Relation between the fluid energy equation and the response of a hot-wire anemometer in time-dependent flows

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Abstract

The behavior of hot-wire and related anemometers working in a constant-temperature mode is analyzed from the point of view of the dynamics of the energy equation in the fluid. In general there is no unique functional relationship between the magnitude of the local flow velocity and the heat rate for the sensor, though approximations may be valid under certain conditions. Some suggestions of sufficiency conditions for extending static calibration to dynamic flows are given, and the question of their validity in practical cases is raised.

1 Introduction

Hot-wire anemometers have been used for a long time in constant temperature operation for instantaneous measurements of the local fluid velocity vector. Numerous textbooks on turbulence [1] (which includes a chapter on turbulence measurements) and on hot-wire anemometry [2–4] have been published, and advantages as well as disadvantages of the methods have been thoroughly discussed. Though usually used for turbulence measurements, the hot-wire has also been employed for other time-dependent velocity fields, as for example in acoustics [5, 6], vortex shedding [7], and oscillations due to vortex generators [8]. However, most standard texts, such as those mentioned above, and reviews [9, 10] describe the operation of hot-wire anemometers without reference to the energy equation in the fluid. Bradshaw [2] uses it only to derive the turbulent transport equation, and even when it is used and solved, as numerically by Bhatia et al. [11] in the context of near-wall corrections, the steady state version is employed. The classical approach is that the dynamics of hot-wires are taken to be in the electronics, in the wire itself or in its supports, but not in the fluid [10, 12–14]).

Recently, some attempts have been made to relate the dynamics of the flow to the anemometer response. Khoo et al. [15,16] have, based on findings by Freymuth [17], tried to couple the electronic circuitry to the heat transfer from the wire, and Kegerise and Spina [18] have studied the dynamic response for different circuitry. Still, the use of the fluid energy equation in dynamic studies of the anemometer response is missing, leaving some nontrivial unanswered questions as to what the hot-wire anemometer really measures. There are results which show that the Nusselt number for laminar, oscillatory flow around a cylinder depends on the frequency [19]. A similar result has also been obtained for a heated cylinder that is oscillated in a cross-flow [20–22].

2 Fluid energy equation

To formulate the problem we consider a hot wire (or some other thermal sensor such as a fluid-submerged or wall-mounted hot-film) with a flow around it. The incompressible energy equation for a fluid with constant properties and without viscous dissipation is

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \alpha \nabla^2 T = 0, \tag{1}$$

where $\mathbf{u}(\mathbf{x}, t)$ and $T(\mathbf{x}, t)$ are the velocity and temperature fields, respectively, in the neighborhood of the wire, and α is the fluid thermal diffusivity; $\mathbf{x}$ is the position vector, and

t is time. The temperature boundary conditions for this equation are: (a) $T = T_s$ at the surface of the sensor (assuming perfect feedback control by which the heating current is varied so that its temperature is maintained constant), and (b) $T = T_\infty$ in the fluid far from it (or at the walls of the wind tunnel or other facility where the experiment is being performed) if proper temperature stabilization is used during experimentation. An initial condition is also needed that is not generally available. Using a normalized temperature $\theta(\mathbf{x}, t) = (T - T_\infty)/\Delta T$, where $\Delta T = T_s - T_\infty$, we have

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta - \alpha \nabla^2 \theta = 0. \tag{2}$$

For this normalized temperature, the boundary conditions are $\theta = 1$ at the sensor surface and $\theta = 0$ far from it. To solve this boundary-value problem $\mathbf{u}$ must be prescribed everywhere. Let $\mathbf{U}$ be the velocity vector at the location of the sensor if it were not present (or, equivalently, several wire diameters from it), so that $U = |\mathbf{U}|$ is the quantity that we want to measure.

The heat rate from the sensor is given by

$$Q(t) = -k \; \Delta T \int_A (\nabla \theta)_s \cdot d\mathbf{A}, \tag{3}$$

where k is the thermal conductivity of the fluid, A is the surface of the sensor, $d\mathbf{A}$ is an element of the surface in a normal direction, and $\Delta T \; (\nabla \theta)_s$ is the normal temperature gradient at the surface. For a perfect constant-temperature hot wire, Q must be the heat generated by electrical heating, the value of which can be deduced from the instantaneous current which can be measured and the wire resistance which is also a constant. In fact the hot-wire anemometer is an inverse problem in the context of Eqs. (2) and (3): to find U if Q is given. Uniqueness issues should then be confronted (for instance, in an infinite fluid Q is invariant to a rotation in $\mathbf{u}$ leading, among other things, to insensitivity of the hot wire to flow direction) but they will not be addressed here.

3 Calibration

Q can be used as a measure of the instantaneous velocity if

$$F(Q) = U, \tag{4}$$

implying a one-to-one functional relationship between Q and U in an instantaneous sense. If this is true, we can then say that the instrument can be calibrated statically and used dynamically. If, however, the relation is of the form (or of related forms with higher derivatives on the left hand side)

$$\tau(Q)\frac{dQ}{dt} + F(Q) = U, \tag{5}$$

a dynamic calibration is needed to determine τ. Yet the hot wire can still be used after appropriate signal processing (a similar relation is normally taken to be valid for a constant-current anemometer in which the thermal dynamics of the wire plays an important role). However, Eqs. (2) and (3) show that in the present case this relation is

$$\frac{dQ}{dt} - k\Delta T \int_A \left[\nabla(\mathbf{u}\cdot\nabla\theta)\right]_s \cdot d\mathbf{A} + k\Delta T \int_A \left[\nabla(\alpha\nabla^2\theta)\right]_s \cdot d\mathbf{A} = 0, \tag{6}$$

which in general cannot be reduced to either Eq. (4) or (5).

Another way of looking at the problem is to say that if θ_1 and θ_2 are two solutions of Eq. (2) for different $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, then $\theta = \theta_1 + \theta_2$ is *not* the solution for $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. In other words the sensor may not, in general, be dynamically calibrated with respect to individual flow frequencies separately and the results used by superposition in a flow where both these frequencies are present. Approximations, however, are possible, but the conditions under which they are valid must be clearly determined.

There are two temperature length scales to be considered: one is the advective scale of the fluctuating motion we are trying to measure, and the other is caused by the interaction between the heated wire and the flow. In reasonably designed experiments, the latter is small and the flow at that scale is laminar leading to a tractable relationship between the flow speed and heat transfer. Temperature gradients associated with the former length scale are due to the non-linear interaction between the fluctuating thermal and velocity fields. This non-linear interaction is highly dynamic and it is unlikely that a calibration procedure can be devised to account for its effects. Fortunately, in many instances the fluctuation intensity is low, and hence we have reason to believe that this interaction is comparably weak. Let us consider the flow and temperature fields as the sum of a steady and a time-dependent part (assuming that such a decomposition is possible), i.e. that

$$\mathbf{u} \;=\; \overline{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}, t), \qquad \theta = \overline{\theta}(\mathbf{x}) + \theta'(\mathbf{x}, t). \tag{7}$$

If $|\mathbf{u}'|/|\overline{\mathbf{u}}|$ and $\theta'/\overline{\theta}$ are both of $O(\epsilon)$ *everywhere*, where ϵ is small, then from Eq. (2) we find that the only term of order $O(\epsilon^2)$ is $\mathbf{u}' \cdot \nabla\theta'$. Neglect of this term enables us to carry our analysis further. Unfortunately, non-linear terms can very quickly grow in size; the approximation may be acceptable for one fluctuation intensity, yet be very poor for a marginally higher intensity. Worse still, theory does not provide us with an error estimate from which a 'critical' intensity can be derived, and consequently we must resort to experimental or numerical data to obtain this information.

4 Velocity and temperature fluctuations

Though statements such as "[i]f the r.m.s. turbulent intensity exceeds about 0.3 of the mean velocity, interpretation of wire readings becomes increasingly uncertain" [2] are frequently found in the literature, it must be stressed that any information derived from experimental and numerical data is inherently case specific. Practical experience has suggested that a intensity of 0.3 may indeed be reasonable as a critical value for some flows, but for near-wall experiments [16,23] reasonable results have been obtained for a turbulence intensity of about 0.4. In acoustics [5], even though the intensity is infinite, the results are not unreasonable.

Possibly this is due to some special feature of the geometry at hand. For example, if $\mathbf{u}$ is nearly perpendicular to $\nabla\theta'$ the fluctuation intensity may be large, but the interaction between the fluctuating fields could be small.

We proceed with our analysis by considering terms of $O(\epsilon^0)$ and substitute Eq. (7) into (2) to get

$$\overline{\mathbf{u}} \cdot \nabla\overline{\theta} - \alpha\nabla^2\overline{\theta} = 0, \tag{8}$$

where $\overline{\theta}$ is 1 at the sensor and 0 far from it. The solution of this equation and obtaining Q from Eq. (3) give, in principle, the static calibration of the sensor, Eq. (4), where F is normally determined experimentally.

Furthermore, we may write

$$\overline{\theta}(\mathbf{x}) = \theta_c(\mathbf{x}) + \Theta(\mathbf{x}) \tag{9}$$

where θ_c is the solution of

$$\nabla^2\theta_c = 0, \tag{10}$$

with the boundary conditions that θ_c is 1 at the sensor surface and 0 far from it. θ_c is the conductive temperature field that would exist if the fluid were not in motion. For a hot-wire mounted far away from any wall, for instance, this temperature field has concentric isotherms and radial heat flux lines, provided we neglect end effects.

Substituting Eq. (9) in (8), we get

$$\mathcal{L}(\Theta) = -\overline{\mathbf{u}} \cdot \nabla\theta_c, \tag{11}$$

where $\mathcal{L} = \overline{\mathbf{u}} \cdot \nabla - \alpha\nabla^2$ is an elliptic partial differential operator with homogeneous Dirichlet boundary conditions for Θ. Typically the source term $-\overline{\mathbf{u}} \cdot \nabla\theta_c$ is positive upstream of the hot wire and negative downstream. Hence we can expect that the heat flux lines connect sources upstream with sinks downstream, which implies that they will be somewhat reminiscent of the velocity streamlines in the vicinity of the wire.

In any case we can represent the solution of Eq. (11) as

$$\Theta(\mathbf{x}) = -\int_V \left(\overline{\mathbf{u}} \cdot \nabla\theta_c\right)(\mathbf{x}') \;\; G(\mathbf{x}, \mathbf{x}') \; d\mathbf{x}', \tag{12}$$

where integration is over the volume of the flow V. G is the Green's function for this operator and geometry.

To $O(\epsilon)$, we have from Eq. (2) that

$$\frac{\partial\theta'}{\partial t} + \overline{\mathbf{u}} \cdot \nabla\theta' - \alpha\nabla^2\theta' = -\mathbf{u}' \cdot \nabla\overline{\theta}, \tag{13}$$

with homogeneous Dirichlet boundary conditions for θ' and with an appropriate initial condition. From the right hand side we can see that the effect of the fluctuating velocity acting on the mean temperature field is a source for the temperature fluctuation θ' and, through Eqs. (3), for the fluctuation in the heat rate. Thus one can also observe that θ' is insensitive to the components of $\mathbf{u}'$ that are orthogonal to $\nabla\overline{\theta}$.

Let us for the moment assume that the $\partial\theta'/\partial t$ term in Eq. (13) is small compared to the others in the equation, so that we have the approximation

$$\mathcal{L}(\theta') = -\mathbf{u}' \cdot \nabla\overline{\theta}, \tag{14}$$

for which an initial condition is no longer needed. The same operator $\mathcal{L}$ as in Eq. (11) appears on the left hand side, and consequently the solution can be written as

$$\theta'(\mathbf{x}, t) = -\int_V \left(\mathbf{u}' \cdot \nabla\overline{\theta}\right)(\mathbf{x}') \;\; G(\mathbf{x}, \mathbf{x}') \; d\mathbf{x}', \tag{15}$$

using the same Green's function. This relation leads us to a dynamic calibration procedure. The simplest is to use the same calibration curve that was used for $\overline{\mathbf{u}}$, Eq. (4). For this to be valid for the time-dependent $\mathbf{u}'$, however, we must have that the effect of $\overline{\mathbf{u}} \cdot \nabla\Theta$ or $\mathbf{u}' \cdot \nabla\Theta$ on Q be negligible. We note that for negligible fluctuation intensity and thermal inertia, the relative error in using the static calibration curve for time-dependent flows is given by the size of $\mathbf{u}' \cdot \nabla\Theta$ compared to $\mathbf{u}' \cdot \nabla\theta_c$.

From Eq. (11) it follows that the size of $\overline{\mathbf{u}} \cdot \nabla\Theta$ depends linearly on the mean velocity field, so that there should be some critical mean velocity for which the error in using the static calibration for time varying flows exceeds a given bound. Hopefully this critical velocity will be large, but there is currently no self-consistent way to establish this. In contrast, we may use the power consumption of the wire with no mean flow as a measure of the strength of the $\nabla\theta_c$ field, and the mean power consumption of the wire at the given mean velocity as a similar gage for the $\nabla(\theta_c + \Theta)$ field. King's law then suggests that for mean velocities of the order of 10 m/s, the $\nabla\Theta$ and $\nabla\theta_c$ fields are of comparable magnitudes. This is of great concern, but due to the very different geometric features of these fields, described briefly above, it is quite possible that the effect of the $\overline{\mathbf{u}} \cdot \nabla\Theta$ on the heat transfer from the wire is considerably less than that of $\overline{\mathbf{u}} \cdot \nabla\theta_c$; the issue still remains open.

5 Discussion and conclusions

The question then arises under what conditions the $\partial\theta'/\partial t$ term can be neglected in Eq. (13). Many users of hot-wire anemometers seem to feel that due to the "fast frequency response of constant-temperature hot-wire anemometers" the device responds to the instantaneous velocity [3]. However, the term in question represents not how fast the wire responds but how fast the fluid does. It is a common misconception (see, for example, Ref. [3]) that for low turbulence intensities we may neglect the $\partial\theta'/\partial t$-term. In fact, the negligibility of the thermal inertia of the fluid is in no way related to the fluctuation intensity, which is seen from the fact that all of the terms in Eq. (13) have the same dependence on the turbulence intensity. The advantage of low turbulence intensity lies, however, in diminishing the interaction between the fluctuating thermal and velocity fields, which is a far more complex issue than the linear thermal inertia. The size of the thermal inertia relative to the other terms in Eq. (13) is proportional to the frequency under consideration. An order of magnitude estimate of thermal inertia compared to the advective term tells us that its relative size is given by

fD/U, where f, D and U are the characteristic frequency, length (the wire diameter) and velocity, respectively. Due to the small wire diameters this is a small quantity except for very large frequencies.

We have reached several conclusions from this cursory look at the fluid energy equation. First, we have found that a hot-wire anemometer is locally insensitive to the components of $\mathbf{u}'$ perpendicular to $\nabla\overline{\theta}$. Furthermore, we have established that static calibration can be used for time-dependent flows provided the following three conditions are satisfied: the fluctuation intensity is low, the thermal inertia of the fluid is negligible, and the effect on the heat flux from the wire of $\overline{\mathbf{u}} \cdot \nabla\Theta$ or $\mathbf{u}' \cdot \nabla\Theta$ is negligibly small. Finally, it should be added that the sufficiency conditions listed above need not be necessary. In fact, at this point the question of what conditions are necessary is still open.

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