Gamma function, gamma density and beta function

§1. *The gamma function*

The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \qquad \alpha > 0.$$

For $\alpha > 1$, integration by parts gives

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}\,dx \\ &= -\lim_{A\to\infty} x^{\alpha-1}e^{-x}\Big|_0^A + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\,dx\end{aligned}$$

so

$$\boxed{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1).} \tag{1}$$

If $\alpha = n \geq 2$ is an integer then by (1)

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \qquad \text{if } n \geq 3 \\ &= \ldots = (n-1)!\Gamma(1) \\ &= (n-1)!\int_0^\infty e^{-x}\,dx.\end{aligned}$$

Hence

$$\boxed{\Gamma(n) = (n-1)!} \tag{2}$$

This means you can think of the gamma function as a way to define factorials for non-integer values.

§2. *The gamma density*

The **gamma density** is

$$\Gamma(x;\alpha,\lambda) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

The book uses the notation $g_n(x)$ for $\Gamma(x;n,\lambda)$.

§3. *Relation between the gamma function and beta function*

The **beta function** is defined by

$$\mathrm{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx.$$

Setting $x = y + \frac{1}{2}$ gives the more symmetric formula

$$\mathrm{B}(\alpha, \beta) = \int_{-1/2}^{1/2} \left(\frac{1}{2} + y\right)^{\alpha-1} \left(\frac{1}{2} - y\right)^{\beta-1} dy.$$

Now let $y = \frac{t}{2s}$ to obtain

$$(2s)^{\alpha+\beta-1}\mathrm{B}(\alpha, \beta) = \int_{-s}^{s} (s+t)^{\alpha-1}(s-t)^{\beta-1}\, dt.$$

Multiply by e^{-2s} then integrate with respect to $s,\ 0 \le s \le A$, to get

$$\mathrm{B}(\alpha, \beta) \int_0^A e^{-2s}(2s)^{\alpha+\beta-1}\, ds = \int_0^A \int_{-s}^{s} e^{-2s}(s+t)^{\alpha-1}(s-t)^{\beta-1}\, dt\, ds.$$

Take the limit as $A \to \infty$ to get

$$\frac{1}{2}\mathrm{B}(\alpha, \beta)\Gamma(\alpha+\beta) = \lim_{A\to\infty} \int_0^A \int_{-s}^{s} e^{-2s}(s+t)^{\alpha-1}(s-t)^{\beta-1}\, dt\, ds.$$

Let $\sigma = s+t,\ \tau = s-t$, so we integrate over

$$R = \{(\sigma, \tau) : \sigma + \tau \le 2A,\ \sigma, \tau \ge 0\}.$$

Since $s = \frac{1}{2}(\sigma+\tau),\ t = \frac{1}{2}(\sigma-\tau)$ the Jacobian determinant of the change of variables is

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

and

$$\frac{1}{2}\mathrm{B}(\alpha, \beta)\Gamma(\alpha+\beta) = \lim_{A\to\infty} \int\int_R \frac{1}{2} e^{-(\sigma+\tau)}\sigma^{\alpha-1}\tau^{\beta-1}\, d\tau\, d\sigma.$$

Thus

$$\begin{aligned} \mathrm{B}(\alpha, \beta)\Gamma(\alpha+\beta) &= \int_0^\infty \int_0^\infty e^{-(\sigma+\tau)}\sigma^{\alpha-1}\tau^{\beta-1}\, d\tau\, d\sigma \\ &= \int_0^\infty \int_0^\infty e^{-\sigma}\sigma^{\alpha-1}e^{-\tau}\tau^{\beta-1}\, d\tau\, d\sigma \\ &= \left(\int_0^\infty e^{-\sigma}\sigma^{\alpha-1}\, d\sigma\right)\left(\int_0^\infty e^{-\tau}\tau^{\beta-1}\, d\tau\right). \end{aligned}$$

So we have:

Theorem

$$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$