Learn about some examples. ([A])

24. How can the Riemann Mapping Theorem help with creating flat maps of the brain? ([HSBSR], [BE]) (**)

25. What is a conformal surface parametrization, and how do you find an optimal one? ([JWYG]) (**)

References

[A] Armitage, "Entire functions that tend to zero on every line," *Amer. Math. Monthly* 114 (2007), no. 3, 251–256

[BN] Bak and Newman, *Complex Analysis*

[Be] Beardon, *Iteration of Rational Functions: Complex Analytic Dynamical Systems*

[BE] Bern and Eppstein, "Optimal Möbius tranformations for information visualization and meshing," *Proceedings of the 7th International Workshop on Algorithms and Data Structures*, Lecture Notes in Computer Science 2125 (2001) 14–25 (also available at http://arxiv.org/abs/cs.CG/0101006)

[Bo] Boas, *Invitation to Complex Analysis*

[Br1] Browder, "Topology in the complex Plane," *Amer. Math. Monthly* 107 (2000), no. 5, 393-401

[Br2] Browder, "Complex numbers and the Ham Sandwich Theorem," *Amer. Math. Monthly* 113 (2006), no. 10, 935-936

[CG] Conway and Guy, *The Book of Numbers*

[CS] Conway and Smith, *On quaternions and octonions: Their geometry, arithmetic and symmetry*

[D] Dunham, *Hyperbolic Art and the Poster Pattern*, http://www.mathaware.org/mam/03/essay1.html

[E] Brilleslyper et al., *Explorations in Complex Analysis*

[Fe] Feeman, *Portraits of the Earth: A Mathematician Looks at Maps*

[FR] Fine and Rosenberger, *The Fundamental Theorem of Algebra*

[Fi] Fisher, *Complex Variables*

[Fo] Forster, *Lectures on Riemann Surfaces*

[Ga] Gamelin, *Complex Analysis*

[Gr] Griffiths, *Introduction to Algebraic Curves*

[Hah] Hahn, *Complex Numbers and Geometry*

[Har] Hardt, ed., *Six Themes on Variation*

[He] Henrici, *Applied and Computational Complex Analysis*

[HSBSR] Hurdal, Stephenson, Bowers, Sumners and Rottenberg, "Cortical surface flattening: a quasi–conformal approach using circle packings," available at http://web.math.fsu.edu/ aluffi/archive/paper144.pdf

[JWYG] Jin, Wang, Yau and Gu, "Optimal global conformal surface parametrization," IEEE Visualization October 10–15 (2004) 267–274

[JS] Jones and Singerman, *Complex Functions: An algebraic and geometric viewpoint*

[KS] Kantor and Solodovnikov, *Hypercomplex Numbers*

[K1] Krantz, *Complex Analysis: The Geometric Viewpoint*

[K2] Krantz, *Geometric Function Theory*

[MH] Marsden and Hoffman, *Basic Complex Analysis*

[M] Mazur, "Number theory as gadfly," *Amer. Math. Monthly* 98 (1991), no. 7, 593-610

[NN] Narasimhan and Nievergelt, *Complex Analysis in One Variable*

[N] Needham, *Visual Complex Analysis*

[Op] Oprea, *The Mathematics of Soap Films: Explorations with Maple*

[Os] Osserman, "Mathematical mapping from Mercator to the millennium," available at http://www.msri.org/people/staff/osserman/papers/recentpubs.html

[Pi] Pijls, "Some properties related to Mercator projection," *Amer. Math. Monthly* 108 (2001), no. 6, 537-543

[R] Range, "Complex analysis: a brief tour into higher dimensions," *Amer. Math. Monthly* 110 (2003), no. 2, 89-108

[S] Springer, *Introduction to Riemann Surfaces*

[SS] Stein and Shakarchi, *Complex Analysis*

[Z] Zemyan, "On the zeroes of the Nth partial sum of the exponential series," *Amer. Math. Monthly* 112 (2005), 891-909

All of the books except Fisher and Stein and Shakarchi are on reserve in the Mathematics Library. Stein and Shakarchi will be on reserve. Some books may be available online through the library, and the library has earlier editions of Bak and Newman and also Narasimhan which include the relevant material.