Math 30650
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SOLVING FIRST ORDER LINEAR CONSTANT COEFFICIENT EQUATIONS

In section 2.1 of Boyce and DiPrima, you learned how to solve a first order linear ordinary differential equation using an integrating factor (typically called μ). Such an equation has the form $y' + p(t)y = g(t)$. This method works for any first order linear ODE.

However, if the equation happens to be constant coefficient and the function g is of a particularly simple form, there is another way to think about the problem. The equation has the form

$$y' + ay = g(t) \tag{1}$$

where a is a constant. You can think of this as a special case of an nth order linear inhomogeneous ODE (with $n = 1$). If you think of it that way, you can solve it the same way you solve higher order constant coefficient linear ODEs. Here's a sketch.

Step 1 Solve the corresponding homogeneous equation

$$y' + ay = 0 \tag{2}$$

by looking for a solution of the form $y = Ce^{rt}$. You find that $r = -a$. So the general solution to (2) is

$$y_c = Ce^{-at}.$$

Now, back to the original equation, (1). The general solution will be of the form

$$y = y_c + y_p$$

where y_p is a particular solution, that is, one solution you will find somehow. Step 2 will apply if $g(t)$ is of a particularly nice form. Suppose

$$g(t) = p(t)e^{-at}$$

where $p(t)$ is a polynomial of degree k.

Step 2 Use the **method of undetermined coefficients**. Look for a particular solution of the form

$$y_p = t(A_0t^k + A_1t^{k-1} + \ldots A_{k-1}t + A_k)e^{-at},$$

that is, t times a general polynomial of degree k, with undetermined coefficients which you need to determine, times an exponential. (You need the factor of t in front because the exponential term solves the homogeneous equation (2).) Plug y_p into the original equation (1). Then equate corresponding terms. This will give you $k + 1$ equations for the $k + 1$ undetermined coefficients $A_0, \ldots, A_k$. Solve these equations to determine the coefficients. Now you have found y_p.

(You can actually handle somewhat more general forms of $g(t)$, any form that can be handled for nth order equations by the method of undetermined coefficients, but this is the form of $g(t)$ which comes up when you are solving a system of the form $\mathbf{x}' = J\mathbf{x}$ where J is a matrix in Jordan canonical form.)

Step 3 The general solution to (1) is

$$y = Ce^{-at} + t(A_0t^k + A_1t^{k-1} + \ldots A_{k-1}t + A_k)e^{-at}$$

where $A_0, \ldots, A_k$ are the coefficients you found in Step 2.