An Approach to Stochastic Integration for Fractional Brownian Motion in a Hilbert Space

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Abstract

A Hilbert-valued stochastic integration is defined for an integrator that is a cylindrical fractional Brownian motion in a Hilbert space. Since the integrator is not a semimartingale for the fractional Brownian motions considered, a different definition of integration is required. Both deterministic and stochastic operator-valued integrands are used. The approach to integration has an analogue with Skorokhod integrals for Brownian motion by the basic use of a derivative of some functionals of Brownian motion. An Itô formula is given for some processes obtained by this stochastic integration.

1 Introduction

Fractional Brownian motion is a family of Gaussian processes that are indexed by the Hurst parameter $H \in (0,1)$. In a finite dimensional Euclidean space these processes were introduced by Kolmogorov [10] and some properties of these processes were given by Mandelbrot and van Ness [13]. Hurst [8], [9] used this approach to describe the long term capacity of reservoirs along the Nile River which was the initial indication that these processes could be used as models of physical phenomena. Mandelbrot [12] used these processes to model some economic data and, most recently, these processes have been noted for models of telecommunication traffic (e.g., [11]). To enhance the analysis and the applicability of these processes, a stochastic calculus has been developed in recent years for these processes in finite dimensional spaces (e.g., [1], [3], [4]). The stochastic calculus given here uses a different approach than the one used in [1], [3] or [4]. Since a fractional Brownian motion, for $H \neq 1/2$, is not a semimartingale, it is necessary to define a stochastic calculus. These processes have a self-similarity in probability law and, for $H \in (1/2, 1)$, a long range dependence property described by the covariance function.

It seems that there is only a very limited amount of work on a stochastic integration for fractional Brownian motion in an infinite dimensional space. In [6], a stochastic integration is defined for deterministic integrands and a (cylindrical) fractional Brownian motion in a Hilbert space. This integration is used in [6] to define solutions of linear partial differential equations with a fractional Brownian motion. The solution of a particular stochastic partial differential equation is given in [7].

2 Main Results

Initially, a stochastic integration is defined where the integrator is a cylindrical fractional Brownian motion in a Hilbert space with $H \in (1/2, 1)$ and the integrand is a deterministic operator-valued function.

Let U, V be separable Hilbert spaces, $(\Omega, \mathcal{F}, P)$ be a complete probability space, and Q be a nonnegative, bounded self-adjoint operator on U. Let B^H denote a U-valued fractional Brownian motion with Hurst parameter H. The existence of B^H (for $H > 1/2$) is proved in [6]. In [6] a stochastic integral is also defined

$$\int_0^T G dB^H$$

for a deterministic function $G : [0, T] \to \mathcal{L}_2(U, V)$, where $\mathcal{L}_2(U, V)$ is the Hilbert space of Hilbert-Schmidt linear operators with norm denoted by $|\cdot|_{\mathcal{L}_2}$. Another definition is given here under less restrictive assumptions than in [6].

Definition 2.1. *Let U be a separable Hilbert space with inner product $\langle \, , \, \rangle_U$, and Q a nonnegative, nuclear, self-adjoint operator on U. A continuous, zero mean, U-valued Gaussian process, $(B^H(t),\, t \in \mathbb{R}_+)$, is said to be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ and associated with the covariance operator Q, if $\mathbb{E}[\langle u, B^H(t)\rangle_U] = 0$ for all $t \in \mathbb{R}_+$ and $u \in U$ and*

$$\mathbb{E}\left[\langle u_1, B^H(s)\rangle_U \langle u_2, B^H(t)\rangle_U\right] = \langle Qu_1, u_2\rangle_U \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right] \tag{2.1}$$

for all $s, t \in \mathbb{R}_+$ and $u_1, u_2 \in U$.

If Q is a nonnegative, self-adjoint bounded linear operator that is not nuclear, then a cylindrical fractional Brownian motion is defined by the formal series,

$$B^H(t) = \sum_{n=1}^{\infty} e_n \beta_n^H(t),$$

where $(e_n,\, n \in \mathbb{N})$ is a complete orthonormal basis in the Hilbert space $Q^{1/2}(U)$ and $(\beta_n^H(t),\, t \in \mathbb{R}_+, n \in \mathbb{N})$ is a sequence of independent, real-valued standard fractional Brownian motions with the Hurst parameter $1/2 < H < 1$.

Remark 2.1. *Proposition 2.2 [6] ensures the existence of a fractional Brownian motion and the existence of a standard cylindrical (i.e., $Q = Id$) fractional Brownian motion for*

$H > 1/2$, but the arguments are valid for arbitrary Q. However B^H takes values in U if and only if Q is a nuclear operator. Otherwise, B^H takes values in the larger Hilbert space U_1, where $U \hookrightarrow U_1$ and the embedding is a Hilbert-Schmidt operator. If Q is a nuclear operator, then a cylindrical fractional Brownian motion is a fractional Brownian motion.

If $H \in (1/2, 1)$ and B^H is a standard cylindrical fractional Brownian motion, then

$$\mathbb{E}\left[\langle \varphi, B^H(s)\rangle_U \langle \psi, B^H(t)\rangle_U\right] = \langle \varphi, \psi\rangle_U \int_0^t \int_0^s \phi_H(u-v)dudv \tag{2.2}$$

for all $\varphi, \psi \in U$ where

$$\phi_H(s) = H(2H-1)|s|^{2H-2}. \tag{2.3}$$

A family of deterministic integrands for a fractional Brownian motion integrator is given in the following definition.

Definition 2.2. *Let $H \in (1/2, 1)$ be fixed. Let $u_a(s) = s^a$ for $a \in \mathbb{R}$ and $I_{0+}^{H-1/2}(f)$ be the $H - 1/2$ fractional integral of f, that is,*

$$\left(I_{0+}^{H-1/2} f\right)(s) = \frac{1}{\Gamma\left(H - \frac{1}{2}\right)} \int_0^s \frac{f(t)}{(s-t)^{3/2-H}} dt.$$

The space $L^2_{\phi_H}([0,T], \mathcal{L}_2(U,V))$, often simply denoted $L^2_{\phi_H}$, is the linear space of $\mathcal{L}_2(U,V)$-valued distributions (or generalized functions) such that $u_{1/2-H} I_{0+}^{H-1/2}(u_{1/2-H} f)$ is square integrable, that is,

$$\int_0^T \left(u_{1/2-H}(s) |I_{0+}^{H-1/2}(u_{1/2-H} f)(s)|_{\mathcal{L}_2}\right)^2 ds < \infty. \tag{2.4}$$

Remark 2.2. *The Hilbert space $L^2_{\phi_H}$ is equivalent to the completion of the pre-Hilbert space of $\mathcal{L}_2(U,V)$-valued bounded Borel measurable functions $F : [0,T] \to \mathcal{L}_2(U,V)$, with the norm induced from the inner product*

$$\int_0^T \int_0^T \langle F(s), G(t)\rangle_{\mathcal{L}_2} \phi_H(s-t) ds dt, \tag{2.5}$$

where ϕ_H is given by (2.3). This equivalence of these Hilbert spaces follows from the completeness of the Lebesgue spaces and a representation of $(y-x)^{2H-2}$ as a hypergeometric function.

It is allowed to let $T = +\infty$ in Definition 2.2 so that $[0, T]$ becomes $[0, +\infty)$.

Fix $(B^H(t),\ t \in \mathbb{R}_+)$ a standard cylindrical fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. For $F \in L^2_{\phi_H}$, the stochastic integral $\int_0^T F dB^H$ can be defined and it is a V-valued Gaussian random variable.

Proposition 2.1. *Let $(B^H(t),\, t \geq 0)$ be a standard cylindrical fractional Brownian motion with $H \in (1/2, 1)$. If $F \in L^2_{\phi_H}([0,T], \mathcal{L}_2(U,V))$ then*

$$\int_0^T F dB^H \tag{2.6}$$

is a V-valued, zero mean Gaussian random variable.

Remark 2.3.

1. *The stochastic integral* (2.6) *does not depend on the choice of U_1, where the cylindrical fractional Brownian motion B^H takes its values and does not depend on the choice of the basis for V.*
2. *The methods allow to define the integration for a more general Gaussian process with the covariance given by* (2.2) *using the function f instead of ϕ_H, where $f : [0,T] \to \mathbb{R}_+$ provided L^2_f from Definition* 2.2 *is a Hilbert space.*

The definition of stochastic integration for deterministic integrands is extended to a natural family of stochastic integrands. This extension uses the ideas of Malliavin calculus (e.g., [14]) as in [2] and [15]. For convenience some details in this framework are repeated.

Let U, V be separable Hilbert spaces and U', V' be the dual spaces (identify U and U' and V and V'). Therefore, if $X : U \to V$, then $X' : V \to U$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and B^H be a standard cylindrical fractional Brownian motion with $H \in (1/2, 1)$. Let $\mathcal{F}_t = \overline{\sigma(B^H_s : s \leq t)}$ and $\mathcal{F} = \mathcal{F}_1$. Fix $(\Omega, \mathcal{F}, P)$ and B^H. To define a stochastic integral with respect to B^H on a finite interval, let $T = 1$.

Initially a family of stochastic processes is introduced.

Definition 2.3. *Let $H \in (1/2, 1)$. The linear space $L^2_H([0,1], \mathcal{L}_2(U,V))$, often denoted L^2_H, is the family of $\mathcal{L}_2(U,V)$-valued generalized processes on $(\Omega, \mathcal{F}, P)$ such that*

i) *for each $\mathcal{L}_2(U,V)$-valued process X, the map $(s,\omega) \mapsto \langle X'(s,\omega)\psi, \varphi\rangle_U$ is $\mathcal{B}([0,1]) \otimes \mathcal{F}$ measurable for all $\varphi \in U$, $\psi \in V$.*

ii)

$$\mathbb{E} \int_0^1 \left(u_{1-2H}(s) | I^{H-1/2}_{0+} (u_{H-1/2} X)(s) |_{\mathcal{L}_2} \right)^2 ds < \infty. \tag{2.7}$$

Remark 2.4. *The norm of L^2_H can be expressed in terms of ϕ_H. Then L^2_H is the linear space of $\mathcal{L}_2(U,V)$-valued generalized processes such that*

$$\mathbb{E} \int_0^1 \int_0^1 \langle X(s), X(t) \rangle_{\mathcal{L}_2} \phi_H(s-t) ds dt < \infty, \tag{2.8}$$

which is the completion of the family of uniformly bounded processes $(X(t),\, t \in [0,1])$ with the inner product (2.8).

It is convenient to introduce a family of elementary random variables that are used in the construction of a stochastic integral.

Definition 2.4. *The linear space $\mathcal{S}$ is the family of smooth, cylindrical, V-valued random variables on $(\Omega, \mathcal{F}, P)$ such that if $F \in \mathcal{S}$, then it has the form*

$$F = \sum_{j=1}^{n} f_j \left(\int_0^1 \gamma_{1j} dB^H, \dots, \int_0^1 \gamma_{n_j j} dB^H \right) \eta_j \tag{2.9}$$

where $\eta_j \in V$, $\gamma_{kj} \in L^2_{\phi_H}([0,1], \mathcal{L}_2(U, \mathbb{R}))$, $f_j \in C_p^\infty(\mathbb{R}^{n_j})$ for $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, n_j\}$ and

$$C_p^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \in C^\infty \text{ and } f \text{ and all of its derivatives have polynomial growth}\}.$$

A derivative along the paths of a fractional Brownian motion plays an important role in the definition of the stochastic integral as it does for the Skorokhod integral of Brownian motion via Malliavin calculus.

Definition 2.5. *The derivative $D : \mathcal{S} \to L^2_H$ is a linear operator which is given for $F \in \mathcal{S}$ in (2.9) by*

$$D_t F = \sum_{j=1}^{n} \sum_{i=1}^{n_j} \frac{\partial f_j}{\partial x_i} \left(\int_0^1 \gamma_{1j} dB^H, \dots, \int_0^1 \gamma_{n_j j} dB^H \right) \eta_j \otimes \gamma_{ij}(t). \tag{2.10}$$

The following result is called an integration by parts formula which plays a basic role in the definition of the stochastic integral. The proof of this result and the proofs of the subsequent results are given in [5].

Theorem 2.1. *If $F, G \in \mathcal{S}$ and $\gamma \in L^2_{\phi_H}([0,1], \mathcal{L}_2(U, \mathbb{R}))$, then*

$$\langle G \otimes \gamma, DF \rangle_{L^2_H} = \mathbb{E}\left[\langle F, G \rangle_V \int_0^1 \gamma dB^H \right] - \langle F \otimes \gamma, DG \rangle_{L^2_H} \tag{2.11}$$

where L^2_H and D are given by Definition 2.3 and Definition 2.5, respectively.

Corollary 2.1. *The derivative operator $D : \mathcal{S} \to L^2_H$ given by (2.10) can be extended in $L^2(\Omega)$ to a closed operator, also denoted D, such that $D : D^{1,2}_H \to L^2_H$ where $D^{1,2}_H = Dom(D)$.*

The stochastic integral is defined as a dual to D.

Definition 2.6. *Let $X \in L^2_H$. The $\mathcal{L}_2(U, V)$-valued generalized process X is integrable with respect to B^H if $F \mapsto \langle X, DF \rangle_{L^2_H}$ is continuous on $\mathcal{S}$ with the $L^2(\Omega)$ norm topology. The stochastic integral $\int_0^1 X dB^H$ is a zero mean, V-valued random variable such that*

$$\langle X, DF \rangle_{L^2_H} = \mathbb{E} \left\langle \int X dB^H, F \right\rangle_V \tag{2.12}$$

for each $F \in \mathcal{S}$.

An explicit class of examples of the stochastic integral (2.12) is given for a family of elementary processes as integrands.

Proposition 2.2. *If $G \in D_H^{1,2}$ and $\gamma \in L^2_{\phi_H}([0,1], \mathcal{L}_2(U,\mathbb{R}))$, then the stochastic integral $\int_0^1 G \otimes \gamma dB^H$ is defined and the following equality is satisfied*

$$\int_0^1 G \otimes \gamma dB^H = G \int_0^1 \gamma dB^H - \int_0^1 \int_0^1 (D_s G)(\gamma(t)) \phi_H(s-t) ds dt \quad \text{a.s.} \tag{2.13}$$

Corollary 2.2. *If $G \in \mathcal{S}$ and $\gamma \in L^2_{\phi_H}([0,1], \mathcal{L}_2(U,\mathbb{R}))$ then $DG \in D_H^{1,2}$ and*

$$D_s \int_0^1 G \otimes \gamma dB^H = \int_0^1 D_s G \otimes \gamma dB^H + G \otimes \gamma(s) \tag{2.14}$$

for $s \in [0,1]$.

Now, a family of processes is defined which are integrable according to Definition 2.6 so that the stochastic integral is defined.

Definition 2.7. *Let $H \in (1/2, 1)$. The space $L_H^{1,2}([0,1], \mathcal{L}_2(U, D_H^{1,2}))$ or simply $L_H^{1,2}$ is the family of $\mathcal{L}_2(U, D_H^{1,2})$-valued generalized processes $(X(t),\, t \in [0,1])$ on $(\Omega, \mathcal{F}, P)$ such that*

i) $X \in L_H^2$,

ii) $X : [0,1] \times \Omega \to \mathcal{L}_2(U, D_H^{1,2})$ *is $\mathcal{B}([0,1]) \otimes \mathcal{F}$ measurable,*

iii) *There is a measurable version of $(D_s X(t),\, s,t \in [0,1])$, that is, the map $(s,t,\omega) \mapsto \langle (D_s X(t))\varphi, \psi \rangle$ is $\mathcal{B}([0,1]^2) \otimes \mathcal{F}$ measurable for each $\mathcal{L}_2(U, D_H^{1,2})$-valued process X and all $\varphi \in U$ and $\psi \in V$, and*

iv)

$$\begin{aligned} |X|^2_{L_H^{1,2}} = \langle X, X \rangle_{L_H^2} + \mathbb{E} \int_0^1 \int_0^1 \int_0^1 \int_0^1 & \langle D_p X(q), D_r X(s) \rangle_{\mathcal{L}_2} \\ & \times \phi_H(p-s) \phi_H(r-q) \\ & \times dp dq dr ds < \infty. \end{aligned} \tag{2.15}$$

The space $L_H^{1,2}$ is a Hilbert space that is the completion of uniformly bounded $\mathcal{L}_2(U, \mathcal{S})$-valued processes using the norm $|\cdot|_{L_H^{1,2}}$. It can be expressed in terms of fractional integrals and a representation of ϕ_H to obtain a Lebesgue integral description in analogy with L_H^2.

The following result verifies that the processes in $L_H^{1,2}$ are integrable and $L_H^{1,2}$ is a natural family of integrands because the stochastic integral satisfies an isometry.

Theorem 2.2. *If $X \in L_H^{1,2}([0,1], \mathcal{L}_2(U, D_H^{1,2}))$, then X is integrable with respect to B^H, so the stochastic integral $\int_0^1 X dB^H$ is a well defined zero mean V-valued random variable in $L^2(\Omega)$. Furthermore, if $X, Y \in L_H^{1,2}$, then*

$$\mathbb{E} \left\langle \int_0^1 X dB^H, \int_0^1 Y dB^H \right\rangle_V = \langle X, Y \rangle_{L_H^{1,2}}. \tag{2.16}$$

In applications of stochastic integration, a change of variables or Itô formula is especially useful. An Itô formula is given for the stochastic integration with a standard cylindrical fractional Brownian motion in a Hilbert space.

Let $t > 0$ be !ed and let "X"s#, $0 \leq s \leq t$# be a V\$valued process with continuous sample paths that satis es the stochastic e%uation

$$X"s\# \;\&\; X"0\# \;'\; \int_0^s a"r\#dr \;'\; \int_0^s b"r\#dB^H"r\# \quad \text{a.s.} \qquad "(.)*\#$$

for $s \in$ +$0, t$, where $X"0\# \in V$ is deterministic, the stochastic integral is de ned by "(.)-# and the V\$valued process "$a"s\#, 0 \leq s \leq t\#$ and the $\mathcal{L}_2"U, V\#$\$valued process "$b"s\#, 0 \leq s \leq t\#$ are $\sigma"B^H"u\#, 0 \leq u \leq t\#$ measurable and satisfy

$$\mathbb{E}\left[|a"r\#|_V^2 \;'\; |D_q a"r\#|_{\mathcal{L}_2}^2\right] \leq M \qquad "(.).\#$$

$$\mathbb{E}\left[|b"r\#|_{\mathcal{L}_2}^2 \;'\; |D_q b"r\#|_{\mathcal{L}_2}^2 \;'\; |D_p D_q b"r\#|_{\mathcal{L}_2}^2\right] \leq M \qquad "(.)/\#$$

where M is a constant that does not depend on $p, q, r \in$ +$0, t$, and $\mathcal{L}_2$ denotes the appropriate space of Hilbert\$0chmidt operators.

Let "$X"s\#, 0 \leq s \leq t\#$ be the process given in "(.)*# *with the assumptions there. Let F 1 $V \to V$ be a twice continuously differentiable function such that F' 1 $V \to \mathcal{L}_2"V, V\#$ and F'' 1 $V^{(2)} \to \mathcal{L}_1"V, V\#$ where F' and F'' are the first and the second derivatives respectively. Then the process* "$F"X"s\#\#, 0 \leq s \leq t\#$ *satisfies the stochastic equation*

$$\begin{aligned} F"X"s\#\# \;\&\; F"X"0\#\# \;'\; &\int_0^s F'"X"r\#\#a"r\#dr \;'\; \int_0^s F'"X"r\#\#b"r\#dB^H"r\# \\ ' &\int_0^s \int_0^t F''"X"p\#\# \int_0^p D_q a"r\#dr\, b"p\#\phi_H"p - q\#dqdp \\ ' &\int_0^s \int_0^t F''"X"p\#\# \int_0^p "D_q b"r\#\#dB^H"r\#b"p\#\phi_H"p - q\#dqdp \\ ' &\int_0^s \int_0^p F''"X"p\#\#b"q\#b"p\#\phi_H"p - q\#dqdp \quad \text{a.s.} \qquad "(.(0\# \end{aligned}$$

If the hypotheses of the theorem are satisfied and the pair of processes "$a"s\#, b"s\#, 0 \leq s \leq t\#$ *is adapted to* "$\sigma"B^H"u\#, 0 \leq u \leq s\#, 0 \leq s \leq t\#$*, then the process* "$F"X"s\#\#, 0 \leq s \leq t\#$ *satisfies*

$$\begin{aligned} F"X"s\#\# \;\&\; F"X"0\#\# \;'\; &\int_0^s F'"X"r\#\#a"r\#dr \;'\; \int_0^s F'"X"r\#\#b"r\#dB^H"r\# \\ ' &\int_0^s \int_0^s F''"X"p\#\# \int_0^p D_q a"r\#dr\, b"p\#\phi_H"p - q\#dqdp \\ ' &\int_0^s \int_0^s F''"X"p\#\# \int_0^p "D_q b"r\#\#dB^H"r\#b"p\#\phi_H"p - q\#dqdp \\ ' &\int_0^s \int_0^p F''"X"p\#\#b"q\#b"p\#\phi_H"p - q\#dqdp \quad \text{a.s.} \qquad "(.()\# \end{aligned}$$

The integrability assumptions (2.18) and (2.19) can be relaxed.

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