# A spatially non-local model for flow in porous media

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**Abstract.** A general mathematical model of steady-state transport driven by spatially non-local driving potential differences is proposed. The porous medium is considered to be a network of short-, medium- and long-range interstitial channels with a continuum of length scales, and the flow rate in each channel is assumed to be linear with the pressure difference between its ends. The flow rate in the model is a functional of the non-local driving pressure difference field. As special cases, the model reduces to familiar forms of transport equations that are commonly used. An important situation arises when the phenomenon is almost, but not quite, locally dependent. The one-dimensional form of the model discussed here can be extended to multiple dimensions, temporal non-locality, and to heat, mass and momentum transfer.

**Key words:** Non-local, non-Darcy

## Nomenclature

| | |
|---|---|
| $f(x', x)$ | material property of locations $x'$ and $x$ |
| $\mathbf{f} = \{f_{ji}\}$ | discrete matrix version of $f(x', x)$ |
| $k$ | material constant |
| $L$ | length of material |
| $\mathcal{L}$ | Riemann-Liouville fractional derivative defined in Eq. (10a) |

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| | |
|---|---|
| $n$ | number of tubes |
| $N$ | number of discrete intervals |
| $p$ | pressure |
| $q$ | flow rate |
| $\mathcal{R}$ | Weyl fractional derivative defined in Eq. (10b) |
| $t$ | time |
| $x$ | spatial coordinate |

*Greek symbols*

| | |
|---|---|
| $\Gamma$ | gamma function |
| $\delta$ | delta distribution |
| $\delta'$ | derivative of $\delta$ |
| $\epsilon$ | spatial scale of non-locality |
| $\delta_\epsilon$ | nascent delta distribution |
| $\delta'_\epsilon$ | derivative of $\delta_\epsilon$ |
| $\mu$, $\nu$ | size of regions |

## 1 Introduction

Constitutive equations for the transport of momentum, heat and mass are usually represented as a flux that is proportional to the gradient of a driving potential. Examples of the above are the Newtonian viscosity relation, Fourier's law and Fick's law, respectively. For physical reasons, however, it is possible sometimes that the quantity being transported be driven by a longer range effect. Radiative heat transfer between two surfaces, where the heat rate depends on the temperatures of possibly widely separated surfaces, is one example of this. Flow in a long channel is another in which the flow rate depends on the pressure difference over a finite distance, and a local pressure gradient cannot be externally imposed. Pressure-driven flow in porous media will be taken here as a specific example of transport so that one can relate to the physics of the problem. The objective of the present investigation is to devise a linear constitutive model for flow in porous media that is driven by *both* local and non-local pressure differences.

Usually a porous medium is thought of as a granular medium through which the fluid flows. One can imagine then a porous bed consisting of tiny grains of sand or small marbles, the spaces between them serving as micro-channels for the flow. Local pressure differences, of the order of the micro-channel length or the distance between the inter-granular spaces, drive the flow. Transport in this kind of medium has been traditionally modeled by Darcy's law where the implicit assumption is that the characteristic micro-channel length distance is small compared to the overall size of the porous medium. This correctly captures the phenomena that occur in porous media that are formed by, for example, grains of sand like in the original experiment of Darcy, where the length of the interstitial micro-channels are small enough to be negligible. The distribution of the micro-channels may be anisotropic or inhomogeneous, so that the permeability may be a tensor, a function of position, or both.

To this picture we would like to add long-range flow by including an embedded network of long tubes or macro-channels through which fluid can also flow. These can be, for example, aquifers and fissures in rocks or blood vessels in human and animal tissues. This situation occurs, for instance, in fractured aquifers [1] and the rocks that contain oil in the Mexican fields of Chicontepec [2]. Only a pressure difference between the ends of the channel will produce a flow, and a local pressure gradient will not. The network of channels can be simple or complicated in geometry, and interconnected or independent from each other. From this perspective, the entire porous medium can be thought of as a network of channels with short-, medium-, and long-distance connections. In this sense, short or long again refers to a comparison with the overall size of the porous medium. In general, the connection lengths in a given porous medium will cover a continuum of channel lengths. There will thus be differences in the lengths of the channels; some may be small compared to the length scale over which the pressure changes, while others may not. Fluid transport in a porous material consisting of both micro- and macro-channels cannot, in general, be modeled with a Darcy-type law, but needs a different model.

A constitutive model will be proposed for flow in a network with a distribution of connection lengths. One-dimensionality will be assumed since it has the advantage of simplifying the algebra so that the basic ideas can be understood from a straightforward, physical perspective, but the proposed model can be easily generalized to multiple dimensions.

The one-dimensional Darcy equation

$$q = -k\frac{\partial p}{\partial x} \tag{1}$$

linearly relates the volume flow rate per unit area $q(x)$ to the gradient of the pressure $p(x)$, where $x$ is a linear coordinate. The inhomogeneous material property $k(x)$ includes the permeability of the porous matrix, coming essentially from the topology of the micro-channel flow paths, and the effect of fluid viscosity. This equation allows calculation of the flow rate if the pressure field is known. Additional relations are needed, however, if the pressure field is not known: one needs to add the conservation of mass equation which gives a relation between the flow rate and the fluid density, and a constitutive relation between the pressure and the density of the fluid.

Many proposed alternatives to Darcy's law have the purpose of including nonlinear effects in Eq. (1). Although variants, such as the Forchheimer equation, are common, only the problem of linear transport will be addressed here. Modifications have also been proposed to include other driving forces: examples are the inclusion of capillary effects [3], and for unsaturated media [4, 5]. However, all these models only take into account transport due to local effects. In other words the flow at a given point in space depends only on the spatial derivative of the driving potential at that point.

There have been a few attempts to include non-local effects. An integral inflow-outflow relationship for a porous slab has been previously suggested [6], but this model is purely kinematic, does not describe the effect of a pressure gradient, and does not readily generalize to configurations other than what can be reduced to a slab. There is also a class of relations based on fractional-order derivatives that have been proposed, mostly for mass transfer, that involve non-local transport [7]. Fractional-order derivatives, it will be remembered, can be written in terms of integrals, and hence involve long-distance effects. These models, however, only represent non-locality with a specific network configuration pattern.

The non-local transport model presented here will be general and will include, as will be shown later, previous models as special cases. There has been much work reported on a similar integral model for an elastic solid with long-range forces [8–10].

## 2 Non-local transport theory

Consider a porous slab in which each pore can be connected with one or more pores, and the connections can be with immediate neighbors or with distant pores. This last property allows for the possibility of including *non-local* transport of fluid due to pressure differences. A network of interstitial channels of different lengths and cross-sectional areas can be responsible for local as well as non-local transport. The analysis will be one-dimensional in the sense that flow rate over the entire cross sectional area of the slab will be considered and the only spatial variation will be with respect to $x$. The pressure field, $p(x)$, is assumed to be given and the transport equation will relate the volume flow rate $q(x)$ to it.

In the steady state, the proposed non-local model is

$$q(x) = \int_{-\infty}^{\infty} f(x', x) \ \{p(x') - p(x)\} \ dx', \tag{2}$$

where the flow rate at $x$ is the sum of the flows rates in all the channels that lead to $x$ which are due to the pressure differences between all other points $x'$ and $x$. Linearity between the flow rate and the pressure difference is assumed. $f(x', x)$ is a flow conductivity that relates the driving pressure difference to the resulting flow. It is a material property that includes the effects of channel length and cross-sectional area of the connections between $x'$ and $x$, one of which is schematically indicated in Fig. 1, as well as the viscosity of the fluid. The model is essentially a functional that maps a pressure field to flow rate at a point.

Several special cases of $f(x', x)$ will be studied in the following two sections. Since the model is a linear functional, superpositions of these *elementary* flows is valid, and may be of practical interest in particular applications in which a linear combination may apply.

## 3 Special cases: local

If the proposed model is general enough, it should have Darcy's law as a special case, as will be demonstrated below. It will also be shown that Darcy's law can be generalized to include higher-order spatial derivatives of the pressure.

### *3.1 Darcy's law*

One way to approach locality is to define the integral kernel $f$ with a support that tends to zero. A natural way to do this is to introduce the $\delta$-distribution [11] (also called generalized or improper function) defined by

$$\int_{-\infty}^{\infty} \delta(x' - x) g(x', x)\ dx' \stackrel{\text{def}}{=} g(x', x)\Big|_{x'=x}.$$

What will be needed will be the derivatives of distributions which can also be defined. That of the $\delta$-distribution is the doublet distribution $\delta'(x' - x)$, where

$$\int_{-\infty}^{\infty} \delta'(x' - x) g(x', x)\ dx' \stackrel{\text{def}}{=} -\frac{\partial g}{\partial x'}\Big|_{x'=x}.$$

Higher-order derivatives are similarly defined. If

$$f(x', x) = k(x)\delta'(x' - x) \tag{3}$$

is substituted in Eq. (2), Eq. (1) is obtained. Thus Darcy's law is a special case of the proposed model.

### *3.2 Higher-order generalization*

If $p(x')$ is a sufficiently smooth function in $x'$, it can be expanded in a Taylor series around $x' = x$ to give

$$p(x') = p(x) + \frac{\partial p}{\partial x'}\Big|_{x'=x}(x'-x) + \frac{1}{2!}\,\frac{\partial^2 p}{\partial x'^2}\Big|_{x'=x}(x'-x)^2 + \ldots$$

Substituting in Eq. (2) gives

$$q = K_1 \frac{\partial p}{\partial x} + K_2 \frac{\partial^2 p}{\partial x^2} + \ldots \tag{4}$$

where

$$K_n(x) = k\,\frac{1}{n!}\int_{-\infty}^{\infty} f(x', x)\;(x'-x)^n\;dx', \quad n = 1, 2, \ldots.$$

are the moments of $f$. If $K_1 \neq 0$, and $K_n = 0$ for $n > 1$, Eq. (4) becomes Darcy's law, Eq. (1), otherwise it is a local generalization with higher-order derivatives of the pressure.

## 4 Special cases: non-local

Non-local effects occur when the flow is in a channel connecting points of the network that are widely separated physically. For this reason, the proposed non-local model includes flow in a single channel, which is the most trivial network possible, as a special case. There are also other non-local possibilities such as almost-local flow and fractional derivatives.

### *4.1 Almost local*

Quite frequently, transport may be *almost* though not quite local. For this, one can use *nascent* delta functions $\delta_\epsilon(x'-x)$ for which

$$\lim_{\epsilon \to 0^+} \delta_\epsilon(x'-x) = \delta(x'-x).$$

These functions can be $C^{\infty}$ like the Gaussian or Lorentz-Cauchy distributions, or non-differentiable like top-hat or triangle. For example, the Gaussian nascent delta function is

$$\delta_\epsilon(x'-x) = \left[\frac{1}{\epsilon\sqrt{\pi}}\right] \exp\left\{-\left(\frac{x'-x}{\epsilon}\right)^2\right\}. \tag{5}$$

Furthermore, the derivative of the Gaussian nascent delta distribution also tends to the derivative of the delta-distribution, so that

$$\lim_{\epsilon\to 0^+} \delta'_\epsilon(x'-x) = \delta'(x'-x).$$

where

$$\delta'_\epsilon(x'-x) = \left[-\frac{2(x'-x)}{\epsilon^3\sqrt{\pi}}\right] \exp\left\{-\left(\frac{x'-x}{\epsilon}\right)^2\right\}. \tag{6}$$

For a porous medium that is almost local, we can take

$$f(x',x) = k(x)\delta'_\epsilon(x'-x), \tag{7}$$

where $\epsilon$ is small but not zero, so that

$$q(x) = k(x) \int_{-\infty}^{\infty} \delta'_\epsilon(x'-x) \ \{p(x') - p(x)\} \ dx'. \tag{8}$$

In principle $\delta'_\epsilon(x'-x)$ can be any nascent doublet function as long as the flow is everywhere from a higher pressure to a lower one (there are some, like the sinc function, that do not satisfy this requirement). The length scale $\epsilon$ characterizes the range or distance to which non-local effects are felt. It will decrease as $\epsilon \to 0^+$, and the transport equation will become Darcy and completely local in the limit, as shown before. In non-dimensional terms, if $L$ is the length scale of the pressure variation, then Darcy's law is recovered as $\epsilon/L \to 0^+$. To be specific, if we define $L = (\partial p/\partial x)/(\partial^2 p/\partial^2 x)$, then $L \to \infty$ as $\partial^2 p/\partial^2 x \to 0$, so that Eq. (4) simplifies to Eq. (1). Thus, non-local effects will be significant if the length scale $L$ is not much larger than the range of the non-locality $\epsilon$, and it is important to use Eq. (8), instead of Eq. (1), for situations in which the range of the non-local transport is not small compared to the length scale of the pressure gradient.

### *4.2 Strongly non-local: two-point flow*

The proposed model includes laminar flow in a channel connecting two different regions in space. If

$$f(x', x) = k\{\delta_\mu(x', x_1) - \delta_\nu(x, x_2)\}\{\delta(x', x_1) - \delta(x, x_2)\}, \tag{9}$$

where $\delta_\mu$ and $\delta_\nu$ are defined by Eq. (5), $\mu$ and $\nu$ being sufficiently small, then there is flow between the two regions $x_1 - \mu \leq x \leq x_1 + \mu$ and $x_2 - \nu \leq x \leq x_2 + \nu$ given by

$$q(x) = \begin{cases} k\ \{p(x_2) - p(x_1)\} & \text{if } x_1 - \mu \leq x \leq x_1 + \mu, \\ k\ \{p(x_1) - p(x_2)\} & \text{if } x_2 - \nu \leq x \leq x_2 + \nu, \\ 0 & \text{otherwise.} \end{cases}$$

The inflow and outflow regions will become points if $\mu$ and $\nu$ tend to zero. Eq. (2) can thus be used to model long-distance flow channels.

### *4.3 Power-law non-locality: fractional derivative*

If $f(x', x) = k\,(x - x')^{\alpha-1}$, then Eq. (2) can be written as

$$q(x) = k\Gamma(\alpha)\ \left[\mathcal{L}_x^\alpha + \mathcal{R}_x^\alpha\right]\{p(x') - p(x)\},$$

where the Riemann-Liouville and the Weyl operators [12] are defined as

$$\mathcal{L}_x^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - x')^{\alpha-1}\ g(x')\ dx', \tag{10a}$$

$$\mathcal{R}_x^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (x - x')^{\alpha-1}\ g(x')\ dx', \tag{10b}$$

respectively. This is similar to the form proposed by Schumer et al. [7], in which a random walk derivation is given of an advection-diffusion equation with fractional spatial derivatives by postulating a probability of the walk step size that decreases as a power law with distance. In the perspective of the present network model of the porous medium, the effect of the connections would be such that the flow decreases in a power-law fashion with separation. Connectivity of the flow network is one of the ways in which it is possible to realize this physically, since there are scale-free networks for which the degree distribution follows a power law [13]; the distribution of channel cross-sectional areas is another. The fractional-order flux model is thus also a special case of Eq. (2).

## 5 Determination of a general $f(x', x)$

Suppose there is a porous medium in the interval $x \in [a, b]$ on which the material property $f(x', x)$ is to be measured. An experiment has to be set up in which the pressure distribution applied along the line and the resulting flow can be measured at every point on the line. To not deal with an infinity of points, it is convenient to discretize $f(x', x)$ in the following way. The interval is divided into $N$ equal parts, each of size $\Delta x = (b - a)/N$, so that the discrete equivalent of Eq. (2) is

$$q_i = \sum_{j=1}^{N+1} f_{ji} \left( p_j - p_i \right) \ \Delta x. \tag{11}$$

The matrix $\mathbf{f} = \{f_{ij}\}$ is the discrete equivalent of $f(x', x)$ in Eq. (2). The intervals $i$ and $j$ are shown in Fig. 2. The diagonal elements $f_{ii}$ of the matrix $\mathbf{f}$ are zero since there is no driving difference of pressure between an interval and itself. Furthermore we can assume that $\mathbf{f}$ is symmetric, which is equivalent to saying that the same pressure difference between $i$ and $j$ produces the same flow rate independent of which is larger. Thus there are $N(N - 1)/2$ independent upper triangular components in $\mathbf{f}$.

In a single experimental run, a given pressure distribution and the resulting flow rate at each interval is measured. Measurement of internal pressures in a liquid-saturated porous medium, which is needed here, is possible in principle using manometers. Substituting in Eq. (11), $N$ equations are obtained. Thus, at least $(N-1)/2$ number of runs with *different* pressure distributions must be made to provide $N(N-1)/2$ number of independent equations from which the matrix $\mathbf{f}$ can be calculated.

## 6 Determination of almost local $f(x', x)$

Consider a slab in $x \in [0, L]$, as shown in Fig. 3, through which there is a pressure driven flow. Assuming that the medium is almost local, the flow rate using Eqs. (6) and (8) can be written as

$$q = k \int_0^L \left[ -\frac{2(x'-x)}{\epsilon^3 \sqrt{\pi}} \right] \exp\left\{ -\left( \frac{x'-x}{\epsilon} \right)^2 \right\} \{p(x') - p(x)\} \; dx'. \tag{12}$$

For an incompressible fluid, $q$ will be constant throughout the medium, and it can also be assumed also that $k$ is constant. The discrete version of this equation is

$$q = k \sum_{j=1}^{N+1} \left[ -\frac{2(x_j - x_i)}{\epsilon^3 \sqrt{\pi}} \right] \exp\left\{ -\left( \frac{x_j - x_i}{\epsilon} \right)^2 \right\} \{p_j - p_i\} \; \Delta x. \tag{13}$$

Measuring $q$, $p_1$, $p_2$, ..., $p_{N+1}$ in several runs, as in Section 5, sufficient algebraic equations can be generated for the unknowns $k$ and $\epsilon$ to be evaluated in a least-square sense.

## 7 Superposition of simple flows

As an example, suppose that there is a porous slab that has fine grains as well as $n$ tubes of known locations embedded within, as schematically shown in Fig. 4. The fine grains can be considered to be a Darcy medium with $f_0(x', x)$ given by Eq. (3), and each one of the tubes represented by $f_i(x', x), \; i = 1, \ldots, n$, where the $f_i(x', x)$ are given by Eq. (9). The combined effect of the grains and the tubes is a sum

$$f = \sum_{i=0}^{n} f_i.$$

## 8 Conclusions

Darcy's law is a local model for flow in porous media which assumes proportionality between the fluid flow rate and the local pressure gradient. A more general model, Eq. (2), is proposed here which allows for the possibility of non-local transport. The material property is a function $f(x', x)$ relating two points $x'$ and $x$. The porous medium is assumed to consist of a network of interconnected channels with a continuum of lengths, and the model is a functional of the pressure difference field. This model includes Darcy's law and flow in a single channel as special cases. Another special case is almost local flow, which is a perturbation of Darcy's law, in which flow at a point is due to interstitial channels that are not only infinitesimal in length but also moderately large. Since the various $f$s discussed here can be superposed, different short- and long-range effects can be simultaneously considered. Techniques of measurement of material properties using flow rate and pressure distribution data have also been indicated.

Experiments have to be carried out to confirm the predictions of the model. For example, the flow in a granular medium with embedded tubes driven by an external pressure difference should have an internal pressure distribution that is not linear. Further work on the model includes time-dependence, which has not be considered here, but which must be taken into account for the dynamics of flow in porous media. Eq. (2) must then be suitably modified to take into account the inertia of the fluid as it accelerates in the channels. Equations with fractional-order time derivatives for the concentration in porous media have been previously postulated [14, 15].

Similar models can also be proposed for other transport phenomena such as momentum, mass and heat transfer, for each replacing the corresponding constitutive relation by an equation of the form of Eq. (2), but with the appropriate flux and driving potential. Non-locality becomes especially important at length scales that are small compared to the size of the medium where the non-bulk nature of the transport becomes significant. The physics of the transport will, of course, be different in each case, and the mechanisms that enable a network representation and non-local transport in each situation would, of course, be different.

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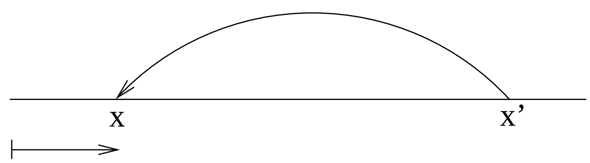


Figure 1: Representation of flow on a line.

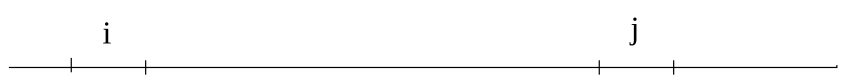


Figure 2: Discrete representation.

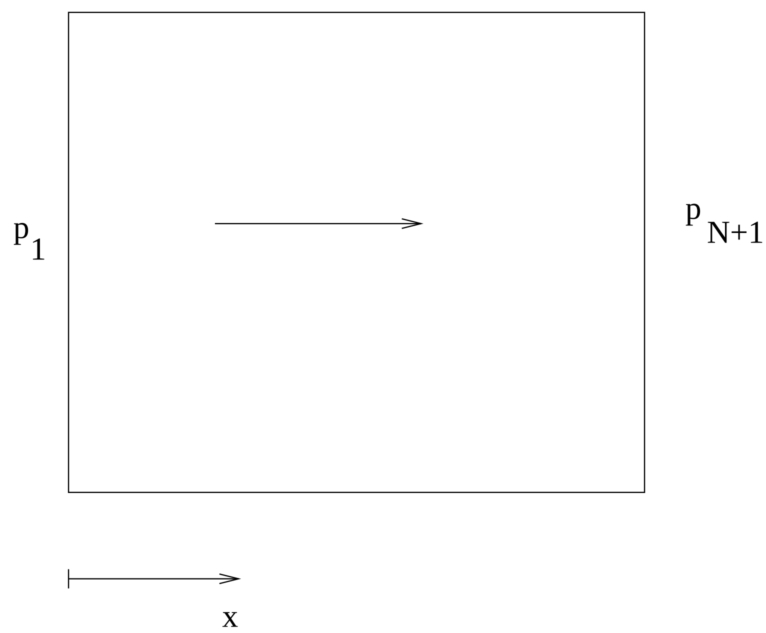


Figure 3: Slab with almost local $f(x', x)$.

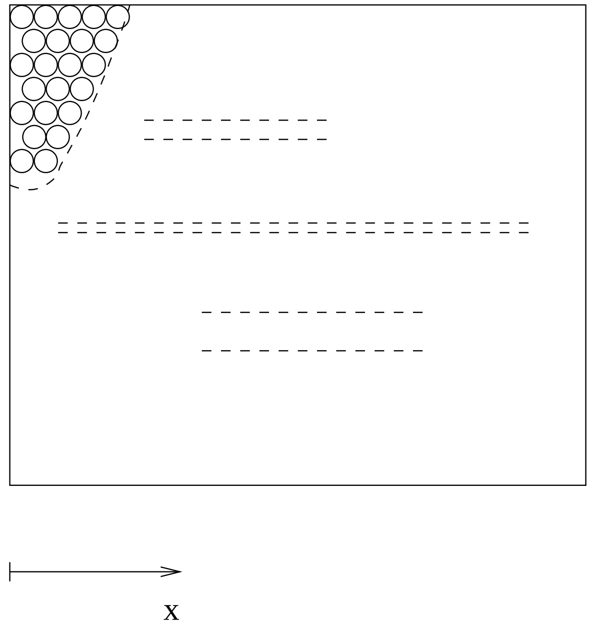


Figure 4: Superposition of grains and tubes.