Some properties of the Riemann integral

Here are proofs of Theorems 3.3.3-3.3.5, Corollary 3.3.6 and Theorem 3.3.7 for any Riemann integrable functions on $[a, b]$. Because the statements in the book are for continuous functions I added $'$ to the number of the theorem or corollary to distinguish it from the corresponding one in the book.

Theorem 3.3.3$'$: If f and g are Riemann integrable on $[a, b]$ and $\alpha, \beta \in \mathbf{R}$ then $\alpha f + \beta g$ is Riemann integrable on $[a, b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx. \tag{1}$$

Proof: (i) If $\alpha \geq 0$, by §2.5 #8

$$\sup_{[c,d]} \alpha f = \alpha \sup_{[c,d]} f$$

for any subinterval $[c, d] \subset [a, b]$. Hence for any partition P of $[a, b]$, $U_P(\alpha f) = \alpha\, U_P(f)$. Also §2.5 #8 holds for the infinimum; for any $S \subset \mathbf{R}$

$$\inf\{\alpha x : x \in S\} = \alpha \inf S \quad \text{if } \alpha \geq 0.$$

Hence

$$\inf_P\{U_P(\alpha f)\} = \inf_P\{\alpha U_P(f)\} = \alpha\ \inf_P\{U_P(f)\} = \alpha \int_a^b f(x)\, dx. \tag{2}$$

Similarly $L_P(\alpha f) = \alpha\, L_P(f)$ so

$$\sup_P\{L_P(\alpha f)\} = \alpha \int_a^b f(x)\, dx. \tag{3}$$

By (2), (3) and the definition of the Riemann integral, αf is Riemann integrable on $[a, b]$ and

$$\int_a^b \alpha\, f(x)\, dx = \alpha \int_a^b f(x)\, dx. \tag{4}$$

(ii) For any $S \subset \mathbf{R}$,

$$\sup_S(-f) = -\inf_S f$$

Hence, $U_P(-f) = -L_P(f)$ so $\inf_P\{U_P(-f)\} = -\sup_P\{L_P(f)\} = -\int_a^b f(x)\, dx$. Similarly, $\sup_P\{L_P(-f)\} = -\int_a^b f(x)\, dx$ so $-f$ is Riemann integrable on $[a, b]$ and

$$\int_a^b -f(x)\, dx = -\int_a^b f(x)\, dx. \tag{5}$$

Combining (4) and (5) shows that (4) holds for any $\alpha \in \mathbf{R}$.

(iii) Because f and g are Riemann integrable on $[a, b]$, for any $\epsilon > 0$ we can find partitions P_1 and P_2 such that

$$\int_a^b f(x)\,dx - \epsilon \le L_{P_1}(f) \le U_{P_1}(f) \le \int_a^b f(x)\,dx + \epsilon \tag{6}$$

and

$$\int_a^b g(x)\,dx - \epsilon \le L_{P_2}(g) \le U_{P_2}(g) \le \int_a^b g(x)\,dx + \epsilon. \tag{7}$$

Also, for any interval $[c, d]$ by §2.5 #9

$$\sup_{[c,d]}(f+g) \le \sup_{[c,d]} f + \sup_{[c,d]} g$$

so for any partition P

$$U_P(f+g) \le U_P(f) + U_P(g) \tag{8}$$

and similarly

$$L_P(f) + L_P(g) \le L_P(f+g). \tag{9}$$

Adding (6) and (7) and using (8), (9) and Lemma 1 shows that if $Q = P_1 \cup P_2$,

$$\begin{aligned} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - 2\epsilon &\le L_{P_1}(f) + L_{P_2}(g) \\ &\le L_Q f + L_Q g \\ &\le L_Q(f+g) \\ &\le U_Q(f+g) \\ &\le U_Q(f) + U_Q(g) \\ &\le U_{P_1}(f) + U_{P_2}(g) \\ &\le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + 2\epsilon. \end{aligned}$$

This holds for every $\epsilon > 0$. Hence

$$\sup_P\{L_P(f+g)\} = \inf_P\{U_P(f+g)\} = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Thus $f + g$ is Riemann integrable on $[a, b]$ and

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \tag{10}$$

The theorem follows from (4), (5) and (10).

Remark: This result says that the Riemann integrable functions on $[a, b]$ form a vector space and integration is a linear operator (transformation) from this vector space to $\mathbf{R}$.

Theorem 3.3.4′: If f and g are Riemann integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$.

Proof: Because $f(x) \le g(x)$, for any partition P of $[a, b]$, $U_P(f) \le U_P(g)$. Hence any lower bound for $\{U_P(f)\}$ is a lower bound for $\{U_P(g)\}$. In particular,

$$\int_a^b f(x)\, dx = \inf_P \{U_P(f)\} \le \inf_P \{U_P(g)\} = \int_a^b g(x)\, dx.$$

Theorem 3.3.5′: If f is Riemann integrable on $[a, b]$ then so is $|f|$ and

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx. \tag{11}$$

Proof: Let $\epsilon > 0$ and let P be a partition of $[a, b]$ such that $U_P(f) - L_P(f) \le \epsilon$. Let $m_i = \inf_{[x_{i-1}, x_i]} f,\ m_i' = \inf_{[x_{i-1}, x_i]} |f|,\ M_i = \sup_{[x_{i-1}, x_i]} f,\ M_i' = \sup_{[x_{i-1}, x_i]} |f|$. There are three cases.

Case (i): If $m_i \ge 0$, then $M_i' = M_i,\ m_i' = m_i$ so

$$M_i' - m_i' = M_i - m_i.$$

Case (ii): If $M_i < 0$ then $M_i' = -m_i,\ m_i' = -M_i$ so

$$M_i' - m_i' = M_i - m_i.$$

Case(iii): If $M_i > 0,\ m_i < 0$ then $M_i' = \max\{M_i, -m_i\}$ and $m_i' \ge 0$ so

$$M_i' - m_i' \le \max\{M_i, -m_i\} < M_i - m_i.$$

In each case

$$M_i' - m_i' \le M_i - m_i$$

so

$$U_P(|f|) - L_P(|f|) \le U_P(f) - L_P(f) \le \epsilon \tag{12}$$

and, by Lemma 3, $|f|$ is integrable. Now (11) follows from Theorems 3.3.3′ and 3.3.4′ since $f(x), -f(x) \le |f(x)|$.

Corollary 3.3.6′: If f is Riemann integrable on $[a, b]$ then

$$\left| \int_a^b f(x)\, dx \right| \le (b - a) \sup_{[a,b]} |f(x)|. \tag{13}$$

Proof: By Theorem 3.3.5′

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$$

Now apply Theorem 3.3.4′ to the right side with $g(x)$ the constant function $\sup_{[a,b]} |f|$.

Theorem 3.3.7′: If f is Riemann integrable on $[a, b]$ and $a < c < b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{14}$$

Proof: For any partition P of $[a, b]$, let P_c be P if c is a point of P and the partition obtained from P by adding the point c otherwise. Let P_1 be the points in P_c which are less than or equal to c, so P_1 is a partition of $[a, c]$, and let P_2 be the points that are greater than or equal to c so P_2 is a partition of $[c, b]$. Then

$$L_P(f) \le L_{P_c}(f) = L_{P_1}(f) + L_{P_2}(f) \le U_{P_1}(f) + U_{P_2}(f) = U_{P_c}(f) \le U_P(f)$$

Hence

$$\sup_P\{L_P(f)\} \le \sup_{P_1}\{L_{P_1}(f)\} + \sup_{P_2}\{L_{P_2}(f)\} \le \inf_{P_1}\{U_{P_1}(f)\} + \inf_{P_2}\{U_{P_2}(f)\} \le \inf_P\{U_P(f)\}.$$

Since the right and left ends are equal to $\int_a^b f(x)\,dx$,

$$\int_a^b f(x)\,dx = \sup_{P_1}\{L_{P_1}(f)\} + \sup_{P_2}\{L_{P_2}(f)\} = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$