Math 30650
Spring, 2012

GROUP PROJECT 2, due Friday, April 20

You have a choice of projects I, II or III. If you choose a project which involves making movies, your group must make an appointment to show me the movies. The time variable t should be used as the parameter for the frames of the movies.

I. Vibrating strings

The purpose of this project is to study the one dimensional wave equation in greater detail and to make movies of vibrating strings. Part B is independent of part A.

A. Do §10.7 #13,14,16-18 in Boyce and DiPrima. In #14, make a movie of $y = \phi(x - at)$ and also of $y = \phi(x + at)$, taking $a = 1$. In 16(d), use the function f from 16(c).

Note: These problems are about the wave equation on the line, not about vibrating strings, so you can't use Fourier series methods.

B. For each of the following initial positions, make a movie of a vibrating string of length 2π (thought of as the interval $0 \leq x \leq 2\pi$ when it is at rest), initial velocity 0 and the given initial position:

1. $\sin x/2$;
2. $\sin x$;
3. $\sin 10x$;
4. the function
$$f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases};$$
5. the function
$$g(x) = \begin{cases} f(2x) & 0 \leq x \leq \pi \\ -f(2(x - \pi)) & \pi \leq x \leq 2\pi \end{cases}$$
where f is the function in 4;

   Hint: First evaluate g at $0, \pi/2, \pi, 3\pi/2,$ and 2π and graph g by hand.
6. several initial positions of your choice.

Hint: be sure to ask whether the your initial positions make sense for vibrating strings.

II. Vibrating drumheads

The purpose of this problem is to study the two dimensional wave equation in the disk $D = \{x^2 + y^2 \leq 1\}$ and to make movies of some vibrating drumheads. The two dimensional wave equation is

$$a^2(u_{xx} + u_{yy}) = u_{tt}$$

where $u = u(x, y, t)$ is the displacement of the point (x, y) at time t. The condition that the edge of the drumhead is fixed is expressed as

$$u(x, y, t) = 0 \qquad \text{if} \qquad x^2 + y^2 = 1.$$

A. *The wave equation in polar coordinates*: Suppose $u(x, y, t)$ solves the wave equation and $v(r, \theta, t) = u(r\cos\theta, r\sin\theta, t)$. Prove that v satisfies

$$v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = a^{-2}v_{tt}. \tag{1}$$

B. *Separation of variables*: Suppose $v(r, \theta, t) = R(r)\Theta(\theta)T(t)$ solves (1) and that v satisfies the boundary condition $v(1, \theta, t) = 0$. Prove that

$$\Theta'' + m^2\Theta = 0 \qquad \text{for some } m = 0, 1, 2, 3, \ldots; \tag{2}$$

$$T'' + \lambda a^2 T = 0 \qquad \text{for some } \lambda; \tag{3}$$

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{m^2}{r^2}\right)R = 0, \qquad R(1) = 0. \tag{4}$$

Hint: Θ must be periodic with period 2π.

C. *Bessel's equation*: Equation (4) is Bessel's equation. It has a regular singular point at the origin, so you expect that there will only be one independent solution which remains bounded as $r \to 0$. This solution is (up to a constant):

$$y(r) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n r^{2n+m}}{2^{2n} n!(m+n)!}.$$

Show that if $y(1) = 0$ then $\lambda > 0$.

D. From now on, assume that $\lambda > 0$. The Bessel function J_m is defined by

$$J_m(r) = \sum_{n=0}^{\infty} \frac{(-1)^n r^{2n+m}}{2^{2n+m} n!(m+n)!}.$$

Verify that $R(r) = J_m(r)$ satisfies the differential equation

$$R'' + \frac{1}{r}R' + \left(1 - \frac{m^2}{r^2}\right)R = 0.$$

The function $J_m(r)$ is known to MATLAB as **besselj**(m,r).

E. Show that $J_m(r\sqrt{\lambda})$ solves (4).

F. Use the results of B and E to show that the separated solutions of the wave equation in polar coordinates which satisfy the boundary condition $v(1, \theta, t) = 0$ are of the form

$$v_{m,n}(r, \theta, t) = J_m(r x_n^{(m)})(A \cos m\theta + B \sin m\theta)(\widetilde{A} \cos t x_n^{(m)} + \widetilde{B} \sin t x_n^{(m)})$$

where $\{x_n^{(m)} : n = 1, 2, ...\}$ are the positive zeros of J_m, listed in increasing order.

G. Find the first two positive zeros of J_0, J_1 and J_2. (**Hint**: Use MATLAB to graph these functions. From the graph, approximate the zero you want to compute. Then use the MATLAB command **fzero**. For example, to find the first positive zero of $J_0(x)$, estimate from the graph that the zero is near 2, then give the commands:

f = @(x) besselj(0,x); **fzero**(f,2)

to find the zero.)

H. For $m = 0, 1, 2$ and $n = 1, 2$ make a movie of the vibrating drumheads $v_{m,n}$ with $A = \widetilde{A} = 1$, $B = \widetilde{B} = 0$. (You want to plot $(r \cos \theta, r \sin \theta, v_{m,n}(r, \theta, t))$ as a function of r and θ for fixed t and then animate with respect to the variable t.)

III. The Gibbs Phenomenon

The purpose of this project is to study the behavior of a Fourier series at a jump discontinuity.

A. Calculate the Fourier series of the functions

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

and

$$g(x) = x, \qquad -\pi \le x \le \pi.$$

Also calculate the Fourier series of at least one other function with a jump discontinuity.

B. For each of the functions in A, use MATLAB to graph the function and the partial sums of the Fourier series with terms of index up to $10, 20, 50, 100$. In the graphs, you should see that the partial sum overshoots the function near a jump discontinuity. Click the mouse at a peak to estimate the overshoot.

C. For the function f of A, you should have found that the partial sums of the Fourier series are

$$f_{2n-1}(x) = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \ldots + \frac{\sin(2n-1)x}{2n-1} \right).$$

The overshoot of the partial sum f_{2n-1} at $x = 0$ is the value of f_{2n-1} at its first positive maximum. To find the location of this maximum, first show that

$$f'_{2n-1}(x) = \frac{4}{\pi} \left(\cos x + \cos 3x + \ldots + \cos(2n-1)x \right). \tag{5}$$

The maximum occurs at a point where $f'_{2n-1}(x) = 0$. To solve this, multiply (5) by $\sin x$ and use the identity

$$\sin x \cos kx = \frac{1}{2}(\sin(k+1)x - \sin(k-1)x)$$

to show that

$$\pi \sin x f'_{2n-1}(x) = 2 \sin 2nx.$$

Conclude that the extrema occur at

$$2nx = \pm\pi, \pm 2\pi, \ldots, \pm(2n-1)\pi$$

and that the first positive maximum occurs at

$$x = \frac{\pi}{2n}.$$

D. From C, you know where the overshoot occurs. Now you need to estimate its behavior as $n \to \infty$. The overshoot is

$$f_{2n-1}\left(\frac{\pi}{2n}\right) = \frac{4}{\pi}\left(\sin\frac{\pi}{2n} + \frac{1}{3}\sin\frac{3\pi}{2n} + \ldots + \frac{1}{2n-1}\sin\frac{(2n-1)\pi}{2n}\right).$$

Show that this is an approximating sum for

$$\frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx. \tag{6}$$

(**Hint**: Divide the interval $0 \le x \le \pi$ into n subintervals of length $\frac{\pi}{n}$, with endpoints $0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{n\pi}{n}$. Evaluate the function $\frac{2}{\pi}\frac{\sin x}{x}$ at the midpoints of the subintervals to form the approximating sums.) Conclude that the limiting value of the overshoot is

$$\lim_{n\to\infty} f_{2n-1}\left(\frac{\pi}{2n}\right) = \frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx.$$

E. Evaluate (6) numerically (for example, by using MATLAB to calculate the integral, then giving the command **double**(ans) to obtain a decimal). This gives the limiting value of the overshoot as $n \to \infty$. Your answer should be about 9% of the jump made at the discontinuity.

F. Does the answer you obtained in E fit with what you observed in the plots in B?