Sets of Measure Zero

Definition: A subset $A \subset \mathbf{R}$ has **measure 0** if

$$\inf_{A \subset \cup I_n} \sum \ell(I_n) = 0$$

where $\{I_n\}$ is a finite or countable collection of open intervals and

$$\ell(a, b) = b - a.$$

In other words, A has measure 0 if for every $\epsilon > 0$ there are open intervals $I_1, I_2, \ldots, I_n, \ldots$ such that $A \subset \cup I_n$ and $\sum \ell(I_n) \leq \epsilon$.

Almost everywhere means except on a set of measure 0.

Examples of sets of measure 0

- A finite, $A = \{x_1, \ldots, x_k\}$
  Proof: Given $\epsilon > 0$, let

$$I_j = (x_j - \frac{\epsilon}{2k}, x_j + \frac{\epsilon}{2k}), \ j = 1, \ldots, k.$$

Then

$$\ell(I_j) = \frac{\epsilon}{k}, \qquad A \subset \bigcup_{j=1}^{k} I_j,$$

and

$$\sum_{j=1}^{k} \ell(I_j) = \epsilon.$$

- A countable, e.g., $A = \mathbf{Q}$, the rational numbers
  Proof: Suppose

$$A = \{x_n\}_{n=1}^{\infty}.$$

Given $\epsilon > 0$, let

$$I_n = (x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n+1}}), \ n = 1, 2, \ldots .$$

Then

$$\ell(I_n) = \frac{\epsilon}{2^n}, \qquad A \subset \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

- The **Cantor set** A is constructed inductively.
  Let

$$A_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}).$$

Obtain A_{n+1} from A_n by removing the open middle third of each remaining interval.
Then

$$A = \bigcap_{n=1}^{\infty} A_n.$$

If you expand numbers in a ternary expansion, so if $x \in [0,1]$ you write

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \text{ where } a_j = 0, 1 \text{ or } 2,$$

then

$$A = \{x \in [0,1] : x \text{ has no 1 in its ternary expansion}\}.$$

The map

$$\sum_{j=1}^{\infty} \frac{a_j}{3^j} \to \sum_{j=1}^{\infty} \frac{a_j/2}{2^j}$$

maps the Cantor set onto the unit interval.

So the Cantor set is uncountable.

To see that it has measure 0, notice that

$$A_n = \bigcup_{k=1}^{2^n} B_{n,k}, \; B_{n,k} = [a_{n,k}, b_{n,k}], \; \ell(B_{n,k}) = \frac{1}{3^n}.$$

Let

$$I_{n,k} = \left(a_{n,k} - \frac{1}{10^n}, b_{n,k} + \frac{1}{10^n} \right)$$

so

$$B_{n,k} \subset I_{n,k} \qquad \text{and} \qquad \ell(I_{n,k}) = \frac{1}{3^n} + \frac{2}{10^n}.$$

Now

$$A \subset A_n \subset \bigcup_{k=1}^{2^n} I_{n,k}, \qquad \sum_{k=1}^{2^n} \ell(I_{n,k}) = \frac{2^n}{3^n} + \frac{2^{n+1}}{10^n} \to 0 \text{ as } n \to \infty$$

so A has measure 0.