Math 30750
Spring, 2017

Guidelines

After studying the description of the project, you should decide who will work on what and on a schedule for completing that work. You may use your textbook, notes, homework and the homework solutions. You may not use other sources. You may discuss the project with me and with members of other groups, but you may not copy any work from another group. Copying work from another group or other sources is a violation of the Honor Code.

The project will be done in two parts, worth a total of 75 points. For each part each group will turn in a report, written jointly by the group. The reports must be readable, literate and informative. When you have completed the project, every member of the group should read the entire project. Be sure to check that the parts are consistent. In particular, make sure that your project does not have any mathematical statements which contradict each other. Along with the report for part 2, each member of the group will turn in a well-written summary worth 10 points of the project, including a summary of the key ideas in the proofs.

Group project

This project is about the Cantor set and Cantor function. You can find a little information about the Cantor set and function on p. 381 of the textbook. At the end you will find some hints which may be helpful.

Group Project, part 1, due Wedensday, April 12

1. (6 points) The purpose of this problem is to construct the Cantor set and establish one of its properties.

   Let
   $$C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}).$$
   Then C_1 is obtained by removing the open middle third from $[0,1]$. It is the union of two closed intervals, each of length $\frac{1}{3}$. Let
   $$C_2 = C_1 \setminus \left((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \right).$$

It is obtained by removing the open middle third from each interval in C_1 and is the union of four closed intervals, each of length $\frac{1}{9}$. Continue this construction, and let C_n be the set constructed at the nth step.

Observe that $\{C_n\}$ is a nested sequence of sets, $C_{n+1} \subset C_n$ for all n.

The *Cantor set* is

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Prove that the sum of the lengths of the intervals excluded in the construction of the Cantor set is 1.

2. (24 points) The purpose of this problem is to prove that every number in $[0, 1]$ has a *ternary expansion* $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where each a_n is one of $0, 1, 2$. In the next problem you will relate this to the Cantor set.

   (a) Suppose $x \in [0, 1]$, and $x = \frac{k}{3^n}$ for some positive integer n and some positive integer $k < 3^n$.

      i. Prove that x has a ternary expansion with $a_m = 0$ for all $m > n$, i.e., the ternary expansion ends in a string of 0's.

      ii. Prove that x also has a ternary expansion with $a_m = 2$ for all $m > n$ if $x > 0$, i.e., the ternary expansion ends in a string of 2's.

      iii. Prove that if $x \in [0, 1)$ has a ternary expansion ending in a string of 0's or one ending in a string of 2's then $x = \frac{k}{3^n}$ for some positive integer n and some positive integer $k < 3^n$.

   (b) Suppose x does not have the form $\frac{k}{3^n}$ for any positive integers k and n. Then x is in one of the open intervals $(0, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 1)$. If x is in the jth interval let $a_1 = j - 1$. Now divide the jth interval into 3 subintervals of length $\frac{1}{9}$. Use this to define a_2. Make this procedure precise, using induction to define a_n for all n. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ is a ternary expansion for x.

   (c) Prove that if two ternary expansions $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{b_n}{3^n}$ are distinct and neither ends in a string of 2's then the numbers they represent are different.

   (d) Conclude that every x not of the form $\frac{k}{3^n}$ has a unique ternary expansion and it does not end in a string of 0's or of 2's.

   (e) Prove that every x of the form $\frac{k}{3^n}$ has exactly two expansions.

3. (4 points) Prove that the Cantor set is the set of $x \in [0, 1]$ such that x has a ternary expansion with no 1's.

4. (3 points) Which points of the Cantor set have ternary expansions ending in a string of 0's or a string of 2's? Prove that these points form a countable set.

Group Project, part 2, due Wednesday, April 26

5. (5 points)

   Prove that the Cantor set is uncountable. Here are two possible methods of doing this. Fill in the details of one of them or use another method.

   One way to do this is to imitate the proof of Theorem 1.3.6.

   Another way is to show that the Cantor function g (see 7 below) maps the Cantor set onto $[0,1]$.

6. (5 points) Is the Cantor set large or small? Your explanation should take into account 1 and 5.

7. (28 points) The purpose of this problem is to define the *Cantor function* and study some of its properties.

   If $x \in [0,1]$ has no 1's in its ternary expansion $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ let $N = \infty$. Otherwise, let N be the index where the first 1 occurs. Let $b_n = \frac{a_n}{2}$ for $n < N$ and $b_N = 1$ if $N < \infty$. The *Cantor function* g is defined by

   $$g(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}.$$

   (a) Prove that the Cantor function is well defined, that is, show that if x has two ternary expansions the value of $g(x)$ does not depend on the expansion.

   (b) Show that

   $$g(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \\ \frac{1}{4} & \text{if } x \in (\frac{1}{9}, \frac{2}{9}) \\ \frac{3}{4} & \text{if } x \in (\frac{7}{9}, \frac{8}{9}) \end{cases}.$$

   (c) Prove that the Cantor function is constant on each interval in the complement of the Cantor set.

   (d) Prove that the Cantor function is differentiable on the complement of the Cantor set with derivative 0 on that set.

   (e) Prove that the Cantor function is monotone increasing.

   (f) Prove that the Cantor function is continuous.

   So the Cantor function is continuous, monotone increasing, maps $[0,1]$ onto $[0,1]$ and is differentiable with derivative 0 on disjoint open subintervals of $[0,1]$ whose lengths add up to 1!

   The Cantor function is not differentiable on $[0,1]$.

Hints

- *Reminder*: (see §6.2), $\sum_{n=1}^{\infty} c_n$ is defined as $\lim_{n\to\infty} S_n$ if this limit exists, where $S_n = \sum_{j=1}^{n} c_j$
- You will probably find the formulas (see Example 6.2.1) for summing a finite geometric progression and a geometric series useful.
- In 2(b), be sure to prove that $\sum_{n=1}^{\infty} \frac{a_n}{3^n} = x$.