Mathematical Methods in Financial Economics

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Lecture Notes 18 & 19

1 Stochastic Calculus and the Black-Scholes-Merton PDE

The main goal of the next few lectures is to introduce the concepts from mathematics needed for understanding and proving the following fundamental result in asset pricing theory. As before, we assume an economy consisting of a stock with initial price S_0 and money that can be invested at the risk-free interest rate r.

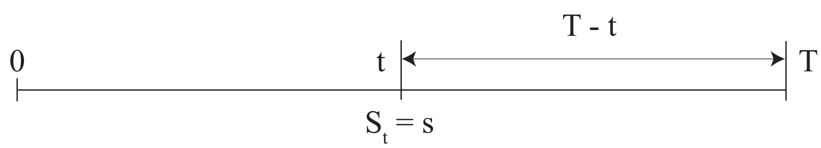
Theorem 1.1 *Suppose that for $t > 0$ the stock price S_t satisfies the* **stochastic differential equation**

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{1.1}$$

where r is the risk-free interest rate, σ is the volatility of the stock and B_t is a **Brownian motion** *stochastic process. Now, consider a European call option for this stock with strike price K and exercise time T. Thus, its value (pay-off) at time T is*

$$V(S_T, T) = [S_T - K]^+ \doteq max\{S_T - K, 0\}. \tag{1.2}$$

Denote by $V(s,t)$ the value of the call option at time t if the price of the stock at that time $S_t = s$.



Then, $V(s,t)$ satisfies the **Black-Scholes-Merton partial differential equation**

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad t \in [0,T], \quad s \geq 0. \tag{1.3}$$

Furthermore, a solution to PDE (1.3) *is given by*

$$V(s,t) = sN(d_+(s, T-t)) - Ke^{-r(T-t)}N(d_-(s,T-t)), \quad 0 \leq t < T, \ s > 0, \tag{1.4}$$

where

$$d_\pm(s,\tau) \doteq \frac{1}{\sigma\sqrt{\tau}}\Big[\ln\frac{s}{K} + (r \pm \frac{1}{2}\sigma^2)\tau\Big], \tag{1.5}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy. \tag{1.6}$$

Finally, this solution to the Black-Scholes-Merton PDE (1.3) *satisfies the* **terminal condition**

$$V(s,T) \doteq \lim_{t \uparrow T} V(s,t) = [s-K]^+, \tag{1.7}$$

and the **boundary conditions**

$$V(0,t) \doteq \lim_{s\downarrow 0} V(s,t) = 0, \quad \textit{for all} \quad t \in [0,T], \tag{1.8}$$

and

$$\lim_{s\uparrow\infty} \Big[V(s,t) - \big(s - e^{-r(T-t)}K \big) \Big] = 0, \quad \textit{for all} \quad t \in [0,T]. \tag{1.9}$$

To fully understand the statement of Theorem 1.1 and present its proof we shall need to develop the following topics.

- *Brownian motion or Wiener stochastic process.*
- *Stochastic differential equations.*
- *Itô's formula.*

Meanwhile, using calculus, the reader can verify some of the statements in Theorem 1.1 by doing the following exercises.

Exercise 1. Check that the standard normal distribution $N(x)$ satisfies the differential equation $N''(x) = -xN'(x)$. Then, use it to show that the function $V(s,t)$ defined by (1.4) is a solution to the Black-Scholes-Merton PDE (1.3).

Exercise 2. Show that the $V(s,t)$ defined by (1.4) satisfies terminal condition (1.7).

Exercise 3. Show that the $V(s,t)$ defined by (1.4) satisfies the boundary conditions (1.8) and (1.9).

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2 Brownian motion or Wiener stochastic process

In 1827 the botanist Robert Brown observed that grains of pollen suspended in fluid perform an erratic movement which came to be known as **Brownian motion**. Assuming that at the beginning of the observation a grain of pollen is at the origin then at the subsequent time its path in the x-directions looks like in Figure 2.

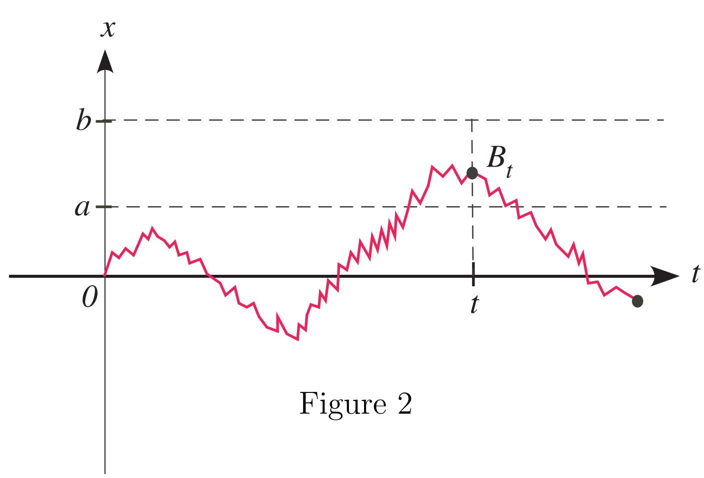

Figure 2

Brownian motion was analyzed by Albert Einstein in 1905, who showed that the probability of the pollen to be in an interval $[a, b]$ at time t is given by the formula

$$P(a \leq B_t \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-\frac{1}{2t}x^2} dx. \tag{2.1}$$

A rigorous mathematical model of Brownian motion was constructed first by Norbert Wiener in 1923. In his book "I am a Mathematician" Wiener describes Brownian motion intuitively as follows:

"To understand the Brownian motion, let us imagine a pushball in a field in which a crowd is milling around. Various people in the crowd will run into the pushball and will move it about. Some will push in one direction and some in another, and the balance of pushes is likely to be tolerably even. Nevertheless, notwithstanding these balanced pushes, the fact remains that they are pushes by individual people and that their balance will be only approximate. Thus, in the course of time, the ball will wander about the field like the drunken man whom we have already mentioned and we shall have a certain irregular motion in which what happens in the future will have very little to do with what has happened in the past."

Exercise 4. Einstein in one of his five famous papers published in 1905, titled "On the Motion of Small Particles Suspended in Liquid at Rest Required by the Molecular-Kinetic Theory of Heat," derived the probability density function

$$f(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}x^2},$$

used in (2.1), as a solution to the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}.$$

Check that indeed $f(x, t)$ solves this equation.

The following definition provides a rigorous description the Brownian motion B_t, which from now on we will denote it by $W(t)$.

Definition. A Brownian motion or Wiener process $\{W(t)\}_{t \geq 0}$ is a family of random variables $W(t) : \Omega \longrightarrow \mathbb{R}$, where Ω is a probability space, satisfying the following properties:

1. $W(0) = 0$.

2. It has continuous paths, i.e. the map $t \longmapsto W(t)$ is continuous from $[0, \infty)$ into $\mathbb{R}$. (More precisely, for each $\omega \in \Omega$ the path $t \longmapsto W(\omega, t)$ is continuous with probability one.)

3. $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$, for $t > s \geq 0$. That is,
$$P(a \leq W(t) - W(s) \leq b) = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{1}{2(t-s)}x^2} dx.$$

4. It has independent increments. That is, for all $0 = t_0 < t_1 < t_2 < \cdots < t_n$ the increments

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \cdots,\ W(t_n) - W(t_{n-1}),$$

are independent random variables.

Other Notation. $W(t)$ is also denoted by W_t. The notation $W(t,\omega)$ and $W_t(\omega)$ stress the fact that for each t it is a random variable defined on a sample sample Ω.

Remark. Although the sample paths are continuous they are **nowhere** differentiable since increments are random (normally distributed). This is indicated by the following formula

$$E\Big[\Big(\frac{W(t+\Delta t) - W(t)}{\Delta t}\Big)^2\Big] = \frac{\Delta t}{(\Delta t)^2} = \frac{1}{\Delta t},$$

which goes to ∞ as $\Delta t \to 0$.

A construction of Brownian motion

The probability space Ω. We begin by introducing the probability space Ω to be used for a construction of Brownian motion. For this, imagine that we toss a **fair** coin infinitely many times. Then, the set of possible outcome can be described by the following **sample space**

$$\Omega = \{\omega = \omega_1\omega_2\omega_3\cdots : \omega_j = H \text{ or } T\}. \tag{2.2}$$

To compute the probability of easy to describe subsets of Ω (**events**) is not difficult. For example, for the event whose outcomes begin with H

$$A_H = \{H\omega_2\omega_3\cdots : \omega_j = H \text{ or } T, j \geq 2\}$$

and the event whose outcomes begin with T

$$A_T = \{T\omega_2\omega_3\cdots : \omega_j = H \text{ or } T, j \geq 2\}$$

we have

$$P(A_H) = P(A_T) = \frac{1}{2}. \tag{2.3}$$

Defining similarly the events A_{HH}, A_{HT}, A_{TH}, and A_{TT}, we have

$$P(A_{HH}) = P(A_{HT}) = P(A_{TH}) = P(A_{TT}) = \Big(\frac{1}{2}\Big)^2. \tag{2.4}$$

More generally, the event whose outcomes ω have the same first k terms their probability is equal to $(1/2)^k$. For example, when $k = 5$ such an event is given by

$$A_{HHTHT} = \{HHTHT\omega_6\omega_7\cdots : \omega_j = H \text{ or } T, j \geq 6\}$$

and

$$P(A_{HHTHT}) = \Big(\frac{1}{2}\Big)^5. \tag{2.5}$$

Furthermore, using the formula

$$P(A^c) = 1 - P(A) \tag{2.6}$$

we are able to compute the probability of the complement of any event A, whose probability is known. Also, knowing the probability of a sequence of disjoint events $A_1, A_2, \cdots$ we are able to compute the probability of their union by using the formula

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots \tag{2.7}$$

The ultimate goal here is to find the largest family $\mathcal{F}$ of subsets of the sample space Ω, which is closed under complementation and countable unions and on which formula (2.7) holds. In such a case the family $\mathcal{F}$ is called a **σ-algebra** (or σ-field) and P is called a **probability measure**. The triple $(\Omega, \mathcal{F}, P)$ is called a **probability space**. Note that

$$P(\Omega) = 1. \tag{2.8}$$

For a discussion about a construction of such probability space we refer the reader to Shreve's Stochastic Calculus II. Next, we shall indicate an alternative approach based on binary expansions. If we encode H with 1 and T with 0, then each element $\omega = \omega_1\omega_2\omega_3 \cdots \in \Omega$ is encoded into a number $\alpha \in [0,1]$ by the formula

$$\alpha = \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}. \tag{2.9}$$

The correspondence $\omega \mapsto \alpha$ defined by formula (2.9) goes from Ω onto $[0,1]$ but it has a defect. It is not one-to-one. For example,

$$\omega = .1000 \cdots \mapsto \alpha = \frac{1}{2} \quad \omega = .0111 \cdots \mapsto \alpha = \frac{1}{2}$$

The problem arises from the recurring 1's, that is, from the elements in the set

$$\Omega_0 = \{\omega_1 \cdots \omega_k 0111 \cdots : \omega_j = 1 \text{ or } 0\}. \tag{2.10}$$

To correct this defect we remove from Ω the set Ω_0. Then, the correspondence

$$\omega \in (\Omega - \Omega_0) \mapsto \alpha = \sum_{n=1}^{\infty} \frac{\omega_n}{2^n} \in (0,1] \tag{2.11}$$

is one-to-one and onto. Furthermore, the thrown away set is countable! Therefore, its probability measure is zero! So, it suffices to define a probability measure on $\Omega - \Omega_0$. This is is done by pulling back via mapping (2.11) all beautiful things that exist on the interval $(0,1]$, that is the Lebesgue measure space.

Exercise 5. Explain in more detail why map (2.11) is one-to-one and onto.

Exercise 6. Show that Ω_0 is countable (can be put into one-to-one with the natural numbers).

Exercise 7. Show that for any $\omega \in \Omega$ we have $P(\{\omega\}) = 0$.

Exercise 8. Show that $P(\Omega_0) = 0$.

Exercise 9. Describe in more detail the construction of the measure space $(\Omega, \mathcal{F}, P)$ by pulling back via mapping (2.11) the Lebesgue measure space on the interval $[0,1]$. Also, you may wish to provide an outline of the construction of the Lebesgue measure space on $[0,1]$.

Random Walk with time step Δt and space step Δx

We shall obtain Brownian motion as a limit of random walks. So, we begin by defining these simpler stochastic processes. For this we shall use our sample space Ω. Recall that the coin is fair, that is, we have

$$P(H) = P(T) = 1/2.$$

Let X_j be the random variable defined by

$$X_j(\omega) = \begin{cases} 1, & \text{if } \omega_j = H; \\ -1 & \text{if } \omega_j = T. \end{cases} \tag{2.12}$$

Note that $X_1, X_2, X_3, \cdots$ is a sequence of **independent binomial** random variables. Therefore,

$$E(X_j) = 0, \quad \text{and} \quad \mathrm{Var}(X_j) = 1. \tag{2.13}$$

Next, let $\Delta x > 0$ be a space step and $\Delta t > 0$ be a time step and imagine the following **random walk** on the x-axis. At the time $t = 0$ we are at the origin, $x = 0$, and flip a coin. If the outcome is heads then we walk a distance of Δx units in the positive direction, and if it is tails then we walk Δx units in the negative direction, both times at the **constant** speed $\Delta x/\Delta t$. At the time $t = \Delta t$ the coin is flipped (**instantaneously**) for the second time and we do the same kind of walk again. That is, we walk Δx units in the positive direction if the outcome is heads and Δx units in the negative direction if the outcome is tails. Flipping the coin for the third, fourth, etc. time we keep walking randomly like this forever. Figure 1 displays our path in the tx-plane if $\omega = HHTHTTTTH\omega_{10}\omega_{11}\cdots$. $W_{j\Delta t}$ is our position at time $t = j\Delta t$.

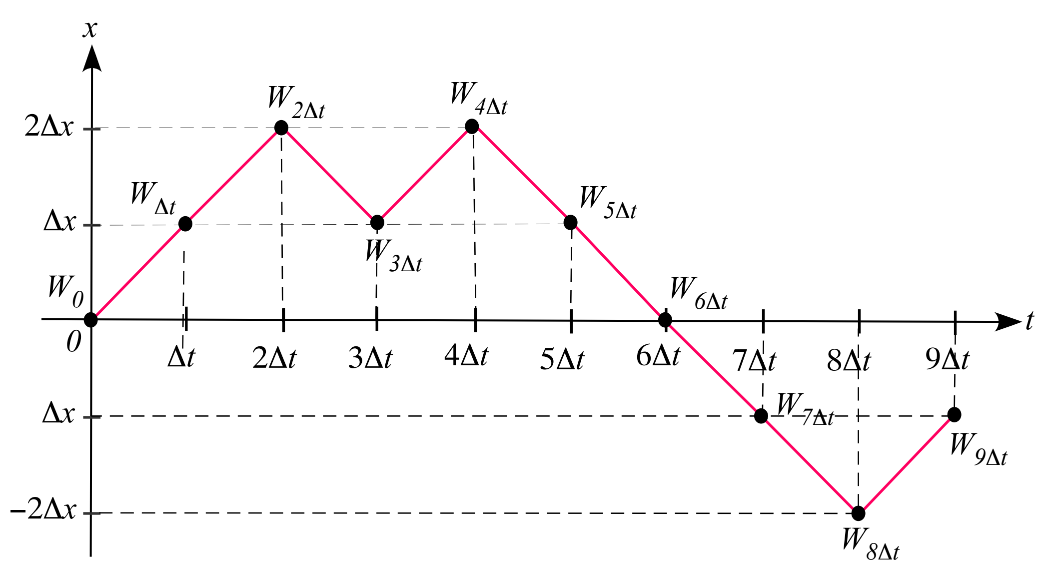

Figure 1

In terms of the random variables X_j we have

$$W_0 = 0 \quad \text{and} \quad W_{k\Delta t} = \sum_{j=1}^{k} X_j \Delta x, \quad k = 1, 2, \cdots. \tag{2.14}$$

At a time t, which may not be a multiple of Δt our position W_t is found by linear interpolation. More precisely, let k be the positive integer such that $k\Delta t \le t < (k+1)\Delta t$. Then

$$W_t = \left(k + 1 - \frac{t}{\Delta t}\right) W_{k\Delta t} + \left(\frac{t}{\Delta t} - k\right) W_{(k+1)\Delta t}, \quad k\Delta t \le t < (k+1)\Delta t. \tag{2.15}$$

The stochastic process $\{W_t\}_{t\ge 0}$ defined by the formulas (2.14) and (2.15) is called a **random walk** with time step Δt and space step Δx.

Exercise 10. Find $P(W_{2\Delta t} = 0)$, and $P(W_{3\Delta t} \ge 0)$.

Properties of Random Walk

1. $W_0(\omega) = 0$ for all $\omega \in \Omega$.
2. For every $\omega \in \Omega$ the path $t \in [0, \infty) \longmapsto W_t(\omega) \in \mathbb{R}$ is **continuous**.
3. If $t = k\Delta t$, then
$$E(W_t) = 0, \quad \text{and} \quad \mathrm{Var}(W_t) = \frac{(\Delta x)^2}{\Delta t} \cdot t.$$
4. If $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and each t_j is a non-negative integer multiple of Δt, then the increments
$$W_{t_1} = W_{t_1} - W_{t_0},\ W_{t_2} - W_{t_1},\ \cdots,\ W_{t_n} - W_{t_{n-1}},$$
are independent random variables.

Properties (1) and (2) are true by construction. To prove (3) we use the fact that X_j are independent and (2.14). Therefore

$$E(W_{k\Delta t}) = E\left(\sum_{j=1}^{k} X_j \Delta x\right) = \sum_{j=1}^{k} E(X_j)\Delta x = \sum_{j=1}^{k} 0 \cdot \Delta x.$$

Also, using the independence of X_j and the fact that $\mathrm{Var}(X_j) = 1$ we obtain

$$\begin{aligned}\mathrm{Var}\Big(W_{k\Delta t}\Big) &= \mathrm{Var}\Big(\sum_{j=1}^{k} X_j \Delta x\Big) = \sum_{j=1}^{k} \mathrm{Var}(X_j)(\Delta x)^2 = \sum_{j=1}^{k} 1 \cdot (\Delta x)^2 \\ &= k(\Delta x)^2 = \frac{(\Delta x)^2}{\Delta t} \cdot k\Delta t = \frac{(\Delta x)^2}{\Delta t} \cdot t.\end{aligned}$$

Finally, property (4) follows from the independence of X_j.

Exercise 11. Are properties (3) and (4) true when the time is not a non-negative integer multiple of Δt?

Brownian Motion as the Limit of Random Walks

Since we are looking for a stochastic process having variance at time t equal to t, property (3) of the random walks constructed above suggest that we should choose the steps Δt and Δx so that

$$\frac{(\Delta x)^2}{\Delta t} = 1.$$

Motivated by this relation, for each positive integer n we constuct the random walk $\{W_t^n\}_{t\geq 0}$ with steps

$$\Delta t = \frac{1}{n}, \quad \text{and} \quad \Delta x = \frac{1}{\sqrt{n}}.$$

For any $t > 0$ and fixed we shall define W_t^n by using formula (2.15). For this let $[nt]$ be the largest positive integer which is less than or equal to nt. Then,

$$\begin{aligned} W_t^n &= \Big([nt] + 1 - nt\Big) W^n_{[nt]\cdot\frac{1}{n}} + \Big(nt - [nt]\Big) W_{([nt]+1)\cdot\frac{1}{n}} \\ &= W^n_{[nt]\cdot\frac{1}{n}} + \Big(nt - [nt]\Big)\Big(W^n_{([nt]+1)\cdot\frac{1}{n}} - W^n_{[nt]\cdot\frac{1}{n}}\Big) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} X_j + \frac{nt - [nt]}{\sqrt{n}} X_{[nt]+1} \end{aligned}$$

or

$$W_t^n = \frac{\sqrt{[nt]}}{\sqrt{n}} \cdot \frac{1}{\sqrt{[nt]}} \sum_{j=1}^{[nt]} X_j + \frac{nt - [nt]}{\sqrt{n}} X_{[nt]+1}. \tag{2.16}$$

Now, observe that

$$\lim_{n\to\infty} \frac{\sqrt{[nt]}}{\sqrt{n}} = \sqrt{t}, \quad \lim_{n\to\infty} \frac{nt - [nt]}{\sqrt{n}} X_{[nt]+1} = 0.$$

Also, by the **Central Limit Theorem** we have

$$\frac{1}{\sqrt{[nt]}} \sum_{j=1}^{[nt]} X_j \longrightarrow Z, \quad \text{as} \quad n \to \infty \quad \text{in probability.}$$

Therefore, by letting n goes to ∞ in equation (2.16) we see that

$$W_t^n \longrightarrow \sqrt{t} \cdot Z = B_t \quad \text{as} \quad n \to \infty \quad \text{in probability.}$$

where B_t satisfies all the properties of a Brownian motion.

Exercise 12. Draw a Brownian path using a computer program.

Exercise 13. State the Central Limit Theorem. Outline its proof if you wish.

Exercise 14. Describe a way for constracting a sequence $Y_0, Y_1, Y_2, \cdots$ of independent normal random variables with expected value qual to 0 and standard deviation equal to 1 on the same probability space.

Exercise 15. Describe a way for constructing Brownian motion, which is different than the one described above. For example, you may try to understant/explain/prove the following result, which is taken from Breiman's Probability book (page 261): If $Y_0, Y_1, Y_2, \cdots$ are independent normal random variables with $E(Y_j) = 0$ and $\sigma(Y_j) = 1$ then

$$X(t) = \frac{1}{\sqrt{\pi}} Y_0 + \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{\sin mt}{m} Y_m$$

is a Brownian motion on $[0, \pi]$.
