Heat transfer in a toroidal natural convection loop

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Fig. 1 shows the convection loop to be analyzed. The working fluid is contained inside a pipe of diameter D so that the perimeter of a cross section is $P = \pi D$, and the transverse area is $A = \pi D^2/4$. The pipe is bent in the form of a torus of mean radius R, the total length of the loop being $L = 2\pi R$. Positions are measured using the angle θ counter-clockwise from the horizontal as shown. The portion of the loop $0 \leq \theta < \pi$ is cooled and that in $\pi \leq \theta < 2\pi$ is heated. On being heated the fluid density decreases due to change in temperature, and there is a buoyancy force that causes it to rise. The positive flow direction is again counter-clockwise.

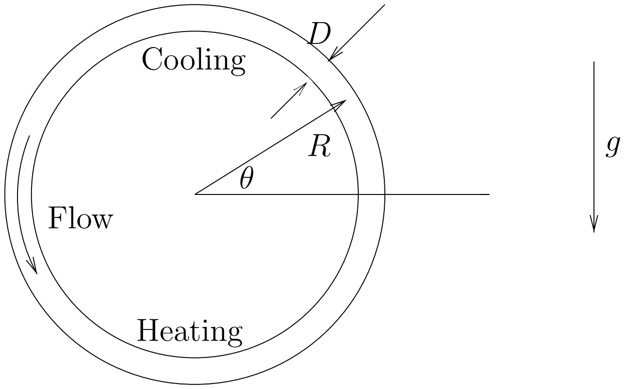

Figure 1: Schematic of toroidal loop.

We will analyze the steady-state governing equations using a one-dimensional approximation. At any position, the sum of the viscous, pressure and gravity forces per unit length must balance, so that

$$P\tau_w + \frac{A}{R}\frac{dp}{d\theta} + \rho' A g \cos\theta = 0, \tag{1}$$

where τ_w is the wall-shear stress, $p(\theta)$ is the pressure, ρ' is the temperature-dependent fluid density, and g the acceleration due to gravity. We will assume the wall shear stress to be proportional to the fluid velocity, i.e. $\tau_w = \alpha u$, where u is the mean velocity; for simplicity we will assume the relation for paraboloidal velocity distributions $\alpha = 8\mu/D$, where μ is the coefficient of viscosity. Taking the fluid density to be linear with temperature, we have $\rho' = \rho\left[1 - \beta(T - T_0)\right]$, where $T(\theta)$ is the temperature, β is the coefficient of thermal expansion, and ρ is the density at a reference temperature T_0. Thus

$$P\alpha u + \frac{A}{R}\frac{dp}{d\theta} + \rho\left[1 - \beta(T - T_0)\right] A g \cos\theta = 0. \tag{2}$$

Integrating around the loop, we get

$$u = a \int_0^{2\pi} T \cos\theta \; d\theta, \tag{3}$$

where

$$a = \frac{\rho D^2 \beta g}{64\pi\mu}. \tag{4}$$

An energy balance per unit length gives

$$\rho c A \; \frac{u}{R} \; \frac{dT}{d\theta} = hP(T_w - T), \tag{5}$$

where c is the specific heat at constant pressure, h is the convective heat transfer coefficient between the fluid and wall (assumed constant), and $T_w(\theta)$ is the known wall temperature. Conduction in the axial direction has been neglected. For a sinusoidal wall temperature corresponding to a heated lower half and a cooled upper half, we can write

$$T_w = T_0 - \Delta T \sin\theta, \tag{6}$$

so that

$$\frac{dT}{d\theta} + bT = b \; \left(T_0 - \Delta T \sin\theta\right), \tag{7}$$

where

$$b = \frac{2hL}{\pi D \rho c u}. \tag{8}$$

Using the condition $T(0) = T(2\pi)$, the solution is

$$T = T_0 + \frac{b \Delta T}{1 + b^2} \left(\cos\theta - b \sin\theta\right). \tag{9}$$

Substituting the temperature field in eq. (3), we get

$$u = \frac{\pi a b \Delta T}{1 + b^2}. \tag{10}$$

which on combining with eq. (8) gives the flow velocity

$$u = \pm \frac{hL}{\pi \rho c D} \; \left(\frac{\pi c \rho^2 D^3 g \beta \Delta T}{hL\mu} - 1 \right)^{1/2}. \tag{11}$$

For $\pi c \rho^2 D^3 g \beta \Delta T > hL\mu$, there are two motions possible, one counter-clockwise and another clockwise; otherwise there is none.

The heat rate over the entire loop is of course zero. In fact

$$\begin{aligned} Q &= R \int_0^{2\pi} hP(T_w - T) \; d\theta \\ &= 0. \end{aligned}$$

The quantity of interest is, however, Q_t which is the heat transferred from the heated wall of the loop to the fluid or, equivalently, from the fluid to the cooled wall. This can be calculated as

$$\begin{aligned}
Q_t &= -R\int_0^\pi hP(T_w - T)\;d\theta \\
&= \frac{hDL\Delta T}{2(1+b^2)} \\
&= \frac{hLD\Delta T}{2}\left(1 - \frac{128hL\mu}{\pi c\rho^2 D^3 \beta g \Delta T}\right).
\end{aligned}$$