

# Nonlinear analysis of transient flow in a piping network

Weihua Cai\* and Mihir Sen†

*Department of Aerospace and Mechanical Engineering  
University of Notre Dame, Notre Dame, IN 46556*

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## Abstract

Mathematical models of piping networks are in the form of a system of nonlinear differential-algebraic equations. Numerical solutions of transients are very time-consuming so that approximations are highly desirable. Different approximation methods are used for these equations and the results compared. Of these, regular perturbation and  $\delta$ -perturbation are valid only for small time. By constructing the homotopy deformation of the algebraic equation, the equations are reduced to a dynamical system which is more tractable. If the auxiliary parameters and functions are properly chosen only three terms of the homotopy analysis method are needed to give a good approximation.

**Keywords** piping networks, perturbation,  $\delta$ -perturbation, homotopy analysis method, nonlinear DAEs

## 1 Introduction

Piping networks are fundamental parts of building climate control, industry processing, chilled water systems and electrical equipment cooling. In general, a piping network consists of pumps, pipes, control valves, and terminal equipments such as heat exchangers, etc. The basic hydraulic equations can be expressed in terms of unknown flow rates and pressure

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\*Currently at Caterpillar, Peoria, IL 61629, e-mail: wcai123@gmail.com

†Corresponding author, e-mail: Mihir.Sen.1@nd.edu, tel. (574) 631-5975

differences. These equations combined with continuity equations constitute a set of nonlinear differential-algebraic equations (NDAEs) that cannot be analytically solved, but must be numerically handled. There are four kinds of methods to obtain steady state solutions: Hardy Cross [1], Newton-Raphson Martin and Peters [2], linear theory Wood and Charles [3], and gradient Todini and Pilati [4] methods. Approximate solution using the perturbation method was also developed by BashaBasha and Kassab [5].

Though the steady state is important, fluid transients are crucial in the startup, shut-down, and in the control of flow in piping networks, and also needs to be solved numerically. However, time-dependent computations for a large network may be very time consuming. Approximations are helpful in the sense that they give a an overall view of the dynamic behavior of the network, and hence may be easily incorporated into a design or control procedure. They are especially useful in flows where changes occur gradually, or if the pipelines are very short Holloway et al. [6], Islam and Chaudhry [7].

There are many methods of approximation that are used for the solution of equations. The most widely used are perturbation methods Dyke [8], Berk and Pfirsch [9], Hyouguchi et al. [10], Chen et al. [11], Holmes [12] which rely on the existence of a small parameter in the problem. Though such a situation is common in applications, the method is inapplicable when a small parameter does not exist. There are alternatives to this such as the  $\delta$ -expansion Basha and Kassab [5], Bender et al. [13], Amore and Aranda [14], and homotopy analysis Kalaba and Tesfatsion [15], Liao [16, 17], Liao and Cheung [18], Li and Liao [19] methods. Recently, the combination of the homotopy technique and perturbation methods has also been proposed He [20, 21, 22].

In this paper, a mathematical model is developed to describe the transient flow in piping networks. The homotopy analysis method is applied to the NDAEs that transforms them into a system of nonlinear differential equations. The advantage is that this is easier to solve numerically, and gives further physical understanding of flow dynamics in piping networks.

## 2 Mathematical model

Analysis of pipe flow is usually based on rigid water-column theory which assumes that the fluid is incompressible and the pipe is rigid. Assuming high Reynolds number flow for fully developed, the one-dimensional momentum equation for flow in a single pipe with constant pressure difference is

$$\frac{dQ^*}{dt^*} + \alpha^* Q^{*n} = \beta^* \Delta p^*. \quad (1)$$

where  $Q^*$  is the volume flow rate,  $t^*$  is time,  $\Delta p^*$  is the driving pressure difference,  $\beta^* = A/\rho L$ ,  $A$  is the cross-sectional area,  $\rho$  is the fluid density, and  $L$  is the length of the pipe. The parameter  $\alpha^*$  incorporates loss coefficients for the pipe as well as the fittings. As valve settings change, so does  $\alpha^*$ , leading to a change in  $Q^*$ .

Fig. 1 shows a very simple piping network with one primary loop and two parallel branches. This is a typical arrangement that is commonly seen in piping networks. Three heat exchangers, representing typical components, are located on the primary loop and the branches as shown. Thus for each branch we have  $\alpha_i^*$ ,  $\beta_i^*$ , and  $Q_i^*$ , where  $i = a, b, c$ , corresponding to the three branches  $A, B, C$ , respectively. The total inlet-to-outlet pressure difference  $\Delta p_t^*$  is taken to be constant. The pressure drops are  $\Delta p^*$  over branches  $A$  and  $B$ , and  $\Delta p_t^* - \Delta p^*$  over  $C$ .

Therefore, the governing equations are

$$\frac{dQ_a^*}{dt^*} + \alpha_a^* Q_a^{*n} = \beta_a^* \Delta p^* , \quad (2a)$$

$$\frac{dQ_b^*}{dt^*} + \alpha_b^* Q_b^{*n} = \beta_b^* \Delta p^* , \quad (2b)$$

$$\frac{dQ_c^*}{dt^*} + \alpha_c^* Q_c^{*n} = \beta_c^* (\Delta p_t^* - \Delta p^*) , \quad (2c)$$

$$Q_c^* = Q_a^* + Q_b^* , \quad (2d)$$

subject to initial conditions on  $Q_i^*(0)$  that satisfy Eq. (2d). These can be written nondimen-

sionally as

$$\frac{dQ_a}{dt} + \alpha_a Q_a^n = \beta_a \Delta p , \quad (3a)$$

$$\frac{dQ_b}{dt} + \alpha_b Q_b^n = \beta_b \Delta p , \quad (3b)$$

$$\frac{dQ_c}{dt} + \alpha_c Q_c^n = \beta_c (\Delta p_t - \Delta p) , \quad (3c)$$

$$c_c Q_c = c_a Q_a + c_b Q_b , \quad (3d)$$

where  $Q_i = Q_i^*/Q_i^*(0)$ ,  $t = t^* \beta_a^* \Delta p^*(0)/Q_a^*(0)$ ,  $\Delta p_t = \Delta p_t^*/\Delta p^*(0)$ ,  $\Delta p = \Delta p^*/\Delta p^*(0)$ ,  $\alpha_i = \alpha_i^* Q_a^*(0) Q_i^*(0)^{n-1}/\beta_a^* \Delta p^*(0)$ ,  $\beta_i = \beta_i^* Q_a^*(0)/\beta_a^* Q_i^*(0)$ ,  $c_i = Q_i^*(0)/Q_a^*(0)$ .

Eqs. (3a)–(3d) are NDAEs which can be easily transformed into a dynamical system.

After substituting Eq. (3d) into Eqs. (3a)–(3c) we have

$$\Delta p = \left( \sum_{i=a,b,c} c_i \beta_i \right)^{-1} \left[ c_c \beta_c \Delta p + \sum_{k=a,b} c_k \alpha_k Q_k^n - c_c^{1-n} \alpha_c \left( \sum_{k=a,b} c_k Q_k \right)^n \right]. \quad (4)$$

Substituting Eq. (4) into Eqs. (3a) and (3b), we get

$$\frac{dQ_a}{dt} = -\alpha_a Q_a^n + \beta'_a \left[ c_c \beta_c \Delta p + \sum_{k=a,b} c_k \alpha_k Q_k^n - c_c^{1-n} \alpha_c \left( \sum_{k=a,b} c_k Q_k \right)^n \right], \quad (5a)$$

$$\frac{dQ_b}{dt} = -\alpha_b Q_b^n + \beta'_b \left[ c_c \beta_c \Delta p + \sum_{k=a,b} c_k \alpha_k Q_k^n - c_c^{1-n} \alpha_c \left( \sum_{k=a,b} c_k Q_k \right)^n \right], \quad (5b)$$

where  $\beta'_k = c_k \beta_k / \sum_{i=a,b,c} c_i \beta_i$ .

### 3 Exact solution for $n = 1$

Eqs. (5) can be written in matrix form as

$$\frac{d\mathbf{Q}}{dt} = \mathbf{A}\mathbf{Q} + \mathbf{B} , \quad (6)$$

where

$$\mathbf{Q} = \begin{pmatrix} Q_a \\ Q_b \end{pmatrix} , \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} ,$$

in which  $a_{11} = \beta'_a(c_a\alpha_a - c_c^{1-n}c_a\alpha_c) - \alpha_a$ ,  $a_{22} = \beta'_b(c_b\alpha_b - c_c^{1-n}c_b\alpha_c) - \alpha_b$ ,  $a_{12} = \beta'_a(c_b\alpha_b - c_c^{1-n}c_b\alpha_c)$ ,  $a_{21} = \beta'_b(c_a\alpha_a - c_c^{1-n}c_a\alpha_c)$ ,  $b_1 = \beta'_a c_c \beta_c \Delta p$  and  $b_2 = \beta'_b c_c \beta_c \Delta p$ . The solution is

$$\mathbf{Q} = e^{\mathbf{A}t}\mathbf{C} + \mathbf{G} , \quad (7)$$

where  $\mathbf{G} = -\mathbf{A}^{-1}\mathbf{B}$ , and  $\mathbf{C} = \mathbf{Q}(0) - \mathbf{G}$ , with  $\mathbf{Q}(0)$  being the initial condition.

## 4 Nonlinear approximations

For  $n \neq 1$ , Eqs. (5) cannot be solved exactly. The following approximate methods will be used.

### 4.1 Regular perturbation

For simplicity, we assume that  $c_a\beta_a = c_b\beta_b$  and identify a small parameter  $\epsilon = c_a\beta_a / \sum_{i=a,b,c} c_i\beta_i \ll 1$ .

1. We write a power series of the form

$$Q_k = Q_{k,0} + \epsilon Q_{k,1} + \epsilon^2 Q_{k,2} + \dots , \quad (8)$$

where  $k = a, b$ .

Collecting  $O(1)$  terms, we have

$$\frac{dQ_{a,0}}{dt} = -\alpha_a Q_{a,0}^n , \quad (9a)$$

$$\frac{dQ_{b,0}}{dt} = -\alpha_b Q_{b,0}^n , \quad (9b)$$

with initial conditions  $Q_{a,0}(0) = 1, Q_{b,0}(0) = 1$ . The solutions can be easily obtained as

$$Q_{a,0} = [(n-1)\alpha_a t + 1]^{1/(1-n)} , \quad (10a)$$

$$Q_{b,0} = [(n-1)\alpha_b t + 1]^{1/(1-n)} . \quad (10b)$$

For  $O(\epsilon)$ ,

$$\begin{aligned} \frac{dQ_{a,1}}{dt} &= -n\alpha_a Q_{a,0}Q_{a,1}^{n-1} + c_a\alpha_a Q_{a,0}^n + c_b\alpha_b Q_{b,0}^n \\ &\quad - c_c^{1-n}\alpha_c(c_a Q_{a,0} + c_b Q_{b,0})^n + c_c\beta_c\Delta p , \end{aligned} \quad (11a)$$

$$\begin{aligned} \frac{dQ_{b,1}}{dt} &= -n\alpha_b Q_{b,0}Q_{b,1}^{n-1} + c_a\alpha_a Q_{a,0}^n + c_b\alpha_b Q_{b,0}^n \\ &\quad - c_c^{1-n}\alpha_c(c_a Q_{a,0} + c_b Q_{b,0})^n + c_c\beta_c\Delta p , \end{aligned} \quad (11b)$$

with initial conditions  $Q_{a,1}(0) = 0, Q_{b,1}(0) = 0$ . These have to be solved numerically.

## 4.2 $\delta$ -perturbation

Section 3 has the solution for  $n = 1$  which can be perturbed Basha and Kassab [5], Bender et al. [13], Amore and Aranda [14]. Assuming  $n = 1 + \delta$ , where  $\delta \ll 1$ , we can expand  $Q$  as a power series in  $\delta$

$$\mathbf{Q} = \mathbf{Q}_0 + \delta\mathbf{Q}_1 + \dots \quad (12)$$

The solution  $\mathbf{Q}_0$  to  $O(1)$  is given in Eq. (7). To  $O(\delta)$ , we have

$$\frac{d\mathbf{Q}_1}{dt} = \mathbf{A}\mathbf{Q}_1 + \mathbf{D} , \quad (13)$$

where the elements of  $\mathbf{D}$  are  $d_k$  given by

$$\begin{aligned} d_k &= \beta'_k [c_k\alpha_k Q_{k,0} \ln Q_{a,0} + c_b\alpha_b Q_{b,0} \ln Q_{b,0} - c_c^{1-n}\alpha_c(c_a Q_{a,0} + c_b Q_{b,0}) \ln(c_a Q_{a,0} + c_b Q_{b,0})] \\ &\quad - \alpha_k Q_{k,0} \ln Q_{k,0} , \end{aligned} \quad (14)$$

where  $k = a$  and  $b$ . Again, this has to be solved numerically.

## 4.3 Homotopy analysis method

To apply this we first construct the *zeroth-order deformation* equations Liao [16, 17], Liao and Cheung [18]

$$(1 - p)\mathcal{L}_i - p h_i H_i(t)\mathcal{N}_i = 0, \quad (15)$$

where  $i = a, b, c, d$  corresponding to each one of the four Eqs. (3a)–(3d).  $h_i$  and  $H_i(t)$  are nonzero auxiliary parameters and functions, respectively, that have to be chosen.  $\mathcal{L}_i$  are auxiliary linear operators that should satisfy  $\mathcal{L}_i(0) = 0$ . We choose

$$\mathcal{L}_{a,b,c} = \left( \frac{d}{dt} + g_i \right) [\phi_{a,b,c} - Q_{a,b,c,0}] , \quad (16a)$$

$$\mathcal{L}_d = \left( \frac{d}{dt} + g_d \right) [\phi_d - \Delta p_0] , \quad (16b)$$

where  $g_i$  are parameters, and  $\phi_i(t, p)$  are homotopy deformations of the unknowns  $Q_a(t)$ ,  $Q_b(t)$ ,  $Q_c(t)$ ,  $\Delta p(t)$ ;  $Q_{a,0}(t)$ ,  $Q_{b,0}(t)$ ,  $Q_{c,0}(t)$ ,  $\Delta p_0(t)$  are initial guesses for the unknowns subject to unit initial condition. The nonlinear operators  $\mathcal{N}_i$  are defined by

$$\mathcal{N}_a = \frac{d\phi_a}{dt} + \alpha_a \phi_a^n - \beta_a \phi_d , \quad (17a)$$

$$\mathcal{N}_b = \frac{d\phi_b}{dt} + \alpha_b \phi_b^n - \beta_b \phi_d , \quad (17b)$$

$$\mathcal{N}_c = \frac{d\phi_c}{dt} + \alpha_c \phi_c^n - \beta_c \Delta p + \beta_c \phi_d , \quad (17c)$$

$$\mathcal{N}_d = c_a \phi_a + c_b \phi_b - c_c \phi_c + (1 - p) . \quad (17d)$$

The homotopy lies in the gradual change in the left hand side operator in Eq. (15) from  $\mathcal{L}_i$  to  $\mathcal{N}_i$  as the embedding parameter  $p$  goes from 0 to 1. When  $p = 0$ , the solutions of Eq. (15) are

$$\phi_{a,b,c}(t, 0) = Q_{a,b,c,0} , \quad (18a)$$

$$\phi_d(t, 0) = \Delta p_0 . \quad (18b)$$

When  $p = 1$ , the equations become the original NDAEs (3a)–(3d).

Differentiating Eq. (15)  $m$  times with respect to  $p$ , evaluating at  $p = 0$ , and dividing by  $m!$ , we get the  $m$ th-order deformation equations

$$\mathcal{L}_{a,b,c} [Q_{a,b,c,m} - \gamma_m Q_{a,b,c,m-1}] = \frac{h_{a,b,c} H_{a,b,c}(t)}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_{a,b,c}}{\partial p^{m-1}} \Big|_{p=0} , \quad (19a)$$

$$\mathcal{L}_d [\Delta p_m - \gamma_m \Delta p_{m-1}] = \frac{h_d H_d(t)}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}_d}{\partial p^{m-1}} \Big|_{p=0} , \quad (19b)$$

in the unknowns  $Q_{a,m}(t)$ ,  $Q_{b,m}(t)$ ,  $Q_{c,m}(t)$ ,  $\Delta p_m(t)$  where

$$\gamma_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} , \quad (20)$$

and the initial conditions are

$$Q_{a,b,c,m}(0) = \Delta p_m(0) = 0 . \quad (21)$$

Eqs. (19) are linear ODEs, and we can integrate them to get the  $m$ th-order deformation.

The  $m = 1$  terms are

$$Q_{a,b,c,1} = h_{a,b,c} e^{-g_{a,b,c}t} \int_0^t H_{a,b,c}(u) e^{g_{a,b,c}u} \mathcal{N}_{a,b,c,0} du , \quad (22a)$$

$$\Delta p_1 = h_d e^{-g_d t} \int_0^t H_d(u) e^{g_d u} \mathcal{N}_{d,0} du , \quad (22b)$$

where

$$\mathcal{N}_{a,0} = \frac{dQ_{a,0}}{dt} + \alpha_a Q_{a,0}^n - \beta_a \Delta p_0 ,$$

$$\mathcal{N}_{b,0} = \frac{dQ_{b,0}}{dt} + \alpha_b Q_{b,0}^n - \beta_b \Delta p_0 ,$$

$$\mathcal{N}_{c,0} = \frac{dQ_{c,0}}{dt} + \alpha_c Q_{c,0}^n - \beta_c \Delta p + \beta_c \Delta p_0 ,$$

$$\mathcal{N}_{d,0} = c_a Q_{a,0} + c_b Q_{b,0} - c_c Q_{c,0} + 1 .$$

The  $m = 2$  terms are

$$Q_{a,b,c,2} = e^{-g_{a,b,c}t} \int_0^t e^{g_{a,b,c}u} \left[ h_{a,b,c} H_{a,b,c}(u) \mathcal{N}_{a,b,c,1} + \frac{dQ_{a,b,c,1}}{dt} + g_{a,b,c} Q_{a,b,c,1} \right] du , \quad (23a)$$

$$\Delta p_2 = e^{-g_d t} \int_0^t e^{g_d u} \left[ h_d H_d(u) \mathcal{N}_{d,1} + \frac{d\Delta p_1}{dt} + g_d \Delta p_1 \right] du , \quad (23b)$$

where

$$\mathcal{N}_{a,1} = \frac{dQ_{a,1}}{dt} + \alpha_a Q_{a,1}^n - \beta_a \Delta p_1 , \quad (24a)$$

$$\mathcal{N}_{b,1} = \frac{dQ_{b,1}}{dt} + \alpha_b Q_{b,1}^n - \beta_b \Delta p_1 , \quad (24b)$$

$$\mathcal{N}_{c,1} = \frac{dQ_{c,1}}{dt} + \alpha_c Q_{c,1}^n + \beta_c \Delta p_1 , \quad (24c)$$

$$\mathcal{N}_{d,1} = c_a Q_{a,1} + c_b Q_{b,1} - c_c Q_{c,1} - 1 . \quad (24d)$$

With initial guesses and proper selection of auxiliary functions, we can integrate Eqs. (22) and (23), so that the  $M$ -term approximation is

$$Q_{a,b,c}(t) \approx \sum_{m=1}^M Q_{a,b,c,m} , \quad (25a)$$

$$\Delta p(t) \approx \sum_{m=1}^M \Delta p_m . \quad (25b)$$

## 5 Results and discussion

In this section, the homotopy analysis results for  $n = 1.02$  and  $2$  will be compared with those using regular and  $\delta$ -perturbation. Numerical solutions are also obtained to be able to evaluate the accuracy of the approximate results. In the calculations, the parameters are taken to be  $\alpha_a^* = 20$ ,  $\alpha_b^* = 10$ ,  $\alpha_c^* = 1$ ,  $\beta_a^* = \beta_b^* = \beta_c^* = 1$ , and all calculations are implemented in MAPLE.

In the homotopy analysis, we choose the initial guesses as

$$Q_{a,b,c,0} = [q_{a,b,c,f} + (1 - q_{a,b,c,f}) \exp(-k_{a,b,c}t)] , \quad (26a)$$

$$\Delta p_0 = [q_{d,f} + (1 - q_{d,f}) \exp(-k_d t)] , \quad (26b)$$

with

$$q_{a,b,f} = \left( q_{d,f} \frac{\beta_{a,b}}{\alpha_{a,b}} \right)^{1/n} , \quad (27a)$$

$$q_{c,f} = \left[ (\Delta p - q_{d,f}) \frac{\beta_c}{\alpha_c} \right]^{1/n} , \quad (27b)$$

$$q_{d,f} = \frac{\beta_c \Delta p}{\alpha_c \left[ \frac{\beta_a}{\alpha_a} + \frac{\beta_b}{\alpha_b} + \frac{\beta_c}{\alpha_c} + 2 \left( \frac{\beta_a \beta_b}{\alpha_a \alpha_b} \right)^{1/n} \right]} . \quad (27c)$$

where  $k_i$  are parameters that need to be selected. We also choose  $H_a(t) = H_b(t) = H_c(t) = e^{-t}$ , and  $H_d(t) = (1+t)e^{-t}$ .

The results of  $n = 1.02$  are shown in Figs. 2, 3, 4, and 5, respectively. As shown in these figures, although the time region of validity of the  $\delta$ -perturbation is larger than that of the regular perturbation, both approximations are valid only in a small initial region of time. On the contrary, the homotopy results fit the numerical results very well in the entire time region, where the  $h, k_i$  and  $g$  are tuning parameters determined by trial and error.

The results of  $n = 2$  are shown in Figs. 6, 7, 8, and 9, respectively. Because of the strong nonlinearity, the results by the  $\delta$ -perturbation are valid in a much smaller initial region of time compared with that of  $n = 1.02$ . The results of the regular perturbation are still valid in a very small initial region of time. The results of the homotopy analysis method, however, are valid in the entire time region and give very good approximations.

As we can see, because of the strong nonlinearity of the system, both regular and  $\delta$  perturbation methods can not give good approximations. It is also not very practical to use these perturbation methods since numerical integration is needed for the higher order evaluation. Homotopy analysis, however, does not have any restrictions, and thus has an advantage compared to the others. It is a very powerful tool that does not assume any small parameter in the problem, but is a way to improve an initial approximation and auxiliary parameters and functions. Based on certain assumptions, this method can give very good results even with a few terms.

## 6 Conclusions

Exact solutions can only be obtained at  $n = 1$ . For nonlinear case  $n > 1$ , the mathematical model of even a very simple piping network are nonlinear differential equations with strong nonlinearity. Complete nonlinear dynamical equations can be obtained after some manipulations. The approximate solutions of the dynamical system obtained by the regular perturbation and the  $\delta$ -perturbation methods are valid only for a small region of time and numerical calculations have to be made for the higher order terms. On the other hand, the

homotopy analysis method provides us with very good approximations. Unlike the perturbation methods, the homotopy analysis are directly performed on the NDAEs and do not need to identify a small parameter in the system. However, the limitations of the homotopy analysis method are also remarkable. There are no rigorous theories directing the selection of the initial approximations, auxiliary linear operators and auxiliary parameters and functions. Parameter tuning has to be performed by trial and error, which limits the application of the method to some nonlinear problems for which numerical solutions cannot be obtained.

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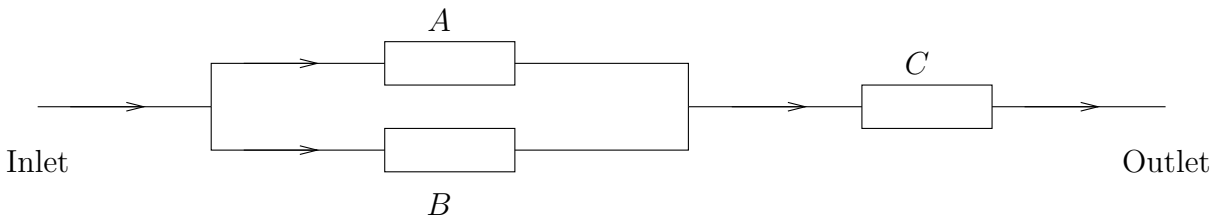


Figure 1: Schematic of simplified network.

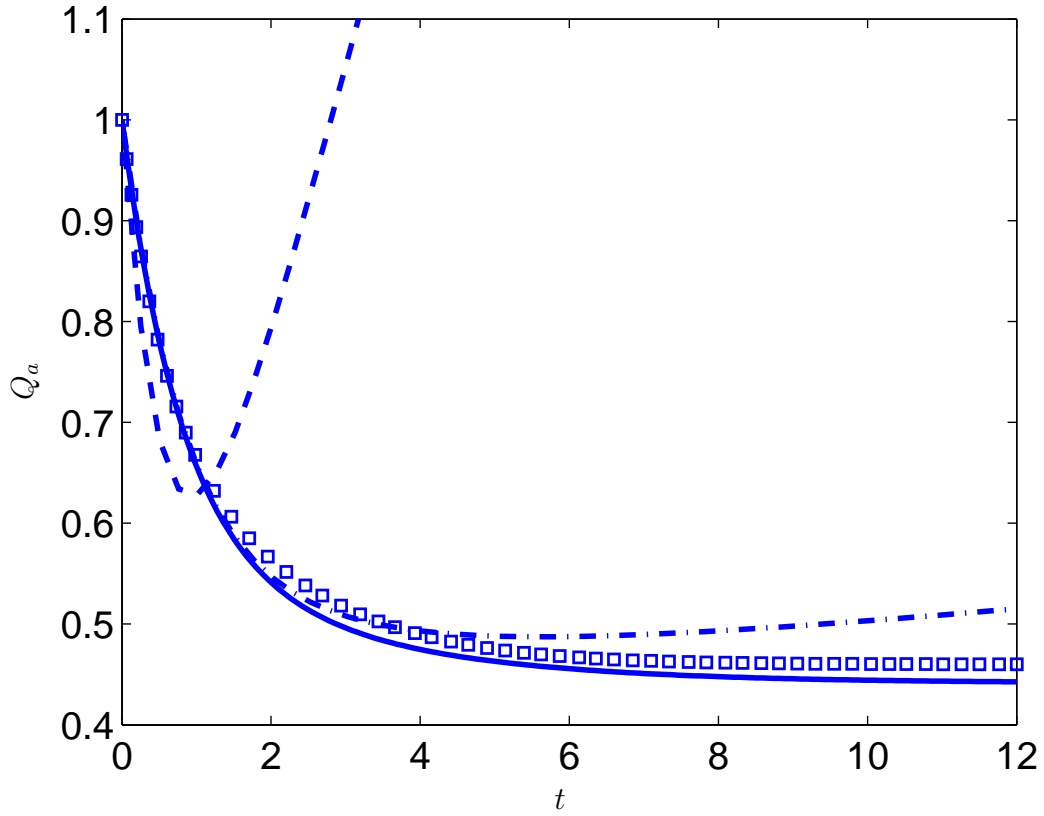


Figure 2: Three-term homotopy approximation of  $Q_a$  at  $n = 1.02$  with  $h_{a,b,c,d} = -0.85$ ,  $g_{a,b,c,d} = 1$ ,  $k_a = k_b = 1.8$ ,  $k_c = 0$  and  $k_d = -0.2$ ; square: homotopy, solid line: numerical, dash-dot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

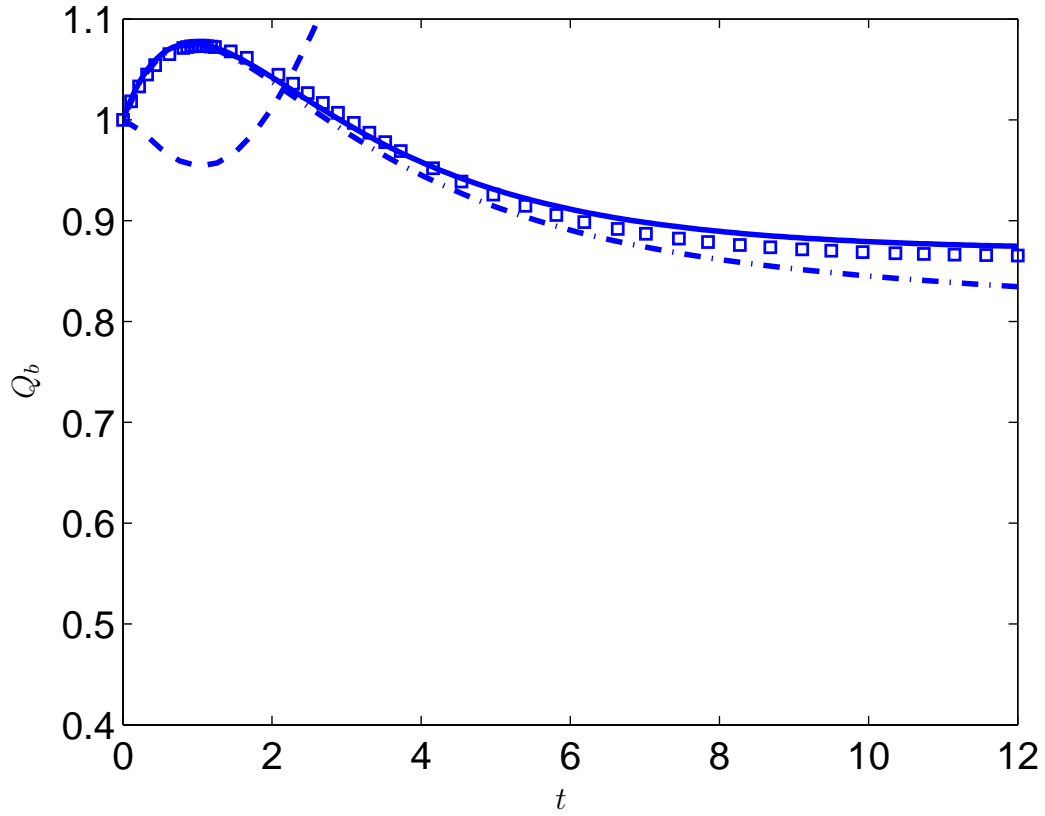


Figure 3: Three-term homotopy approximation of  $Q_b$  at  $n = 1.02$  with  $h_{a,b,c,d} = -0.95$ ,  $g_{a,b,c,d} = 0.7$ ,  $k_a = k_b = 1.5$ ,  $k_c = 0$  and  $k_d = -0.4$ ; square: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

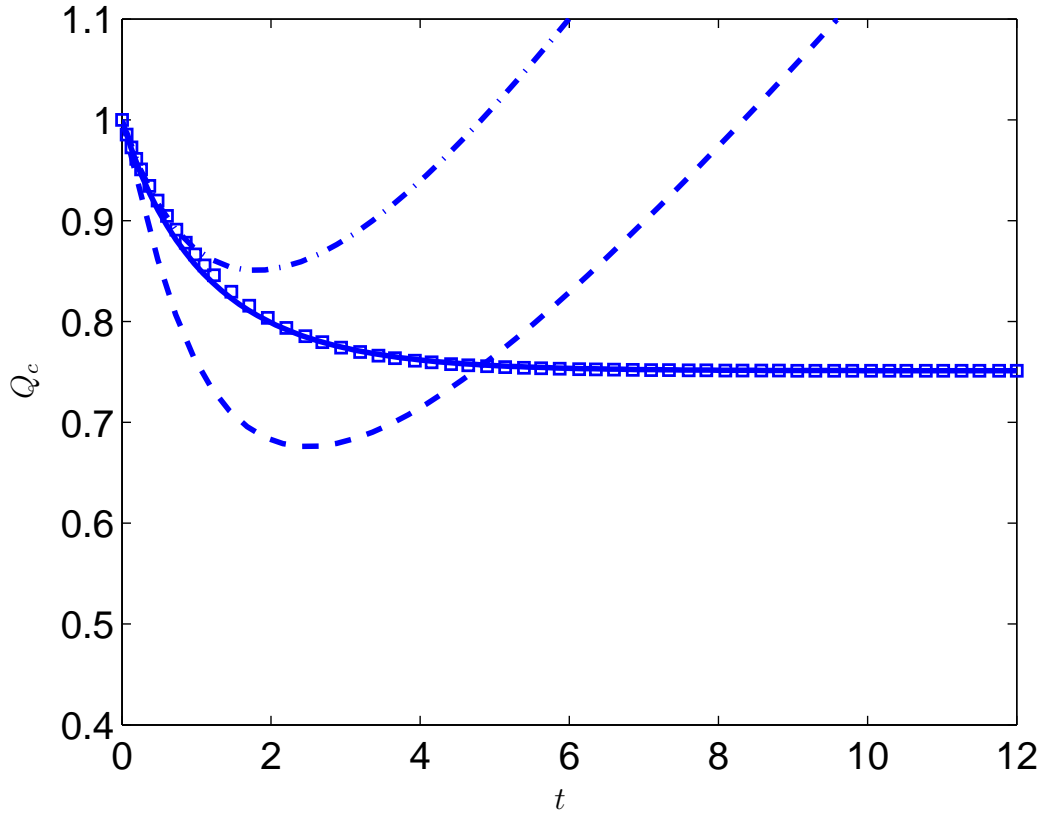


Figure 4: Three-term homotopy approximation of  $Q_c$  at  $n = 1.02$  with  $h_{a,b,c,d} = -0.6$ ,  $g_{a,b,c,d} = 1.5$ ,  $k_a = k_b = 0$ ,  $k_c = 0.42$  and  $k_d = -0.4$ ; square: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

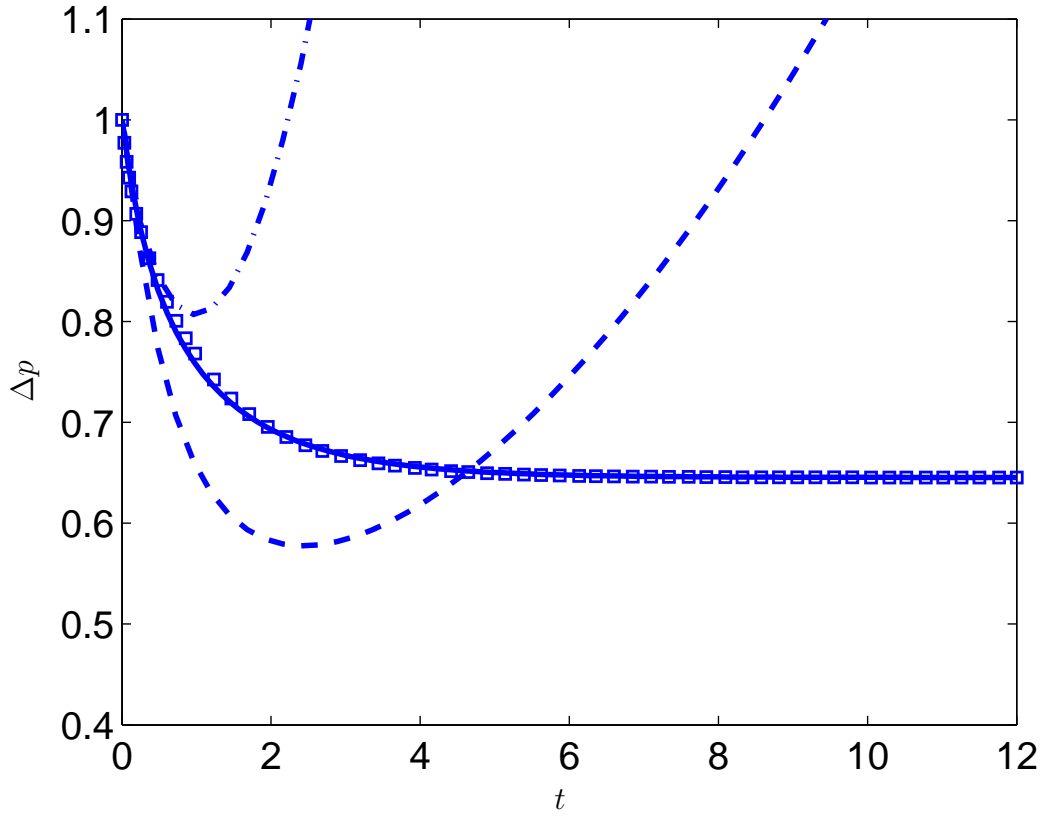


Figure 5: Three-term homotopy approximation of  $\Delta p$  at  $n = 1.02$  with  $h_{a,b,c,d} = -0.2$ ,  $g_{a,b,c,d} = 2.1$ ,  $k_a = k_b = 2.5$ ,  $k_c = 1.4$  and  $k_d = 0.44$ ; square: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

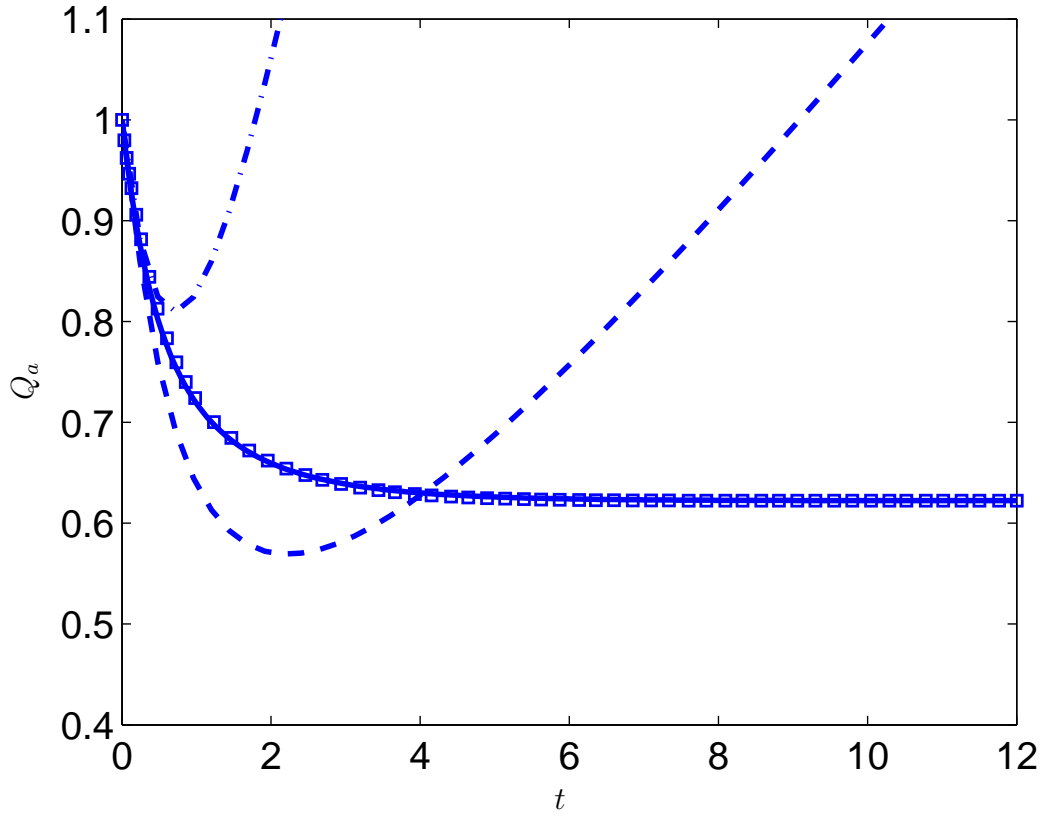


Figure 6: Three-term homotopy approximation of  $Q_a$  at  $n = 2$  with  $h_{a,b,c,d} = -1$ ,  $g_{a,b,c,d} = 2$ ,  $k_a = k_b = 20$ ,  $k_c = 0$  and  $k_d = 0$ ; square: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

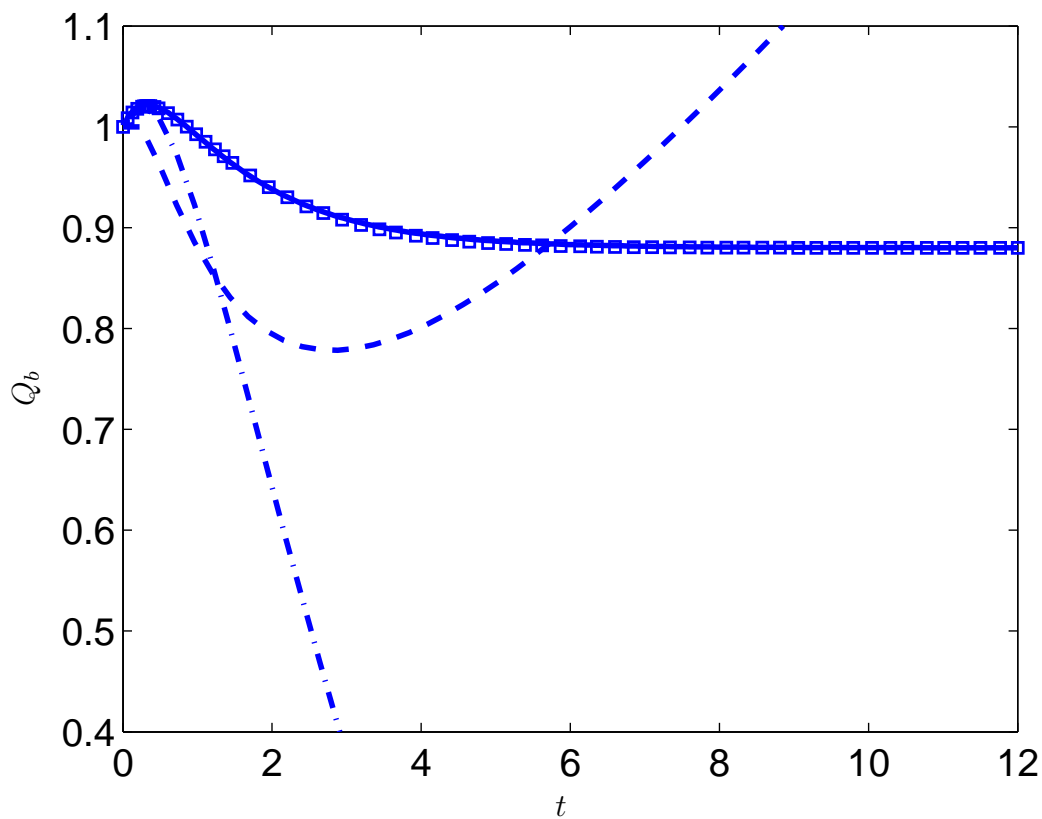


Figure 7: Three-term homotopy approximation of  $Q_b$  at  $n = 2$  with  $h_{a,b,c,d} = -1.1$ ,  $g_{a,b,c,d} = 2$ ,  $k_a = k_b = 10$ ,  $k_c = 0$  and  $k_d = 0$ ; square: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

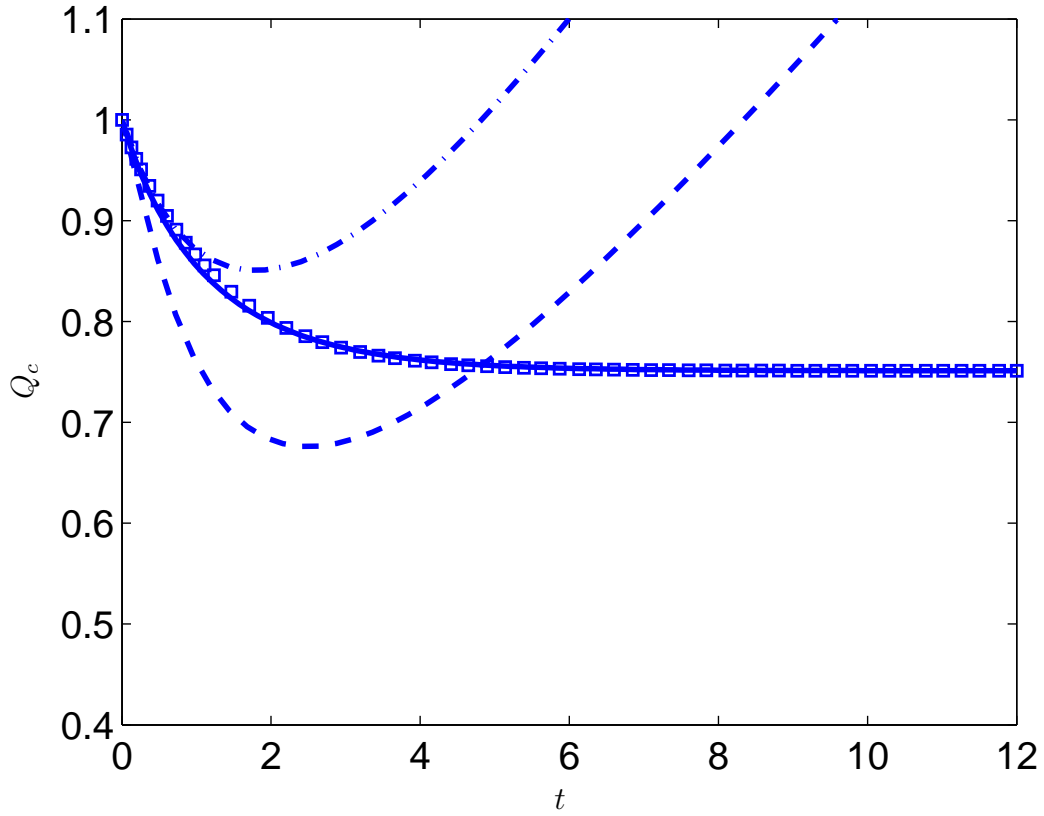


Figure 8: Three-term homotopy approximation of  $Q_c$  at  $n = 2$  with  $h_{a,b,c,d} = -0.9$ ,  $g_{a,b,c,d} = 4$ ,  $k_a = k_b = 5$ ,  $k_c = 0.82$  and  $k_d = 1$ ; square: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.

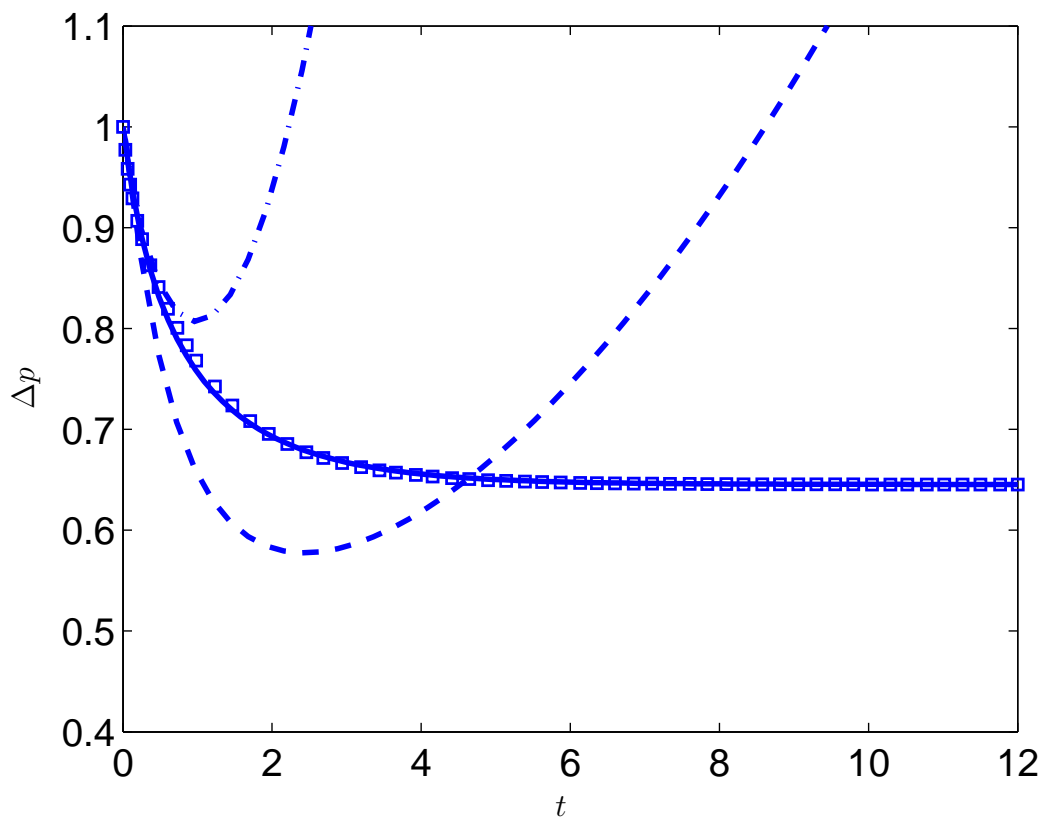


Figure 9: Three-term homotopy approximation of  $\Delta p$  at  $n = 2$  with  $h_{a,b,c,d} = -0.5$ ,  $g_{a,b,c,d} = 9$ ,  $k_a = k_b = 2$ ,  $k_c = 1.6$  and  $k_d = 0.8$ ; cross: homotopy, solid line: numerical, dashdot line: regular perturbation, and dashed line:  $\delta$ -perturbation.