

MODEL THEORY AND THE TANNAKIAN FORMALISM

MOSHE KAMENSKY

ABSTRACT. We draw the connection between the model theoretic notions of internality and the binding group on one hand, and the Tannakian formalism on the other.

More precisely, we deduce the fundamental results of the Tannakian formalism by associating to a Tannakian category a first order theory, and applying the results on internality there. We also formulate the notion of a differential tensor category, and a version of the Tannakian formalism for differential linear groups, and show how the same techniques can be used to deduce the analogous results in that context.

INTRODUCTION

The aim of this paper is to exhibit the analogy and relationship between two seemingly unrelated theories. On the one hand, the Tannakian formalism, giving a duality theory between affine group schemes (or, more generally, gerbs) and a certain type of categories with additional structure, the Tannakian categories. On the other hand, a general notion of internality in model theory, valid for an arbitrary first order theory, that gives rise to a definable Galois group. The analogy is made precise by deriving (a weak version of) the fundamental theorem of the Tannakian duality (Theorem 2.8) using the model theoretic internality.

The Tannakian formalism assigns to a group \mathcal{G} over a field \mathbb{k} , its category of representations $\mathcal{Rep}_{\mathcal{G}}$. In the version we are mainly interested in, due to Saavedra (Saavedra-Rivano [24]), the group is an affine group scheme over a field. A similar approach works with groups in other categories, the first due to Krein and Tannaka, concerned with locally compact topological groups. Another example is provided in Section 4. In the algebraic case, the category is the category of algebraic finite-dimensional representations. This is a \mathbb{k} -linear category, but the category structure alone is not sufficient to recover the group. One therefore considers the additional structure given by the tensor product. The Tannakian formalism says that \mathcal{G} can be recovered from this structure, together with the forgetful functor to the category of vector-spaces. The other half of the theory is a description of the tensor categories that arise as categories of representations: any tensor category satisfying suitable axioms is of the form $\mathcal{Rep}_{\mathcal{G}}$, provided it has a “fibre functor” into the category of vector spaces. Our main references for this subject are the first three sections of Deligne and Milne [4] and Deligne [3].

In model theory, internality was discovered by Zilber as a tool to study the structure of strongly minimal structure (Zil’ber [27]). Later, Poizat realised (in Poizat [23]) that this notion can be used to treat the Galois theory of differential equations. The definable Galois correspondence outlined in Theorem 1.6 has its origins there. Later, the theory was generalised to larger classes of theories (Hrushovski [10], Hart

and Shami [7], etc.), and applied in various contexts (e.g., Pillay [19] extended the differential Galois theory to arbitrary “D-groups” definable in DCF).

In appendix B of Hrushovski [8], internality was reformulated in a way that holds in an arbitrary theory. One is interested in the group of automorphisms G of a definable set Q over another definable set C . A set Q is internal to another set C if, after extending the base parameters, any element of Q is definable over the elements of C . The idea is that under this condition, Q is close enough to C so that the G has a chance to be definable, but the requirement that a base extension is required prevents it from being trivial. The theorem is that indeed G is the group of points of a (pro-) definable group (see Theorem 1.6). This theory is reformulated again in Hrushovski [9], where the same construction is described as an abstract duality theory between definable groupoids in a theory \mathcal{T} , and certain expansions of it, called internal covers. It is this formulation that we use.

The main results of the paper appear in Sections 3 and 4. In Section 3 we apply internality to prove the fundamental result on Tannakian categories (Theorem 2.8). This is done by constructing, for a tensor category \mathcal{C} an internal cover $T_{\mathcal{C}}$ of $\text{ACF}_{\mathbb{k}}$. Models of this theory correspond, roughly, with fibre functors on \mathcal{C} . The theory of internality provides a definable group in $\text{ACF}_{\mathbb{k}}$, and this is the group corresponding to \mathcal{C} . The other parts of the theory follow from the Galois theory, and from the abstract duality theory of Hrushovski [9].

The result we prove is weaker than the original Theorem 2.8 in the following ways. First, 2.8(1) states that a certain functor is representable by an affine algebraic group, but we only prove that its restriction to the category of fields agrees with such a group (see also Question 0.1.3 below). Second, our proof works only in characteristic 0. Finally, 2.8(2) is covered, in part, by the general model theoretic statement 1.10, but the rest of the proof is only sketched, since it is not significantly different from the proof found in Deligne and Milne [4].

On the other hand, the proof has the advantage that it is simple and more “geometric” than the algebraic one. It also has the advantage that the method is applicable in a more general context. One such application, concerning usual Galois theory, is briefly discussed in Remark 3.14. A more detailed application appears in Section 4, where we define the notion of a differential tensor category, and explain how the same model theoretic approach gives an analogous theorem (Theorem 4.5.5) there (though the method again gives a weaker result, in the same way as in the algebraic case). A similar result, using a somewhat different language, was first obtain using algebraic methods in Ovchinnikov [17, 18]. It seems obvious that similar formalisms are possible in other contexts (e.g., difference fields, real closed fields). We note that, though model theory provides a general method of proof of such results, the right notion of a “tensor category” is new, and cannot be deduced directly from this method.

The first two sections contain background material. Section 1 contains some general notions from model theory, as well as a statement of internality result, and some auxiliary remarks on it. Section 2 gives a short review of Tannakian categories, with enough terminology and results to state the main theorem. Both sections are provided with the hope that the paper will be accessible to a wide audience (both within model theory and outside it), and that the information contained there will provide the reader with enough information to at least get a feeling for the nature of the results.

We mention also that there is a result in the other direction: It is possible to define the analogue of internal covers and recover a group object in a general categorical context, and recover the model theoretic result from it. This result will appear separately.

0.1. Questions. Several questions (some vague) remain unanswered in the paper. The main ones, from my point of view, are the following.

0.1.1. As described in Remarks 3.12 and 3.13, the main results of Deligne [3] have very natural model theoretic translations. It seems reasonable to expect that there is a model theoretic proof, especially of the main theorem (which translates to having no new structure on $\text{ACF}_{\mathbb{k}}$), but I could not find it. The case when \mathcal{C} is neutral appears to be easier from the algebraic point of view (and is proven already in Deligne and Milne [4]), but model theoretically I don't know how to do even this case.

0.1.2. Conversely, the results and methods of Deligne and Milne [4] and Deligne [3] seem to suggest that one can do “model theory” inside a Tannakian category (and perhaps more generally in a tensor category). For example, as can be seen from 3.10, the statement of 2.8(2) can be viewed as saying that \mathcal{C} has elimination of quantifiers and elimination of imaginaries. I think it would be interesting to find out if this makes sense, and make it precise.

0.1.3. What is a general model theoretic machinery to prove results as in 3.7, but in full generality? (rather than just for fields). This can also be asked purely in terms of functors (though in this context there is probably no answer): Assume there is a functor \mathcal{F} from \mathbb{k} -algebras to sets (or groups), and a scheme (or group scheme) \mathcal{G} with a map $\mathcal{G} \rightarrow \mathcal{F}$ that is a bijection on fields. What are conditions on \mathcal{F} that allow us to deduce that this map is an isomorphism?

Acknowledgements. I would like to thank Ehud Hrushovski for suggesting this approach, and for help along the way. I would also like to thank Anand Pillay for his interest and useful remarks.

1. MODEL THEORY AND INTERNALITY

In this section, we recall some notions from model theory. Some elements of the model theoretic language are briefly sketched in 1.1 through 1.3. These appear in most texts on model theory, for example Marker [13] or Pillay [20].

In the rest of the section, we recall the main model theoretic tool that we use, namely internality, and the definable Galois group. We shall, for the most part, follow the presentation and terminology from Hrushovski [9].

The only new results in this section appear after the statement of the theorem (Theorem 1.6), though most of them are implicit in Hrushovski [9].

1.1. Basic terminology. We briefly recall that a *theory* is collection of statements (axioms) written in a fixed formal language, and a *model* of the theory \mathcal{T} is a structure consisting of an interpretation of the symbols in the language of \mathcal{T} , in which all the axioms of \mathcal{T} hold.

For example, the theory $\mathcal{T} = \text{ACF}$ of algebraically closed fields can be written in a language containing symbols $(0, 1, +, -, \cdot)$, and a model of this theory is a particular algebraically closed field.

A *formula* is written in the same formal language, but has *free variables*, into which elements of the model can be plugged. For the example of ACF above, any finite collection of polynomial equations and inequalities can be viewed as a formula, but there are other formulas, involving quantifiers. Any such formula $\phi(x_1, \dots, x_n)$ (where x_1, \dots, x_n contain all free variables of ϕ) thus determines a subset $\phi(\mathcal{M})$ of \mathcal{M}^n , for any model \mathcal{M} , namely, the set of all tuples \bar{a} for which $\phi(\bar{a})$ holds. Two formulas ϕ and ψ are *equivalent* if $\phi(\mathcal{M}) = \psi(\mathcal{M})$ for all models \mathcal{M} . An equivalence class under this relation is called a *definable set* (So, in this paper, “definable” is without parameters.)

If \mathcal{M} is the model of some theory, the set of all statements (in the underlying language) that are true in \mathcal{M} is a theory $\mathcal{T}(\mathcal{M})$, and \mathcal{M} is a model of $\mathcal{T}(\mathcal{M})$.

We will assume all our theories to be multi-sorted, i.e., the variables of a formula can take values in any number of disjoint sets. In fact, if \mathbf{X} and \mathbf{Y} are sorts, we view $\mathbf{X} \times \mathbf{Y}$ as a new sort. In particular, any definable set is a subset of some sort. By a statement such as “ $\mathbf{a} \in \mathcal{M}$ ” we will mean that \mathbf{a} is an element of one of the sorts, interpreted in \mathcal{M} .

1.2. Definable closure and automorphisms. A *definable function* f from a definable set \mathbf{X} to another definable set \mathbf{Y} is a definable subset of $\mathbf{X} \times \mathbf{Y}$ that determines the graph of a function $f_{\mathcal{M}} : \mathbf{X}(\mathcal{M}) \rightarrow \mathbf{Y}(\mathcal{M})$ for each model \mathcal{M} . If \mathcal{M} is a model, $A \subseteq \mathbf{X}(\mathcal{M})$ any subset, and $\mathbf{b} \in \mathbf{Y}(\mathcal{M})$, then \mathbf{b} is *definable* over A if there is a definable function $f : \mathbf{X}^n \rightarrow \mathbf{Y}$ for some n , such that $f(\bar{a}) = \mathbf{b}$ for some $\bar{a} \in A^n$ (note that even though we are working in \mathcal{M} , it is \mathcal{T} that should think that f is a function, rather than just $\mathcal{T}(\mathcal{M})$). A subset is *definably closed* if it is closed under definable functions. The *definable closure* $\text{dcl}(A)$ of a subset $A \subseteq \mathcal{M}$ is the smallest definably closed subset of \mathcal{M} containing A . We denote by $\mathbf{Y}(A)$, the set of all elements in $\mathbf{Y}(\mathcal{M})$ definable over A .

We note that if \mathcal{T} is complete, then the definable closure $\mathbf{X}(0)$ of the empty set does not depend on the model. For general \mathcal{T} , we denote by $\mathbf{X}(0)$ the set of definable subsets of \mathbf{X} containing one element.

More generally, a formula over A is a formula $\phi(x, \mathbf{a})$, where $\phi(x, \mathbf{y}) \subseteq \mathbf{X} \times \mathbf{Y}$ is a regular formula, and $\mathbf{a} \in \mathbf{Y}(A)$. It defines a subset of $\mathbf{X}(\mathcal{M})$ for any model \mathcal{M} containing A , in the same way as regular formulas do.

An automorphism of a model \mathcal{M} is a bijection φ from \mathcal{M} to itself such that the induced bijection $\varphi^n : \mathcal{M}^n \rightarrow \mathcal{M}^n$ preserves any definable subset $\mathbf{X}(\mathcal{M}) \subseteq \mathcal{M}^n$. If $A \subseteq \mathcal{M}$ is a set of parameters, the automorphism φ is *over* A if it fixes all elements of A . The group of all automorphisms of \mathcal{M} over A is denoted by $\text{Aut}(\mathcal{M}/A)$. It is clear that $\mathbf{X}(A)$ is fixed pointwise by any automorphism over A . The model \mathcal{M} is called *homogeneous* if, conversely, for any $A \subseteq \mathcal{M}$ of cardinality smaller than the cardinality of \mathcal{M} , for any definable set \mathbf{X} , any element of $\mathbf{X}(\mathcal{M})$ fixed by all of $\text{Aut}(\mathcal{M}/A)$ is in $\mathbf{X}(A)$ (computed in $\mathcal{T}(\mathcal{M})$). Homogeneous models are known to exist (cf. Sacks [25, Ch. 20]).

1.3. Imaginaries and interpretations. A *definable equivalence relation* on a definable set \mathbf{X} is a definable subset of $\mathbf{X} \times \mathbf{X}$ that determines an equivalence relation in any model. The theory *eliminates imaginaries* if any equivalence relation has a quotient, i.e., any equivalence relation can be represented as $f(x) = f(y)$ for some definable function f on \mathbf{X} .

If \mathcal{T} eliminates imaginaries, an *interpretation* i of another theory \mathcal{T}_0 in \mathcal{T} specified by giving, for each sort \mathbf{X}_0 of \mathcal{T}_0 a definable set $\mathbf{X} = i(\mathbf{X}_0)$ of \mathcal{T} , and for each atomic relation $\mathbf{Y}_0 \subseteq \mathbf{X}_0$ of \mathcal{T}_0 a definable subset $\mathbf{Y} = i(\mathbf{Y}_0)$ of \mathbf{X} , such that for any model \mathcal{M} of \mathcal{T} , the sets $\mathbf{X}(\mathcal{M})$ form a model \mathcal{M}_0 of \mathcal{T}_0 (when interpreted as a \mathcal{T}_0 structure in the obvious way). It follows that any definable set \mathbf{Z}_0 of \mathcal{T}_0 determines a definable set $i(\mathbf{Z}_0)$ in \mathcal{T} , and \mathcal{M}_0 can be viewed as a subset of \mathcal{M} (there is a universal interpretation of any theory \mathcal{T} in a theory \mathcal{T}^{eq} that eliminates imaginaries. An interpretation of \mathcal{T}_0 in a general theory \mathcal{T} is then defined to be an interpretation of \mathcal{T}_0 in \mathcal{T}^{eq} in the sense already defined.)

If \mathcal{M} is a model of a theory \mathcal{T} , and $\mathbf{A} \subseteq \mathcal{M}$, we denote by $\mathcal{T}_{\mathbf{A}}$ the theory obtained by adding constants for \mathbf{A} to the language, and the axioms satisfied for \mathbf{A} in \mathcal{M} (in particular, $\mathcal{T}_{\mathbf{A}}$ is complete). There is an obvious interpretation of \mathcal{T} in $\mathcal{T}_{\mathbf{A}}$. The model \mathcal{M} is in a natural way a model of $\mathcal{T}_{\mathbf{A}}$, and for any definable set \mathbf{X} of \mathcal{T} , $\mathbf{X}(\mathbf{A})$ in the sense of \mathcal{T} is identified with $\mathbf{X}(0)$ in $\mathcal{T}_{\mathbf{A}}$.

1.4. Internal covers. An interpretation i of \mathcal{T}_0 in \mathcal{T} is *stably embedded* if any subset of the sorts of \mathcal{T}_0 definable in \mathcal{T} with parameters from \mathcal{T} is also definable in \mathcal{T}_0 with parameters from \mathcal{T}_0 . More precisely, given definable sets \mathbf{Y}_0 in \mathcal{T}_0 , and \mathbf{X} and $\mathbf{Z} \subseteq \mathbf{X} \times i(\mathbf{Y}_0)$ in \mathcal{T} , there are definable sets \mathbf{W}_0 and $\mathbf{Z}_0 \subseteq \mathbf{W}_0 \times \mathbf{Y}_0$ in \mathcal{T}_0 , such that \mathcal{T} implies that for each $x \in \mathbf{X}$, $\mathbf{Z}_x = i(\mathbf{Z}_0)_w$ for some $w \in i(\mathbf{W}_0)$.

Definition 1.5. A stably embedded interpretation i of \mathcal{T}_0 in \mathcal{T} is an *internal cover* (of \mathcal{T}_0) if there is a stably embedded interpretation π of \mathcal{T} in \mathcal{T}_0 , such that $\pi \circ i$ is the identity.

The theory \mathcal{T} is also called an internal cover of \mathcal{T}_0 (if i is understood).

If \mathbf{Q} is a definable set in \mathcal{T} (an internal cover of \mathcal{T}_0), the identity map on $\pi(\mathbf{Q})$ comes, by assumption, from some bijection from \mathbf{Q} to $i(\pi(\mathbf{Q}))$ in \mathcal{T} , definable with parameters. Hence there is a set of parameters \mathbf{A} , such that $\text{dcl}(\mathcal{M}_0 \cup \mathbf{A}) = \mathcal{M}$ for any model \mathcal{M} of \mathcal{T} . Such a set \mathbf{A} will be called a set of *internality parameters*, and will always be taken to be definably closed. We denote by \mathbf{A}_0 the restriction of \mathbf{A} to \mathcal{T}_0 . (In the language of Hrushovski [9], this implies that the corresponding groupoid in \mathcal{T}_0 is equivalent to a groupoid with objects over \mathbf{A}_0 .)

We note that by compactness, a finite number of elements of \mathbf{A} suffices to define all elements of a given sort of \mathcal{T} over \mathcal{M}_0 . Thus, if \mathcal{T} has a finite number of sorts that do not come from \mathcal{T}_0 over which everything is definable, then \mathbf{A} can be taken to be generated by one tuple \mathbf{a} . In general, \mathbf{A} can be thought of as an element of a pro-definable set.

The following formulation of the internality theorem is closest in language and strength to the one in Hrushovski [9, prop. 1.5]. This is the language we would like to use later, but in the strength we actually require (namely, for ω -stable theories), it was already proved in Poizat [23].

Theorem 1.6 (Hrushovski [8, Appendix B], Hrushovski [9, prop. 1.5]). *Let \mathcal{T} be an internal cover of \mathcal{T}_0 . There is a pro-group \mathbf{G} in \mathcal{T} , together with a definable action $\mathfrak{m}_{\mathbf{Q}} : \mathbf{G} \times \mathbf{Q} \rightarrow \mathbf{Q}$ of \mathbf{G} on every definable set \mathbf{Q} of \mathcal{T} , such that for any model \mathcal{M} of \mathcal{T} , $\mathbf{G}(\mathcal{M})$ is identified with $\text{Aut}(\mathcal{M}/\mathcal{M}_0)$ through this action (with \mathcal{M}_0 the restriction of \mathcal{M} to \mathcal{T}_0).*

Furthermore, given a set of internality parameters \mathbf{A} (whose restriction to \mathcal{T}_0 is \mathbf{A}_0), there is a Galois correspondence between \mathbf{A} -definable pro-subgroups of \mathbf{G} and definably closed subsets $\mathbf{A}_0 \subseteq \mathbf{B} \subseteq \mathbf{A}$. Such a subset $\mathbf{B}_{\mathbf{H}}$ for a subgroup \mathbf{H} is

always of the form $\mathbf{C}(\mathcal{A})$, where \mathbf{C} is an \mathcal{A} -definable ind-set in \mathcal{T} , and \mathbf{H} is the subgroup of \mathbf{G} fixing $\mathbf{C}(\mathcal{M})$ pointwise. If \mathbf{H} is normal, then $\mathbf{G}/\mathbf{H}(\mathcal{M})$ is identified with $\text{Aut}(\mathbf{C}(\mathcal{M})/\mathcal{M}_0)$.

Here, a *pro-group* is a filtering inverse system of definable groups.

1.7. In fact, the result is slightly stronger. With notation as above, the assumption that \mathcal{T}_0 is stably embedded implies that any automorphism of \mathcal{M}_0 fixing \mathcal{A}_0 can be extended (uniquely) to an automorphism of \mathcal{M} fixing \mathcal{A} . In other words, we have a split exact sequence

$$0 \rightarrow \mathbf{G}(\mathcal{M}) \rightarrow \text{Aut}(\mathcal{M}/\mathcal{A}_0) \rightarrow \text{Aut}(\mathcal{M}_0/\mathcal{A}_0) \rightarrow 0 \quad (1)$$

where \mathbf{G} is as provided by Theorem 1.6.

More generally, we have the following interpretation of $\mathbf{G}(\mathcal{B})$.

Proposition 1.8. *Let $\mathcal{B}_0 \subseteq \mathcal{M}_0$ be a definably closed set containing \mathcal{A}_0 , and let $\mathcal{B} = \text{dcl}(\mathcal{A} \cup \mathcal{B}_0)$. Then $\mathbf{G}(\mathcal{B}) = \text{Aut}(\mathcal{B}/\mathcal{B}_0)$.*

Proof. Any $g \in \mathbf{G}(\mathcal{B})$ acts as an automorphism, preserves \mathcal{B} as a set (since \mathcal{B} is definably closed), and fixes \mathcal{B}_0 pointwise, so $\mathbf{G}(\mathcal{B}) \subseteq \text{Aut}(\mathcal{B}/\mathcal{B}_0)$.

Conversely, since \mathcal{B}_0 is definably closed and contains \mathcal{A}_0 , $\mathcal{B} \cap \mathcal{M} = \mathcal{B}_0$, and so any automorphism of \mathcal{B} over \mathcal{B}_0 extends to an automorphism of \mathcal{B} over \mathcal{M}_0 . Since \mathcal{M}_0 is stably embedded, this automorphism extends to an automorphism of \mathcal{M} . Thus, any element of $\text{Aut}(\mathcal{B}/\mathcal{B}_0)$ is represented by some $g \in \mathbf{G}(\mathcal{M})$. Since it is fixed by any automorphism fixing \mathcal{B} pointwise, it is in fact in $\mathbf{G}(\mathcal{B})$. \square

1.9. We make the assumption that a set \mathcal{A} as above can be found (in some model) such that the corresponding set \mathcal{A}_0 is (the definable closure of) the empty set. This is not a real assumption in the current context, since the results will hold in general for the theory with parameters from \mathcal{A} .

Likewise, the interpretation of \mathcal{T}_0 in \mathcal{T} factors through a maximal extension \mathcal{T}_1 of \mathcal{T}_0 (i.e., \mathcal{T}_1 is the theory, in \mathcal{T} , of the definable sets coming from \mathcal{T}_0). We assume from now on that $\mathcal{T}_1 = \mathcal{T}_0$.

Combining the two assumptions, we get that $\mathcal{B}_0 \mapsto \mathcal{B} = \text{dcl}(\mathcal{A} \cup \mathcal{B}_0)$ gives an equivalence between definably closed sets \mathcal{B}_0 of \mathcal{T}_0 and definably closed sets \mathcal{B} containing \mathcal{A} of \mathcal{T} . In particular, we have a definable group $\mathbf{G}_{\mathcal{A}}$ in \mathcal{T}_0 . The following proposition says that all definable group action of $\mathbf{G}_{\mathcal{A}}$ in \mathcal{T}_0 come from the canonical action of \mathbf{G} in \mathcal{T} .

Proposition 1.10. *Assume 1.9, and let $\alpha : \mathbf{G}_{\mathcal{A}} \times \mathbf{D} \rightarrow \mathbf{D}$ be a definable group action in \mathcal{T}_0 . There is a definable set $\mathbf{X}_{\mathbf{D}}$ in \mathcal{T} , and an \mathcal{A} -definable isomorphism of $\mathbf{G}_{\mathcal{A}}$ actions from $\mathbf{X}_{\mathbf{D}}$ to \mathbf{D} . If \mathbf{D} and \mathbf{E} are two such $\mathbf{G}_{\mathcal{A}}$ -sets, and $f : \mathbf{D} \rightarrow \mathbf{E}$ is a definable map of $\mathbf{G}_{\mathcal{A}}$ sets in \mathcal{T}_0 , it also comes from a definable map $F : \mathbf{X}_{\mathbf{D}} \rightarrow \mathbf{X}_{\mathbf{E}}$.*

Proof. The proof of Theorem 1.6 produces a definable \mathbf{G} -torsor \mathbf{X} in \mathcal{T} , which is 0-definable with our assumptions. There is an \mathcal{A} -definable isomorphism $f_b : \mathbf{G} \rightarrow \mathbf{G}_{\mathcal{A}}$, which, composed with α , defines an \mathcal{A} -definable action $c_b : \mathbf{G} \times \mathbf{D} \rightarrow \mathbf{D}$. We set $\mathbf{X}_{\mathbf{D}} = \mathbf{X} \times_{\mathbf{G}} \mathbf{D}$, i.e., $\mathbf{X}_{\mathbf{D}}$ is the set of pairs $(x, d) \in \mathbf{X} \times \mathbf{D}$, up to the equivalence $(gx, d) = (x, c_b(g, d))$ for $g \in \mathbf{G}$. Since $\mathbf{G}_{\mathcal{A}}$ is in \mathcal{T}_0 , $f_{hb}(hgh^{-1}) = f_b(g)$ for all $h, g \in \mathbf{G}$, so the equivalence relation is invariant under the action of \mathbf{G} , and hence $\mathbf{X}_{\mathbf{D}}$ is definable without parameters.

If $b \in \mathbf{X}(\mathcal{A})$ is any element (known to exist), the map $[b, d] \mapsto d$ is an isomorphism of $\mathbf{G}_{\mathcal{A}}$ actions from $\mathbf{X}_{\mathbf{D}}$ to \mathbf{D} . The proof for maps is similar. \square

2. TANNAKIAN CATEGORIES

In this section we review, without proofs, the definitions and basic properties of Tannakian categories. The proofs, as well as more details (originally from Saavedra-Rivano [24]) can be found in the first two sections of Deligne and Milne [4]. The section contains no new results (though the terminology is slightly different, and follows in part Mac Lane [12, ch. VII]).

Let \mathcal{G} be an affine algebraic group (or, more generally, an affine groups scheme) over a field \mathbb{k} . The category $\mathcal{R}\text{ep}_{\mathcal{G}}$ of finite-dimensional representations of \mathcal{G} over \mathbb{k} , admits, in addition to the category structure, a tensor product operation on the objects. With this structure, and with the forgetful functor into the category of vector spaces over \mathbb{k} it satisfies the axioms of a (neutralised) Tannakian category (see Example 2.7). The main theorem of Saavedra-Rivano [24] (Theorem 2.8) asserts that categories satisfying these axioms, summarised below, are in fact precisely of the form $\mathcal{R}\text{ep}_{\mathcal{G}}$, and that furthermore, the group \mathcal{G} can be recovered from the (tensor) category structure.

2.1. Monoidal categories. Recall that a *symmetric monoidal category* is given by a tuple $(\mathcal{C}, \otimes, \phi, \psi)$, where:

- (1) \mathcal{C} is a category
- (2) \otimes is a functor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, $(\mathbf{X}, \mathbf{Y}) \mapsto \mathbf{X} \otimes \mathbf{Y}$ (in other words, the operation is functorial in each coordinate separately).
- (3) ϕ is a collection of functorial isomorphisms

$$\phi_{\mathbf{X}, \mathbf{Y}, \mathbf{Z}} : (\mathbf{X} \otimes \mathbf{Y}) \otimes \mathbf{Z} \xrightarrow{\sim} \mathbf{X} \otimes (\mathbf{Y} \otimes \mathbf{Z})$$

one for each triple $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ of objects of \mathcal{C} . It is called the *associativity constraint*.

- (4) ψ is a collection of functorial isomorphisms $\psi_{\mathbf{X}, \mathbf{Y}} : \mathbf{X} \otimes \mathbf{Y} \xrightarrow{\sim} \mathbf{Y} \otimes \mathbf{X}$, called the *commutativity constraint*.

The commutativity constraint is required to satisfy $\psi_{\mathbf{X}, \mathbf{Y}} \circ \psi_{\mathbf{Y}, \mathbf{X}} = \text{id}_{\mathbf{X} \otimes \mathbf{Y}}$ for all objects \mathbf{X}, \mathbf{Y} . In addition, ϕ and ψ are required to satisfy certain “pentagon” and “hexagon” identities, which ensure that any two tensor expressions computed from the same set of objects are *canonically* isomorphic.

Finally, \mathcal{C} is required have an *identity object*: this is an object $\mathbf{1}$, together with an isomorphism $\mathbf{u} : \mathbf{1} \rightarrow \mathbf{1} \otimes \mathbf{1}$, such that $\mathbf{X} \mapsto \mathbf{1} \otimes \mathbf{X}$ is an equivalence of categories. It follows (Deligne and Milne [4, prop. 1.3]) that $(\mathbf{1}, \mathbf{u})$ is unique up to a unique isomorphism with the property that \mathbf{u} can be uniquely extended to isomorphisms $\mathbf{l}_{\mathbf{X}} : \mathbf{X} \rightarrow \mathbf{1} \otimes \mathbf{X}$, commuting with ϕ and ψ .

We will usually drop ϕ and ψ from the notation, and choose a particular identity object $\mathbf{1}$. We will also drop the adjective “symmetric”.

2.2. Rigidity. A monoidal category (\mathcal{C}, \otimes) is *closed* if for any two objects \mathbf{X} and \mathbf{Y} , there is an object $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ and a functorial isomorphism $\text{Hom}(\mathbf{Z} \otimes \mathbf{X}, \mathbf{Y}) \xrightarrow{\sim} \text{Hom}(\mathbf{Z}, \underline{\text{Hom}}(\mathbf{X}, \mathbf{Y}))$. In this case, the *dual* $\check{\mathbf{X}}$ of \mathbf{X} is defined to be $\underline{\text{Hom}}(\mathbf{X}, \mathbf{1})$. The symmetry constraint and the functoriality determine, for each object \mathbf{X} , a map $\mathbf{X} \rightarrow \check{\check{\mathbf{X}}}$ and for any four objects $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2$, a map

$$\underline{\text{Hom}}(\mathbf{X}_1, \mathbf{Y}_1) \otimes \underline{\text{Hom}}(\mathbf{X}_2, \mathbf{Y}_2) \rightarrow \underline{\text{Hom}}(\mathbf{X}_1 \otimes \mathbf{X}_2, \mathbf{Y}_1 \otimes \mathbf{Y}_2)$$

\mathcal{C} is said to be *rigid* if it is closed, and all these maps are isomorphisms (The latter requirement can be viewed as a formal analogue of the objects being finite-dimensional.)

We note that setting $\mathbf{Y}_1 = \mathbf{X}_2 = \mathbf{1}$ in the above map, we get an isomorphism $\check{\mathbf{X}} \otimes \mathbf{Y} \rightarrow \underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$. In particular, a map $f : \mathbf{X} \rightarrow \mathbf{Y}$ corresponds to a global section $\mathbf{1} \rightarrow \underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$, hence to a map $\check{f} : \mathbf{1} \rightarrow \check{\mathbf{X}} \otimes \mathbf{Y}$.

2.3. The rank of an object. Let $f : \mathbf{X} \rightarrow \mathbf{X}$ be an endomorphism in a rigid category. As above, it corresponds to a morphism $\check{f} : \mathbf{1} \rightarrow \check{\mathbf{X}} \otimes \mathbf{X}$. Composing with the commutativity followed by the evaluation we get a morphism $\mathrm{Tr}_{\mathbf{X}}(f) : \mathbf{1} \rightarrow \mathbf{1}$ called the *trace* of f . The trace map $\mathrm{Tr}_{\mathbf{X}} : \mathrm{End}(\mathbf{X}) \rightarrow \mathbb{k} := \mathrm{End}(\mathbf{1})$ is multiplicative (with respect to the tensor product), and $\mathrm{Tr}_{\mathbf{1}}$ is the identity. The *rank* of an object \mathbf{X} is defined to be $\mathrm{Tr}_{\mathbf{X}}(\mathrm{id}_{\mathbf{X}})$. Thus it is an element of \mathbb{k} .

2.4. Tensor categories. A *tensor category* is a rigid monoidal category \mathcal{C} which is abelian, with \otimes additive in each coordinate. It follows that \otimes is exact in each coordinate, that $\mathbb{k} = \mathrm{End}(\mathbf{1})$ is a commutative ring, that \mathcal{C} has a natural \mathbb{k} -linear structure, where \otimes is \mathbb{k} -bilinear, and $\mathrm{Tr}_{\mathbf{X}}$ is \mathbb{k} -linear.

Given an object \mathbf{X} of a tensor category \mathcal{C} , let $\mathcal{C}_{\mathbf{X}}$ be the full subcategory of \mathcal{C} whose objects are isomorphic to sub-quotients of finite sums of tensor powers of \mathbf{X} (including $\mathbf{X}^{\otimes -1} = \check{\mathbf{X}}$ and its powers). The object \mathbf{X} is called a *tensor generator* for \mathcal{C} if any object of \mathcal{C} is isomorphic to an object of $\mathcal{C}_{\mathbf{X}}$. For most of what follows we will restrict our attention to categories that have a tensor generator. In general, \mathcal{C} is a filtered limit of categories of this form.

2.5. Tensor functors. Let \mathcal{C} and \mathcal{D} be monoidal categories. A *tensor functor* from \mathcal{C} to \mathcal{D} is a pair (\mathcal{F}, χ) , where $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is a functor, and χ is a collection of isomorphisms $\chi_{\mathbf{X}, \mathbf{Y}} : \mathcal{F}(\mathbf{X}) \otimes \mathcal{F}(\mathbf{Y}) \xrightarrow{\sim} \mathcal{F}(\mathbf{X} \otimes \mathbf{Y})$, compatible with the constraints, such that (\mathcal{F}, χ) takes identity objects to identity objects.

When \mathcal{C} and \mathcal{D} are abelian, we assume \mathcal{F} to be additive. In this case, \mathcal{F} makes $\mathbb{k}_{\mathcal{D}}$ into a $\mathbb{k}_{\mathcal{C}}$ -algebra, and \mathcal{F} is automatically $\mathbb{k}_{\mathcal{C}}$ -linear, and in this sense preserves the trace. In particular, $\mathrm{rk}(\mathbf{X}) = \mathrm{rk}(\mathcal{F}(\mathbf{X}))$.

If (\mathcal{F}, χ) and (\mathcal{G}, δ) are two tensor functors from \mathcal{C} to \mathcal{D} , a map from (\mathcal{F}, χ) to (\mathcal{G}, δ) is a map of functors that commutes with χ and δ . We denote by $\mathrm{Aut}(\mathcal{F})$ the group of automorphisms of \mathcal{F} as a tensor functor.

2.6. Tannakian categories. A *neutral Tannakian category* over a field \mathbb{k} is a tensor category \mathcal{C} with $\mathrm{End}(\mathbf{1}) = \mathbb{k}$, which admits an exact tensor functor ω into the category $\mathcal{V}\mathrm{ec}_{\mathbb{k}}$ of finite-dimensional \mathbb{k} -vector spaces (with the usual tensor structure). Such a functor is called a *fibre functor*. It is automatically faithful, and is said to neutralise \mathcal{C} .

Given a fibre functor ω and a (commutative) \mathbb{k} -algebra A , the functor $\omega \otimes_{\mathbb{k}} A$ (with $(\omega \otimes_{\mathbb{k}} A)(\mathbf{X}) = \omega(\mathbf{X}) \otimes_{\mathbb{k}} A$) is again an exact tensor functor (into a tensor subcategory of the category $\mathcal{V}\mathrm{ec}_A$ of projective finitely presented A -modules). We thus get a functor $A \mapsto \mathrm{Aut}(\omega \otimes_{\mathbb{k}} A)$ from the category of \mathbb{k} -algebras to groups. This functor is denoted by $\underline{\mathrm{Aut}}_{\mathbb{k}}^{\otimes}(\omega)$.

Example 2.7 (representations of a group). If \mathcal{G} is an affine group scheme over \mathbb{k} , the category $\mathcal{C} = \mathcal{R}\mathrm{ep}_{\mathcal{G}}$ of (finite-dimensional) representations of \mathcal{G} over \mathbb{k} , with the usual tensor product (and constraints) forms a monoidal category. It is rigid, with

$\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ the space of all linear maps between \mathbf{X} and \mathbf{Y} , and is abelian, hence it is a tensor category. The trace of a map coincides with the image in \mathbb{k} of the usual trace. Finally, \mathcal{C} is neutralised by the forgetful functor ω .

If A is a \mathbb{k} -algebra, and $\mathfrak{g} \in \mathcal{G}(A)$, the action of \mathcal{G} exhibits \mathfrak{g} as an automorphism of $\omega \otimes_{\mathbb{k}} A$. This determines a map (of functors) $\mathcal{G} \rightarrow \underline{\text{Aut}}_{\mathbb{k}}^{\otimes}(\omega)$. The main theorem states that this map is an isomorphism, and that this example is the most general one:

Theorem 2.8 (Saavedra). *Let \mathcal{C} be a neutral Tannakian category, and let $\omega : \mathcal{C} \rightarrow \text{Vec}_{\mathbb{k}}$ be a fibre functor.*

- (1) *The functor $\underline{\text{Aut}}_{\mathbb{k}}^{\otimes}(\omega)$ is representable by an affine group scheme \mathcal{G} over \mathbb{k} .*
- (2) *The fibre functor ω factors through a tensor equivalence $\omega : \mathcal{C} \rightarrow \text{Rep}_{\mathcal{G}}$.*
- (3) *If $\mathcal{C} = \text{Rep}_{\mathcal{H}}$ for some affine group scheme \mathcal{H} , then the natural map $\mathcal{H} \rightarrow \mathcal{G}$ given in Example 2.7 is an isomorphism.*

In the following section we present a proof of a slightly weaker statement (as explained in the introduction), using the model theoretic tools of Section 1.

3. THE THEORY ASSOCIATED WITH A TENSOR CATEGORY

In this section, we associate a theory with any Tannakian category, and use Theorem 1.6 to prove the main Theorem 2.8, in the case that the base field has characteristic 0 (and with the other caveats mentioned in the introduction).

3.1. Let $(\mathcal{C}, \otimes, \phi, \psi)$ be a tensor category over a field \mathbb{k} of characteristic 0, such that the rank of every object is a natural number. Let \mathcal{K} be an extension field of \mathbb{k} . The theories $\mathcal{T}_{\mathcal{C}}$ and $\widetilde{\mathcal{T}}_{\mathcal{C}}$ (depending on \mathcal{K} , which is omitted from the notation) are defined as follows:

3.1.1. $\mathcal{T}_{\mathcal{C}}$ has a sort \mathbf{L} , as well as a sort $V_{\mathbf{X}}$ for every object \mathbf{X} of \mathcal{C} , and a function symbol $v_f : V_{\mathbf{X}} \rightarrow V_{\mathbf{Y}}$ for any morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$. There are a binary operation symbol $+_{\mathbf{X}}$ on each $V_{\mathbf{X}}$, and a function symbol $\cdot_{\mathbf{X}} : \mathbf{L} \times V_{\mathbf{X}} \rightarrow V_{\mathbf{X}}$ for each object \mathbf{X} . The sort \mathbf{L} contains a constant symbol for each element of \mathcal{K} .

3.1.2. The theory says that \mathbf{L} is an algebraically closed field, whose field operations are given by $+_1$ and \cdot_1 . The restriction of the operation to the constant symbols is given by the field structure on \mathcal{K} .

It also says that $+_{\mathbf{X}}$ and $\cdot_{\mathbf{X}}$ determine a vector space structure over \mathbf{L} on every $V_{\mathbf{X}}$, and $V_{\mathbf{X}}$ has dimension $\text{rk}(\mathbf{X})$ over \mathbf{L} . Each v_f is an \mathbf{L} -linear map.

3.1.3. The \mathbb{k} -linear category structure is reflected in the theory: $v_{\text{id}_{\mathbf{X}}}$ is the identity map for each \mathbf{X} , $v_f = 0$ if and only if $f = 0$, $v_{f \circ g} = v_f \circ v_g$, if $\mathfrak{a} \in \mathbb{k}$ is viewed as an element of $\text{End}(\mathbf{X})$, then $v_{\mathfrak{a}}$ is multiplication by $\mathfrak{a} \in \mathcal{K}$. Also, $v_{f+g} = v_f + v_g$, and v_f is injective or surjective if and only if f is.

3.1.4. For any two objects \mathbf{X} and \mathbf{Y} there is a further function symbol $\mathfrak{b}_{\mathbf{X}, \mathbf{Y}} : V_{\mathbf{X}} \times V_{\mathbf{Y}} \rightarrow V_{\mathbf{X} \otimes \mathbf{Y}}$. The theory says that $\mathfrak{b}_{\mathbf{X}, \mathbf{Y}}$ is bilinear, and the induced map $V_{\mathbf{X}} \otimes_{\mathbf{L}} V_{\mathbf{Y}} \rightarrow V_{\mathbf{X} \otimes \mathbf{Y}}$ is an isomorphism. Here $\otimes_{\mathbf{L}}$ is the usual tensor product of vector spaces over \mathbf{L} (in a given model). This statement is first order, since the spaces are finite-dimensional.

The theory also says that $v_{\psi_{\mathbf{X}, \mathbf{Y}}}(\mathfrak{b}_{\mathbf{X}, \mathbf{Y}}(x, y)) = \mathfrak{b}_{\mathbf{Y}, \mathbf{X}}(y, x)$, and similarly for ϕ .

If $u \in V_{\mathbf{X}}$ and $v \in V_{\mathbf{Y}}$ (either terms or elements in some model), we write $u \otimes v$ for $\mathfrak{b}_{\mathbf{X}, \mathbf{Y}}(u, v)$.

3.1.5. The theory $\widetilde{\mathcal{T}}_{\mathcal{C}}$ is the expansion of $\mathcal{T}_{\mathcal{C}}$ by an extra sort $P_{\mathbf{X}}$, for every object \mathbf{X} , together with a surjective map $\pi_{\mathbf{X}} : V_{\mathbf{X}} \rightarrow P_{\mathbf{X}}$ identifying $P_{\mathbf{X}}$ with the projective space associated with $V_{\mathbf{X}}$. It also includes function symbols $p_f : P_{\mathbf{X}} \rightarrow P_{\mathbf{Y}}$ and $d_{\mathbf{X},\mathbf{Y}} : P_{\mathbf{X}} \times P_{\mathbf{Y}} \rightarrow P_{\mathbf{X} \otimes \mathbf{Y}}$ for any objects \mathbf{X}, \mathbf{Y} and morphism f of \mathcal{C} , and the theory says that they are the projectivisations of the corresponding maps v_f and $b_{\mathbf{X},\mathbf{Y}}$.

Proposition 3.2. *The theory $\mathcal{T}_{\mathcal{C}}$ is stable. In particular, \mathbf{L} is stably embedded. In each model, \mathbf{L} is a pure algebraically closed field (possibly with additional constants).*

Proof. A choice of basis for each vector space identifies it with a power of \mathbf{L} . All linear and bilinear maps are definable in the pure field structure on \mathbf{L} . Since \mathbf{L} is stable and stability is not affected by parameters, so is $\mathcal{T}_{\mathcal{C}}$. \square

Proposition 3.3. *The theories $\mathcal{T}_{\mathcal{C}}$ and $\widetilde{\mathcal{T}}_{\mathcal{C}}$ eliminate quantifiers (possibly after naming some constants). The theory $\widetilde{\mathcal{T}}_{\mathcal{C}}$ eliminates imaginaries.*

Proof. That $\widetilde{\mathcal{T}}_{\mathcal{C}}$ eliminates imaginaries is precisely the statement of Hrushovski [9, prop. 4.2] (see also the proof of Proposition 4.5.6). The proof of quantifier elimination is similar: since all sorts are interpretable (with parameters) in ACF, all definable sets are boolean combinations of Zariski-closed subsets. A Zariski-closed subset of $V_{\mathbf{X}}$ or $P_{\mathbf{X}}$ is the set of zeroes of polynomials, which are elements of the symmetric algebra on $\widetilde{V}_{\mathbf{X}}$. But $\widetilde{V}_{\mathbf{X}}$ is identified with $V_{\widetilde{\mathbf{X}}}$, and the tensor algebra on it, as well as the action on $V_{\mathbf{X}}$ or $P_{\mathbf{X}}$ is definable without quantifiers. \square

3.4. Models and fibre functors. Models of $\mathcal{T}_{\mathcal{C}}$ are essentially fibre functors over algebraically closed fields. More precisely, we have the following statements.

Proposition 3.4.1. *Let \mathcal{M} be a model of $\mathcal{T}_{\mathcal{C}}$, and let \mathcal{A} be a subset of \mathcal{M} such that for any object \mathbf{X} , $V_{\mathbf{X}}(\mathcal{A})$ contains a basis of $V_{\mathbf{X}}(\mathcal{M})$ over $\mathbf{L}(\mathcal{M})$. Then $\mathbf{X} \mapsto V_{\mathbf{X}}(\mathcal{A})$ determines a fibre functor over the field extension $\mathbf{L}(\mathcal{A})$ of \mathbf{K} .*

Proof. We first note that since (by definition) $\mathbf{L}(\mathcal{A})$ is definably closed, it is indeed a field. Likewise, since each $V_{\mathbf{X}}(\mathcal{A})$ is definably closed, it is a vector space over $\mathbf{L}(\mathcal{A})$, and the $V_{\mathbf{X}}(\mathcal{A})$ are closed under the linear maps that come from \mathcal{C} . Thus $\mathbf{X} \mapsto V_{\mathbf{X}}(\mathcal{A})$ defines a \mathbf{k} -linear functor from \mathcal{C} to the category of vector spaces over $\mathbf{L}(\mathcal{A})$.

The map $b_{\mathbf{X},\mathbf{Y}}$ determines a map $c_{\mathbf{X},\mathbf{Y}} : V_{\mathbf{X}}(\mathcal{A}) \otimes_{\mathbf{L}(\mathcal{A})} V_{\mathbf{Y}}(\mathcal{A}) \rightarrow V_{\mathbf{X} \otimes \mathbf{Y}}(\mathcal{A})$. Since the map $V_{\mathbf{X}}(\mathcal{A}) \otimes_{\mathbf{L}(\mathcal{A})} V_{\mathbf{Y}}(\mathcal{A}) \rightarrow V_{\mathbf{X}}(\mathcal{M}) \otimes_{\mathbf{L}(\mathcal{M})} V_{\mathbf{Y}}(\mathcal{M})$ is injective, so is $c_{\mathbf{X},\mathbf{Y}}$, and so the whole statement will follow from knowing that the dimension of $V_{\mathbf{X}}(\mathcal{A})$ over $\mathbf{L}(\mathcal{A})$ is $\text{rk}(\mathbf{X})$. Since $V_{\mathbf{X}}(\mathcal{A})$ contains a basis over $\mathbf{L}(\mathcal{M})$, we know that it is at least $\text{rk}(\mathbf{X})$.

For any $\mathbf{a}_1, \dots, \mathbf{a}_n \in V_{\mathbf{X}}(\mathcal{A})$, the set of all $(x_1, \dots, x_n) \in \mathbf{L}^n$ such that $\sum x_i \mathbf{a}_i = 0$ is an \mathcal{A} -definable subspace of \mathbf{L}^n . Since \mathbf{L} is stably embedded, it is $\mathbf{L}(\mathcal{A})$ definable. Hence it has an $\mathbf{L}(\mathcal{A})$ -definable basis (Milne [16, Lemma 4.10]). This shows that the dimension is correct. \square

Proposition 3.4.2. *Let \mathcal{A} be a subset of a model \mathcal{M} of $\mathcal{T}_{\mathcal{C}}$ such that $\text{dcl}(\mathcal{A} \cup \mathbf{L}(\mathcal{M})) = \mathcal{M}$, and such that $\mathbf{L}(\mathcal{A})$ is algebraically closed. Then $\text{dcl}(\mathcal{A})$ is a model. In particular, $\mathbf{X} \mapsto V_{\mathbf{X}}(\mathcal{A})$ determines a fibre functor for \mathcal{C} over $\mathbf{L}(\mathcal{A})$.*

Proof. We only need to prove that each $V_{\mathbf{X}}(\mathcal{A})$ contains a basis over $\mathbf{L}(\mathcal{M})$. By assumption, there is an \mathcal{A} -definable map f from a subset \mathbf{U} of \mathbf{L}^m to $V_{\mathbf{X}}^n$, where

$n = \text{rk}(\mathbf{X})$, such that $f(\mathbf{u})$ is a basis of $V_{\mathbf{X}}(\mathcal{M})$ for all $\mathbf{u} \in \mathbf{U}(\mathcal{M})$. Since \mathbf{U} is $\mathbf{L}(\mathcal{A})$ -definable, and $\mathbf{L}(\mathcal{A})$ is algebraically closed, $\mathbf{U}(\mathcal{A})$ contains a point \mathbf{u} , and $f(\mathbf{u})$ is thus a basis in $V_{\mathbf{X}}(\mathcal{A})$. \square

Proposition 3.4.3. *Conversely, if \mathcal{K}_1 is a field extension of \mathcal{K} , $\omega : \mathcal{C} \rightarrow \text{Vec}_{\mathcal{K}_1}$ is a fibre functor, and $\bar{\mathcal{K}}$ is an algebraically closed field containing \mathcal{K}_1 , the assignment $V_{\mathbf{X}} \mapsto \omega(\mathbf{X}) \otimes_{\mathcal{K}_1} \bar{\mathcal{K}}$ determines a model \mathcal{M}_{ω} of $\mathcal{T}_{\mathcal{C}}$, such that $\mathcal{A} = \omega \subseteq \mathcal{M}$ satisfies $\text{dcl}(\mathcal{A} \cup \mathbf{L}(\mathcal{M}_{\omega})) = \mathcal{M}_{\omega}$, and $\mathbf{L}(\mathcal{A}) = \mathcal{K}_1$.*

In particular, $\mathcal{T}_{\mathcal{C}}$ is consistent if and only if \mathcal{C} is has a fibre functor.

Proof. Since $\bar{\omega} = \omega \otimes_{\mathcal{K}_1} \bar{\mathcal{K}}$ is itself a fibre functor, it is clear that \mathcal{M}_{ω} is a model ($\mathbf{b}_{\mathbf{X}, \mathbf{Y}}$ is defined to be the composition of the canonical pairing $\bar{\omega}(\mathbf{X}) \times \bar{\omega}(\mathbf{Y}) \rightarrow \bar{\omega}(\mathbf{X}) \otimes_{\bar{\mathcal{K}}} \bar{\omega}(\mathbf{Y})$ with the structure map $\mathbf{c}_{\mathbf{X}, \mathbf{Y}} : \bar{\omega}(\mathbf{X}) \otimes_{\bar{\mathcal{K}}} \bar{\omega}(\mathbf{Y}) \rightarrow \bar{\omega}(\mathbf{X} \otimes \mathbf{Y})$). Also, it is clear that $\text{dcl}(\mathcal{A} \cup \mathbf{L}(\mathcal{M}_{\omega})) = \mathcal{M}_{\omega}$. To prove that $\mathbf{L}(\mathcal{A}) = \mathcal{K}_1$, we note that any automorphism of $\bar{\mathcal{K}}$ over \mathcal{K}_1 extends to an automorphism of the model by acting the second term. \square

3.5. From now on we assume that \mathcal{C} has a fibre functor ω , and denote the theory of the model \mathcal{M}_{ω} constructed in Proposition 3.4.3 by \mathcal{T}_{ω} (so \mathcal{T}_{ω} is a completion of $\mathcal{T}_{\mathcal{C}}$).

We also assume that the category \mathcal{C} has a tensor generator (2.4). The general results are obtained as a limit of this case, in the usual way. (This is not really necessary, since, as mentioned in 1.4, the theory works in general, but it makes it easier to apply standard model theoretic results about definable, rather than pro-definable set.) We fix a tensor generator \mathbf{Q} .

Claim 3.6. *The theory $\mathcal{T}_{\mathcal{C}}$ is an internal cover of \mathbf{L} , viewed as an interpretation of $\text{ACF}_{\mathcal{K}}$.*

Proof. \mathbf{L} is stably embedded in $\mathcal{T}_{\mathcal{C}}$ since $\mathcal{T}_{\mathcal{C}}$ is stable. Given a model \mathcal{M} of $\mathcal{T}_{\mathcal{C}}$, any element of $V_{\mathbf{X}}(\mathcal{M})$ is a linear combination of elements of a basis \mathbf{a} for $V_{\mathbf{X}}(\mathcal{M})$ over $\mathbf{L}(\mathcal{M})$, hence is definable over \mathbf{a} and $\mathbf{L}(\mathcal{M})$. We note that if \mathbf{Q} is a tensor generator, then a basis for $V_{\mathbf{Q}}$ determines a basis for any other $V_{\mathbf{X}}$ (This is clear for tensor products, direct sums, duals and quotients. For a sub-object $\mathbf{Y} \rightarrow \mathbf{X}$, the set of all tuples \mathbf{x} in \mathbf{L} such that $\sum x_i \mathbf{a}_i \in V_{\mathbf{Y}}$ is \mathbf{a} -definable, hence has a point in $\mathbf{L}(\mathbf{a})$.) \square

3.7. **Proof of 2.8(1).** Let \mathbf{G} be the definable group in $\mathcal{T}_{\mathcal{C}}$ corresponding to it as an as an internal cover of \mathbf{L} (Theorem 1.6). By Proposition 3.4.3, ω determines a definably closed subset \mathcal{A} of an arbitrarily large model \mathcal{M} , such that $\text{dcl}(\mathcal{A} \cup \mathbf{L}(\mathcal{M})) = \mathcal{M}$ and $\mathbf{L}(\mathcal{A}) = \mathbb{k}$. Given a field extension \mathcal{K} , we may assume that $\mathbf{L}(\mathcal{M})$ contains \mathcal{K} .

For each object \mathbf{X} , $V_{\mathbf{X}}(\mathcal{A} \cup \mathcal{K}) = \omega(\mathbf{X}) \otimes_{\mathbb{k}} \mathcal{K}$ (again by Proposition 3.4.3), It follows by Proposition 1.8 that for such a field \mathcal{K} , $\mathbf{G}(\text{dcl}(\mathcal{A} \cup \mathcal{K})) = \text{Aut}(\text{dcl}(\mathcal{A} \cup \mathcal{K})/\mathcal{K})$, and since by the definition of the theory, such automorphisms are the same as automorphisms of $\omega \otimes_{\mathbb{k}} \mathcal{K}$ as a tensor functor, we get that $\mathbf{G}(\text{dcl}(\mathcal{A} \cup \mathcal{K})) = \underline{\text{Aut}}^{\otimes}(\omega)(\mathcal{K})$.

On the other hand, \mathcal{A} satisfies Assumption 1.9, hence we get a definable group $\mathbf{G}_{\mathcal{A}}$ in \mathbf{L} (i.e., in ACF), such that $\mathbf{G}_{\mathcal{A}}(\mathcal{K}) = \mathbf{G}(\text{dcl}(\mathcal{A} \cup \mathcal{K}))$. By Poizat [22, § 4.5], any such group is algebraic. \square

Remark 3.8. By inspecting the proof of Proposition 1.8 and the construction of \mathbf{G} in this particular case, it is easy to extend the above proof to the case when \mathcal{K} is

an integral domain. However, I don't know how to use the same argument for more general \mathcal{K} .

3.9. Proof of 2.8(3). Let $\mathcal{C} = \mathcal{R}ep_{\mathcal{H}}$, where \mathcal{H} is an algebraic group over \mathbb{k} , which we view as a definable group \mathbf{H} in $\text{ACF}_{\mathbb{k}}$. Let \mathbf{G} be the definable internality group in \mathcal{T}_{ω} and \mathcal{A} the definable subset corresponding to the (forgetful) fibre functor ω , all as in 3.7. We obtain a definable group $\mathbf{G}_{\mathcal{A}}$, and the action of \mathcal{H} on its representations determines a map of group functors from \mathbf{H} to $\mathbf{G}_{\mathcal{A}}$, which by Beth definability is definable and therefore algebraic. This homomorphism is injective (on points in any field extension) since \mathbf{H} has a faithful representation. We identify \mathbf{H} with its image in $\mathbf{G}_{\mathcal{A}}$.

By the Galois correspondence (Theorem 1.6), to prove that $\mathbf{H} = \mathbf{G}_{\mathcal{A}}$ it is enough to show that any \mathcal{A} -definable element in $\mathcal{T}_{\mathcal{C}}^{\text{eq}}$ fixed by $\mathbf{H}(\mathcal{M})$ is also fixed by $\mathbf{G}_{\mathcal{A}}(\mathcal{M})$, i.e., is 0-definable. By Proposition 3.3, $\mathcal{T}_{\mathcal{C}}^{\text{eq}} = \tilde{\mathcal{T}}_{\mathcal{C}}$. If $v \in V_{\mathbf{X}}(\mathcal{A})$ is fixed by \mathbf{H} , then the map $\mathbf{L} \rightarrow V_{\mathbf{X}}$ given by $1 \mapsto v$ is a map of \mathbf{H} representations, hence is given by a function symbol f , so $v = f(1)$ is 0-definable. Likewise, if $\mathfrak{p} \in \mathcal{P}_{\mathbf{X}}(\mathcal{A})$ is fixed by \mathbf{H} , then it corresponds to a sub-representation $\mathfrak{l}_{\mathfrak{p}} \subset V_{\mathbf{X}}$ (over \mathbb{k}), hence is given by a predicate, so again \mathfrak{p} is 0-definable.

This proves that the map from \mathbf{H} to $\mathbf{G}_{\mathcal{A}}$ is bijective on points in any field. Any such algebraic map (in characteristic 0) is an isomorphism (Waterhouse [26, § 11.4]). \square

3.10. Proof of 2.8(2) (sketch). Let $\mathcal{T}_{\mathcal{C}}$, \mathbf{G} , \mathcal{A} and $\mathbf{G}_{\mathcal{A}}$ be as above, and let $\mathcal{V}ec_{\mathbf{L}}$ be the category of definable \mathbf{L} -vector spaces interpretable in $\mathcal{T}_{\mathcal{C}}$, with definable linear maps between them as morphisms. This is clearly a tensor category, and $\mathbf{X} \mapsto V_{\mathbf{X}}$ is an exact faithful \mathbb{k} -linear tensor functor from \mathcal{C} to $\mathcal{V}ec_{\mathbf{L}}$.

On the other hand, a representation of $\mathbf{G}_{\mathcal{A}}$ is given by some definable action in ACF , and by Proposition 1.10, this action comes from some action of \mathbf{G} on a definable set in $\mathcal{T}_{\mathcal{C}}$, which must therefore be a vector space over \mathbf{L} . Conversely, each linear action of \mathbf{G} on an object in $\mathcal{V}ec_{\mathbf{L}}$ gives a representation of $\mathbf{G}_{\mathcal{A}}$ after taking \mathcal{A} -points, and similarly for morphisms. Thus, we get an equivalence (of \mathbb{k} -linear tensor categories) between $\mathcal{V}ec_{\mathbf{L}}$ and $\mathcal{R}ep_{\mathbf{G}_{\mathcal{A}}}$.

It remains to show that the functor $\mathbf{X} \mapsto V_{\mathbf{X}}$ from \mathcal{C} to $\mathcal{V}ec_{\mathbf{L}}$ is full and surjective on isomorphism classes, i.e., that any definable \mathbf{L} -vector space is essentially of the form $V_{\mathbf{X}}$, and that any map between is of the form v_f . To prove the first, we consider the group \mathbf{G} and its torsor \mathbf{X} . These are defined by some 0-definable subspaces of powers of $V_{\mathcal{Q}}$ (where \mathcal{Q} is a tensor generator.) By examining the explicit definition of \mathbf{G} and \mathbf{X} , one concludes that these 0-definable subspaces in fact come from \mathcal{C} . We omit the proof, since it is very similar to the original algebraic one. Once this is known, given any other representation \mathbf{D} , the set $\mathbf{X}_{\mathbf{D}}$ in Proposition 1.10 is explicitly given by tensor operations, and so comes from \mathcal{C} as well. Then proof for morphisms is again similar. \square

Corollary 3.11. *Quantifier elimination in $\mathcal{T}_{\mathcal{C}}$ holds without parameters: any definable set is quantifier-free without parameters.*

Proof. This holds anyway with parameters from the definable closure of the empty set. We now know that any such parameter is given by a term in the language. \square

Remark 3.12. In Hrushovski [9], an equivalence is constructed between internal covers of a complete theory \mathcal{T} , and connected definable groupoids in \mathcal{T} . It follows

from Theorem 2.8 there that if $\mathcal{T}_{\mathcal{C}}$ is consistent and the induced structure on \mathbf{L} is precisely $\mathrm{ACF}_{\mathbb{k}}$, then the associated groupoid is connected. It also follows that $\mathcal{T}_{\mathcal{C}}$ itself is complete. It then follows from lemma 2.4 in the same paper that any two models of $\mathcal{T}_{\mathcal{C}}$ are isomorphic. Given two fibre functors of \mathcal{C} (over some extension field), extending both to a model of $\mathcal{T}_{\mathcal{C}}$ thus shows that the fibre functors are locally isomorphic. Conversely, if $\mathcal{T}_{\mathcal{C}}$ induces new structure on \mathbf{L} , it is easy to construct fibre functors which are not locally isomorphic (cf. Deligne and Milne [4, § 3.15]).

The main result of Deligne [3] can be viewed as stating that the induced structure on \mathbf{L} is indeed $\mathrm{ACF}_{\mathbb{k}}$. More precisely, given a fibre functor ω over an (affine) scheme \mathcal{S} over \mathbb{k} , Deligne constructs a groupoid scheme $\underline{\mathrm{Aut}}_{\mathbb{k}}^{\otimes}(\omega)$, with object scheme \mathcal{S} . The main result of Deligne [3] (Theorem 1.12) states that this groupoid scheme is faithfully flat over $\mathcal{S} \times \mathcal{S}$ (hence connected). A \mathcal{K} -valued point of \mathcal{S} (for some field \mathcal{K}) determines a fibre functor $\omega_{\mathcal{K}}$ over \mathcal{K} . Viewing this fibre functor as subset of a model, a choice of a basis for a tensor generator determines a type over \mathbf{L} , and therefore an object of the groupoid corresponding to $\mathcal{T}_{\mathcal{C}}$ (choosing a different basis amounts to choosing a different object which is isomorphic over \mathcal{K} in the groupoid). It is clear from definition of $\underline{\mathrm{Aut}}_{\mathbb{k}}^{\otimes}(\omega)$ that this process gives an equivalence from this groupoid to the groupoid associated with the internal cover $\mathcal{T}_{\mathcal{C}}$. Thus, Deligne's theorem implies that this groupoid is connected, and therefore that the induced structure on \mathbf{L} is that of $\mathrm{ACF}_{\mathbb{k}}$. It seems plausible that there should be a direct model theoretic proof of this result, but I could not find it. We note that the non-existence of new structure also directly implies (by the existence of prime models) Deligne [3, corollary 6.20], which states that \mathcal{C} has a fibre functor over the algebraic closure of \mathbb{k} .

We note also that any definable groupoid in $\mathrm{ACF}_{\mathbb{k}}$ is equivalent to a groupoid scheme (any such groupoid is equivalent to one with a finite set of objects, and the automorphism group of each object is algebraic). The category \mathcal{C} of representations of the (gerb associated with the) groupoid is a tensor category satisfying $\mathrm{End}(\mathbf{1}) = \mathbb{k}$ (Deligne and Milne [4, section 3]), and it is easy to see (using the same methods as in 3.9) that the groupoid associated with the internal cover $\mathcal{T}_{\mathcal{C}}$ is equivalent to the original one. Therefore, we obtain that any internal cover of $\mathrm{ACF}_{\mathbb{k}}$ has (up to equivalence) the form $\mathcal{T}_{\mathcal{C}}$ for some Tannakian category over \mathbb{k} .

Remark 3.13. We interpret model theoretically two additional result of Deligne [3]. In section 7, it is shown that in our context (i.e., $\mathrm{char}(\mathbb{k}) = 0$ and $\mathrm{rk}(\mathbf{X})$ is a natural number), \mathcal{C} always has a fibre functor. In light of Proposition 3.4.3, this result asserts that $\mathcal{T}_{\mathcal{C}}$ is consistent for any such \mathcal{C} .

In section 8, Deligne defines, for a Tannakian category \mathcal{C} (and even somewhat more generally), a \mathcal{C} -group $\pi(\mathcal{C})$, called the *fundamental group* of \mathcal{C} (a \mathcal{C} -group is a commutative Hopf algebra object in $\mathrm{Ind}(\mathcal{C})$; similarly for affine \mathcal{C} -schemes). The basic idea is that the concepts associated with neutral Tannakian categories, also make sense for “fibre functors” into other tensor categories, and $\pi(\mathcal{C})$ is obtain by reconstructing a group from the identity functor.

Thus, $\pi(\mathcal{C})$ comes with an action on each object, commuting with all morphisms. It thus represents the tensor automorphisms of the identity functor from \mathcal{C} to itself, in the sense that for any affine \mathcal{C} -scheme \mathbf{X} , the action identifies $\pi(\mathcal{C})(\mathbf{X})$ with the group of tensor automorphisms of the functor $\mathbf{A} \mapsto \mathbf{A} \otimes \mathbf{X}$ (from \mathcal{C} to the tensor category of vector bundles over \mathbf{X}). It has the property that for any “real” fibre

functor ω over a scheme \mathcal{S} , applying ω to the action identifies $\omega(\pi(\mathcal{C}))$ with the group scheme $\underline{\text{Aut}}_{\mathcal{S}}^{\otimes}(\omega)$ constructed.

Using either the explicit construction, or the properties above, it is clear that in terms of the theory $\mathcal{T}_{\mathcal{C}}$, $\pi(\mathcal{C})$ is nothing but the definable automorphism group (more precisely, the Hopf algebra object that defines it maps to an ind-definable Hopf algebra in $\mathcal{T}_{\mathcal{C}}$, which is definably isomorphic to the ind-definable Hopf algebra of functions on the internality group).

Remark 3.14. One can recover in a similar manner (and somewhat more easily) Grothendieck’s approach to usual Galois theory (cf. Grothendieck et al. [6, Exposé V.4]). Briefly, given a category \mathcal{C} with a “fibre functor” \mathcal{F} into the category of finite sets, satisfying conditions (G1)–(G6), one constructs as above a theory $\mathcal{T}_{\mathcal{C}}$ with a sort $V_{\mathbf{X}}$ for each object, and a function symbol for each morphism. $\mathcal{T}_{\mathcal{C}}$ is then the theory of \mathcal{F} viewed as a structure in this language. Since every sort is finite, they are all internal to $\mathbf{2}$ (the co-product of the terminal object $\mathbf{1}$ with itself). Conditions (G1)–(G5) ensure that any definable set in fact comes from \mathcal{C} , and (G2), (G5) ensure that $\mathcal{T}_{\mathcal{C}}$ has elimination of imaginaries. The Galois objects P_i that appear in the proof are precisely the 1-types of internality parameters that appear in the model theoretic construction of Galois group.

4. DIFFERENTIAL TANNAKIAN CATEGORIES

Our purpose in this section is to define differential tensor categories, and to give a model theoretic proof of the basic theorem, corresponding such categories, endowed with a suitably defined fibre functor, with linear differential algebraic groups. The method is completely analogous to that in the previous section.

Throughout this section, \mathbb{k} is a field of characteristic 0 .

4.1. Prolongations of abelian categories. We assume that in a tensor category (\mathcal{C}, \otimes) , the functor \otimes is exact; this is automatic if \mathcal{C} is rigid (see Deligne and Milne [4, prop. 1.16].)

Definition 4.1.1. Let \mathcal{C} be a \mathbb{k} -linear category. The *prolongation* $\mathcal{P}(\mathcal{C})$ of \mathcal{C} is defined as follows: The objects are exact sequences $\mathbf{X} := 0 \rightarrow \mathbf{X}_0 \xrightarrow{i_{\mathbf{X}}} \mathbf{X}_1 \xrightarrow{\pi_{\mathbf{X}}} \mathbf{X}_0 \rightarrow 0$ of \mathcal{C} , and the morphisms between such objects are morphisms of exact sequences whose two \mathbf{X}_0 parts coincide.

An exact functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ gives rise to an induced functor $\mathcal{P}(F) : \mathcal{P}(\mathcal{C}_1) \rightarrow \mathcal{P}(\mathcal{C}_2)$. We denote by Π_i ($i = 0, 1$) the functors from $\mathcal{P}(\mathcal{C})$ to \mathcal{C} assigning \mathbf{X}_i to the object $0 \rightarrow \mathbf{X}_0 \xrightarrow{i_{\mathbf{X}}} \mathbf{X}_1 \xrightarrow{\pi_{\mathbf{X}}} \mathbf{X}_0 \rightarrow 0$ of $\mathcal{P}(\mathcal{C})$ (thus there is an exact sequence $0 \rightarrow \Pi_0 \xrightarrow{i_{\Pi}} \Pi_1 \xrightarrow{\pi_{\Pi}} \Pi_0 \rightarrow 0$.) $\Pi_i(\mathbf{X})$ is also abbreviated as \mathbf{X}_i , and \mathbf{X} is said to be over \mathbf{X}_0 (and similarly for morphisms.)

Remark 4.1.2. We note that $\mathcal{P}(\mathcal{C})$ can be viewed as the full subcategory of the category of “differential objects” in \mathcal{C} , consisting of objects whose homology is 0 . A differential object is a pair (\mathbf{X}, ϕ) where \mathbf{X} is an object of \mathcal{C} and ϕ is an endomorphism of \mathbf{X} with $\phi^2 = 0$. A morphism is a morphism in \mathcal{C} that commutes with ϕ , and the homology is $\ker(\phi)/\text{im}(\phi)$. This is the same as the category of $\mathbb{k}[\epsilon]$ -modules in \mathcal{C} , in the sense of Deligne and Milne [4, p. 155] (where $\epsilon^2 = 0$). The advantage of this category is that it is again \mathbb{k} -linear. However, I don’t know how to extend the tensor structure (defined below) to this whole category (in particular, the tensor structure defined there does not seem to coincide with ours).

4.1.3. Let \mathbf{A} and \mathbf{B} be two objects over \mathbf{X}_0 . Their *Yoneda sum* $\mathbf{A} \star \mathbf{B}$ is a new object over \mathbf{X}_0 , defined as follows (this is the addition in Yoneda's description of $\text{Ext}^1(\mathbf{X}_0, \mathbf{X}_0)$): the combined map $\mathbf{X}_0 \times \mathbf{X}_0 \rightarrow \mathbf{A}_1 \times \mathbf{B}_1$ factors through $\mathbf{A}_1 \times_{\mathbf{X}_0} \mathbf{B}_1$, and together with the map $\mathbf{X}_0 \xrightarrow{1, -1} \mathbf{X}_0 \times \mathbf{X}_0$ gives rise to a map $f : \mathbf{X}_0 \rightarrow \mathbf{A}_1 \times_{\mathbf{X}_0} \mathbf{B}_1$. Let \mathbf{W}_1 be the co-kernel of this map. The map f composed with the projection from $\mathbf{A}_1 \times_{\mathbf{X}_0} \mathbf{B}_1$ to \mathbf{X}_0 is 0, so we obtain an induced map $p : \mathbf{W}_1 \rightarrow \mathbf{X}_0$. The diagonal inclusion Δ of \mathbf{X}_0 in \mathbf{W}_1 together with p give rise to an exact sequence $0 \rightarrow \mathbf{X}_0 \xrightarrow{\Delta} \mathbf{W}_1 \xrightarrow{p} \mathbf{X}_0 \rightarrow 0$, which is the required object.

For any object \mathbf{A} of $\mathcal{P}(\mathcal{C})$, we denote by $\mathsf{T}(\mathbf{A})$ the object obtained by negating all arrows that appear in \mathbf{A} .

4.1.4. *Tensor structure.* Let $(\mathcal{C}, \otimes, \phi_0, \psi_0)$ be a tensor category. An object \mathbf{X}_0 of \mathcal{C} gives rise to a functor from $\mathcal{P}(\mathcal{C})$ to itself, by tensoring the exact sequence pointwise. Since we assumed \otimes to be exact, this functor also commutes with Yoneda sums: $(\mathbf{A} \star \mathbf{B}) \otimes \mathbf{X}_0$ is canonically isomorphic with $(\mathbf{A} \otimes \mathbf{X}_0) \star (\mathbf{B} \otimes \mathbf{X}_0)$. Also, $\mathsf{T}(\mathbf{A}) \otimes \mathbf{X}_0$ is isomorphic to $\mathsf{T}(\mathbf{A} \otimes \mathbf{X}_0)$.

We endow $\mathcal{P}(\mathcal{C})$ with a monoidal structure. The tensor product $\mathbf{A} \otimes \mathbf{B}$ of the two $\mathcal{P}(\mathcal{C})$ objects \mathbf{A} and \mathbf{B} is defined as follows: After tensoring the first with \mathbf{B}_0 and the second with \mathbf{A}_0 , we obtain two objects over $\mathbf{A}_0 \otimes \mathbf{B}_0$. We now take their Yoneda sum.

We shall make use of the following exact sequence.

Lemma 4.1.5. *For any two objects \mathbf{A} and \mathbf{B} of $\mathcal{P}(\mathcal{C})$, there is an exact sequence*

$$0 \rightarrow (\mathbf{A} \otimes \mathsf{T}(\mathbf{B}))_1 \xrightarrow{i} \mathbf{A}_1 \otimes \mathbf{B}_1 \xrightarrow{\pi} (\mathbf{A} \otimes \mathbf{B})_1 \rightarrow 0 \quad (2)$$

where π is the quotient of the map obtained from the maps $\pi_{\mathbf{A}} \otimes 1$ and $1 \otimes \pi_{\mathbf{B}}$, and i is the restriction of the map obtained from the maps $i_{\mathbf{A}} \otimes 1$ and $-1 \otimes i_{\mathbf{B}}$.

Proof. Exactness in the middle follows directly from the definitions. We prove that π is surjective, the injectivity of i being similar. We shall use the Mitchell embedding theorem (cf. Freyd [5]), which reduces the question to the case of abelian groups.

We in fact prove that already the map

$$\mathbf{A}_1 \otimes \mathbf{B}_1 \xrightarrow{\pi} \mathbf{A}_0 \otimes \mathbf{B}_1 \times_{\mathbf{A}_0 \otimes \mathbf{B}_0} \mathbf{A}_1 \otimes \mathbf{B}_0 =: \mathbf{U}$$

is surjective. Let y be an element of \mathbf{U} , and let y_1 and y_2 be its two projections to the components of \mathbf{U} . Since the map $\mathbf{A}_1 \otimes \mathbf{B}_1 \xrightarrow{\pi_{\mathbf{A}} \otimes 1} \mathbf{A}_0 \otimes \mathbf{B}_1$ is surjective, y_1 can be lifted to an element \widetilde{y}_1 of $\mathbf{A}_1 \otimes \mathbf{B}_1$. We have that

$$(\pi_{\mathbf{A}} \otimes 1)((1 \otimes \pi_{\mathbf{B}})(\widetilde{y}_1)) = (1 \otimes \pi_{\mathbf{B}})((\pi_{\mathbf{A}} \otimes 1)(\widetilde{y}_1)) = (1 \otimes \pi_{\mathbf{B}})(y_1) = (\pi_{\mathbf{A}} \otimes 1)(y_2)$$

Let $z = (1 \otimes \pi_{\mathbf{B}})(\widetilde{y}_1) - y_2$. Since z is killed by $\pi_{\mathbf{A}} \otimes 1$, it comes from an element, also z , of $\mathbf{A}_0 \otimes \mathbf{B}_0$. Let \tilde{z} be a lifting of z to $\mathbf{A}_0 \otimes \mathbf{B}_1$, and denote by \tilde{z} also its image in $\mathbf{A}_1 \otimes \mathbf{B}_1$ under the inclusion $i_{\mathbf{A}} \otimes 1$. Then $\widetilde{y}_1 - \tilde{z}$ is a lifting of y . \square

4.1.6. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be three objects of $\mathcal{P}(\mathcal{C})$. The associativity constraint ϕ_0 of \mathcal{C} gives rise to an isomorphism of $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$ with the quotient of

$$\mathbf{A}_1 \otimes \mathbf{B}_0 \otimes \mathbf{C}_0 \times_{\mathbf{A}_0 \otimes \mathbf{B}_0 \otimes \mathbf{C}_0} \mathbf{A}_0 \otimes \mathbf{B}_1 \otimes \mathbf{C}_0 \times_{\mathbf{A}_0 \otimes \mathbf{B}_0 \otimes \mathbf{C}_0} \mathbf{A}_0 \otimes \mathbf{B}_0 \otimes \mathbf{C}_1$$

that identifies the three natural inclusions of $\mathbf{A}_0 \otimes \mathbf{B}_0 \otimes \mathbf{C}_0$, and similarly for $\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$. We thus get an associativity constraint ϕ on $\mathcal{P}(\mathcal{C})$, over ϕ_0 .

Likewise, the commutativity constraint ψ_0 induces a commutativity constraint ψ on $\mathcal{P}(\mathcal{C})$ over ψ_0 .

Proposition 4.1.7. *The data $(\mathcal{P}(\mathcal{C}), \otimes, \phi, \psi)$ as defined above forms a symmetric monoidal category, and Π_0 is a monoidal functor. It is rigid if \mathcal{C} is rigid.*

Proof. We define the additional data. Verification of the axioms reduces, as in Lemma 4.1.5, to the case of abelian groups, where it is easy.

Let $u : \mathbf{1}_0 \rightarrow \mathbf{1}_0 \otimes \mathbf{1}_0$ be an identity object of \mathcal{C} . We set $\mathbf{1} = 0 \rightarrow \mathbf{1}_0 \rightarrow \mathbf{1}_0 \oplus \mathbf{1}_0 \rightarrow \mathbf{1}_0 \rightarrow 0$. For any object \mathbf{A} of $\mathcal{P}(\mathcal{C})$, $\mathbf{1} \otimes \mathbf{A}$ is identified via u with

$$0 \rightarrow \mathbf{A}_0 \rightarrow (\mathbf{A}_1 \times_{\mathbf{A}_0} (\mathbf{A}_0 \oplus \mathbf{A}_0)) / \mathbf{A}_0 \rightarrow \mathbf{A}_0 \rightarrow 0$$

This object is canonically isomorphic (over \mathcal{C}) to \mathbf{A} , and so $\mathbf{1}$ acquires a structure of an identity object.

Assume that \mathcal{C} is rigid. For an object \mathbf{A} of $\mathcal{P}(\mathcal{C})$, we set $\check{\mathbf{A}}$ to be the dual exact sequence $0 \rightarrow \check{\mathbf{A}}_0 \xrightarrow{i_{\check{\mathbf{A}}}} \check{\mathbf{A}}_1 \xrightarrow{\pi_{\check{\mathbf{A}}}} \check{\mathbf{A}}_0 \rightarrow 0$. We define an evaluation map $\mathbf{A} \otimes \check{\mathbf{A}} \rightarrow \mathbf{1}$ as follows: We need to define two maps from $\mathbf{A}_0 \otimes \check{\mathbf{A}}_1 \times \mathbf{A}_1 \otimes \check{\mathbf{A}}_0$ to $\mathbf{1}_0$, that agree on the two inclusions of $\mathbf{A}_0 \otimes \check{\mathbf{A}}_0$, and such that the resulting map restricts to the evaluation on $\mathbf{A}_0 \otimes \check{\mathbf{A}}_0$.

To construct the first map, we consider the exact sequence (2), for $\mathbf{B} = \check{\mathbf{A}}$. We claim that the evaluation map on $\mathbf{A}_1 \otimes \check{\mathbf{A}}_1$ restricts to 0 when composed with i . To prove this, it is enough to show that the pair of maps obtained from $\text{ev}_{\mathbf{A}_1}$ by composition with $i_{\mathbf{A}} \otimes 1$ and $-1 \otimes i_{\check{\mathbf{A}}}$ comes from a map $\mathbf{A}_0 \otimes \check{\mathbf{A}}_0 \rightarrow \mathbf{1}_0$. However, under the adjunction, this pair of maps corresponds to $(i_{\mathbf{A}}, -\pi_{\mathbf{A}})$, and so comes from the identity map on \mathbf{A}_0 . It follows that $\text{ev}_{\mathbf{A}_1}$ induces a map on $(\mathbf{A} \otimes \check{\mathbf{A}})_1$, which is the required map. The second map is obtained by projecting to $\mathbf{A}_0 \otimes \check{\mathbf{A}}_0$, and using the evaluation map on \mathbf{A}_0 . By definition, this second map commutes with the projections to $\mathbf{A}_0 \otimes \check{\mathbf{A}}_0$ and the second coordinate of $\mathbf{1}$, restricting to the evaluation on \mathbf{A}_0 . To prove that the first map restricts to the evaluation as well, we note that there is a commutative diagram

$$\begin{array}{ccc} (\mathbf{A} \otimes \check{\mathbf{A}})_1 & \xrightarrow{i} & \mathbf{A}_1 \otimes \check{\mathbf{A}}_1 \\ \downarrow \pi_{\mathbf{A} \otimes \check{\mathbf{A}}} & & \downarrow \pi \\ \mathbf{A}_0 \otimes \check{\mathbf{A}}_0 & \xrightarrow{i_{\mathbf{A} \otimes \check{\mathbf{A}}}} & (\mathbf{A} \otimes \check{\mathbf{A}})_1 \end{array}$$

where i is the (restriction of the) map obtained from the two maps $i_{\mathbf{A}} \otimes 1$ and $1 \otimes i_{\check{\mathbf{A}}}$.

Since $\pi_{\mathbf{A} \otimes \check{\mathbf{A}}}$ is surjective, it is therefore enough to prove that the maps $\text{ev}_{\mathbf{A}_1} \circ i$ and $\text{ev}_{\mathbf{A}_0} \circ \pi_{\mathbf{A}}$ coincide. This is indeed the case, since they both correspond to the inclusion of \mathbf{A}_0 in \mathbf{A}_1 . \square

4.2. Differential tensor categories.

Definition 4.2.1. A *differential structure* on a tensor category \mathcal{C} is a tensor functor \mathcal{D} from \mathcal{C} to $\mathcal{P}(\mathcal{C})$ which is a section of Π_0 . If \mathcal{D}_1 and \mathcal{D}_2 are two differential structures on \mathcal{C} , a morphism from \mathcal{D}_1 to \mathcal{D}_2 is a morphism of tensor functors that induces the identity morphism under Π_0 . A *differential tensor category* is a tensor category together with a differential structure.

Let \mathcal{D} be a differential structure on \mathcal{C} . Since \mathcal{D} is a section of Π_0 , it is determined by $\partial = \Pi_1 \circ \mathcal{D}$. In other words, on the abelian level, it is given by a functor $\partial : \mathcal{C} \rightarrow \mathcal{C}$, together with an exact sequence $0 \rightarrow \text{Id} \rightarrow \partial \rightarrow \text{Id} \rightarrow 0$. However, this description does not include the tensor structure. We also note that ∂ is necessarily exact.

4.2.2. Let $(\mathcal{C}, \mathcal{D})$ be a differential tensor category, let $\partial = \Pi_1 \circ \mathcal{D}$, and let $A = \text{End}(\mathbf{1})$. Recall that for any object \mathbf{X} , $\text{End}(\mathbf{X})$ is an A -algebra. The functor ∂ defines another ring homomorphism $\partial_1 : A \rightarrow \text{End}(\partial(\mathbf{1}))$. Given $\mathbf{a} \in A$, the morphism $\partial_1(\mathbf{a}) - \mathbf{a}$ in $\text{End}(\partial(\mathbf{1}))$ restricts to 0 on $\mathbf{1}$, and thus induces a morphism from $\mathbf{1}$ to $\partial(\mathbf{1})$. Similarly, its composition with the projection $\partial(\mathbf{1}) \rightarrow \mathbf{1}$ is 0, so it factors through $\mathbf{1}$. We thus get a new element \mathbf{a}' of A .

Claim 4.2.3. *The map $\mathbf{a} \mapsto \mathbf{a}'$ of 4.2.2 is a derivation on A .*

Proof. We need to show that given elements $\mathbf{a}, \mathbf{b} \in A$, the maps $\partial(\mathbf{a}\mathbf{b}) - \mathbf{a}\mathbf{b}$ and $(\partial(\mathbf{a}) - \mathbf{a})\mathbf{b} + \mathbf{a}(\partial(\mathbf{b}) - \mathbf{b})$ coincide on $\mathbf{1}$. This follows from the formula $\partial(\mathbf{a}\mathbf{b}) - \mathbf{a}\mathbf{b} = \partial(\mathbf{a})(\partial(\mathbf{b}) - \mathbf{b}) + (\partial(\mathbf{a}) - \mathbf{a})\mathbf{b}$, together with the fact that $\partial(\mathbf{a})(\partial(\mathbf{b}) - \mathbf{b})$ induces $\mathbf{a}(\partial(\mathbf{b}) - \mathbf{b})$ on $\mathbf{1}$. \square

Example 4.2.4. Let \mathcal{C} be the tensor category $\text{Vec}_{\mathbb{k}}$ of finite dimensional vector spaces over a field \mathbb{k} . Given a derivative $'$ on \mathbb{k} , we construct a differential structure on \mathcal{C} as follows: For a vector space \mathbf{X} , define $\mathbf{d}(\mathbf{X}) = \mathfrak{D} \otimes \mathbf{X}$, where \mathfrak{D} is the vector space with basis $\mathbf{1}, \partial$, and \otimes is the tensor product with respect to the *right* vector space structure on \mathfrak{D} , given by $\mathbf{1} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{1}$ and $\partial \cdot \mathbf{a} = \mathbf{a}' \cdot \mathbf{1} + \mathbf{a} \cdot \partial$. The exact sequence $\mathcal{D}(\mathbf{X})$ is defined by $x \mapsto \mathbf{1} \otimes x$, $\mathbf{1} \otimes x \mapsto 0$ and $\partial \otimes x \mapsto x$, for any $x \in \mathbf{X}$. If $T : \mathbf{X} \rightarrow \mathbf{Y}$ is a linear map, $\mathbf{d}(T) = \mathbf{1} \otimes T$. We shall write x for $\mathbf{1} \otimes x$ and ∂x for $\partial \otimes x$. The structure of a tensor functor is obtained by sending $\partial(x \otimes y) \in \mathbf{d}(\mathbf{X} \otimes \mathbf{Y})$ to the image of $\partial(x) \otimes y \oplus x \otimes \partial(y)$ in $(\mathcal{D}(\mathbf{X}) \otimes \mathcal{D}(\mathbf{Y}))_1$.

Claim 4.2.5. *The constructions in 4.2.4 and in 4.2.2 give a bijective correspondence between derivatives on \mathbb{k} and isomorphism classes of differential structures on $\text{Vec}_{\mathbb{k}}$.*

Proof. If \mathcal{D}_1 and \mathcal{D}_2 are two differential structures, then $\mathcal{D}_1(\mathbf{1}_0)$ and $\mathcal{D}_2(\mathbf{1}_0)$ are both identity objects, and are therefore canonically isomorphic to the same object $\mathbf{1}$. If \mathcal{D}_1 and \mathcal{D}_2 are isomorphic, then the maps $\mathbf{d}_i : \text{End}(\mathbf{1}_0) \rightarrow \text{End}(\mathbf{1})$ are conjugate, and therefore equal, since $\text{End}(\mathbf{1})$ is commutative.

It is clear from the definition that the derivative on \mathbb{k} obtained from the differential structure associated with a derivative is the original one. Conversely, if \mathcal{D}_1 and \mathcal{D}_2 are two differential structures that give the same derivative on \mathbb{k} , then we may identify $\mathcal{D}_1(\mathbf{1}_0)$ and $\mathcal{D}_2(\mathbf{1}_0)$. Under this identification, we get that the maps \mathbf{d}_i are the same. But the functors \mathcal{D}_i are determined by \mathbf{d}_i . \square

4.2.6. We now come to the definition of functors between differential tensor categories. For simplicity, we shall only define (and use) *exact* such functors.

Let $\omega : \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between abelian categories. There is an induced functor $\mathcal{P}(\omega) : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{P}(\mathcal{D})$, given by applying ω to each term. If \mathcal{C} and \mathcal{D} are tensor categories, the structure of a tensor functor on ω gives rise to a similar structure on $\mathcal{P}(\omega)$ (again, since ω is exact.) If $t : \omega_1 \rightarrow \omega_2$ is a (tensor) morphism of functors, we likewise get an induced morphism $\mathcal{P}(t) : \mathcal{P}(\omega_1) \rightarrow \mathcal{P}(\omega_2)$.

Definition 4.2.7. Let $(\mathcal{C}_1, \mathcal{D}_1)$ and $(\mathcal{C}_2, \mathcal{D}_2)$ be two differential tensor categories.

- (1) A *differential tensor functor* from \mathcal{C}_1 to \mathcal{C}_2 is an exact tensor functor ω from \mathcal{C}_1 to \mathcal{C}_2 , together with an isomorphism of tensor functors $r : \mathcal{P}(\omega) \circ \mathcal{D}_1 \rightarrow \mathcal{D}_2 \circ \omega$, such that the “restriction” $\Pi_0 \odot r : \omega \rightarrow \omega$, obtained by composing with Π_0 on $\mathcal{P}(\mathcal{C}_2)$, is the identity.
- (2) A morphism between two such differential tensor functors (ω_1, r_1) and (ω_2, r_2) is a morphism t between them as tensor functors such that the following diagram (of tensor functors and tensor maps between them) commutes:

$$\begin{array}{ccc}
 \mathcal{P}(\omega_1) \circ \mathcal{D}_1 & \xrightarrow{r_1} & \mathcal{D}_2 \circ \omega_1 \\
 \downarrow \mathcal{P}(t) \odot \mathcal{D}_1 & & \downarrow \mathcal{D}_2 \odot t \\
 \mathcal{P}(\omega_2) \circ \mathcal{D}_1 & \xrightarrow{r_2} & \mathcal{D}_2 \circ \omega_2
 \end{array} \tag{3}$$

where $\mathcal{D}_2 \odot t$ is the map from $\mathcal{D}_2 \circ \omega_1$ to $\mathcal{D}_2 \circ \omega_2$ obtained by applying \mathcal{D}_2 to t “pointwise”.

4.2.8. Given a differential tensor functor ω , we denote by $\text{Aut}^\partial(\omega)$ the group of automorphisms of ω .

If \mathcal{C} is a differential tensor category, and $\mathbb{k} = \text{End}(\mathbf{1})$ is a field, a \mathbb{k} -linear differential tensor functor ω into $\mathcal{V}\text{ec}_{\mathbb{k}}$ (with the induced differential structure) is called a *differential fibre functor*. Analogously to the algebraic case, we will say that \mathcal{C} is a *neutral differential Tannakian category* if such a fibre functor exists, and that ω neutralises \mathcal{C} . Since we will not define more general differential Tannakian categories, the adjective “neutral” will be dropped.

As in the algebraic case, given a differential fibre functor ω on \mathcal{C} , we denote by $\underline{\text{Aut}}^\partial(\omega)$ the functor from differential \mathbb{k} -algebras to groups assigning to a differential algebra A the group $\text{Aut}^\partial(A \otimes \omega)$.

4.2.9. Given a \mathbb{k} vector space V , the map $d : V \rightarrow \mathfrak{D} \hat{\otimes} V$ given by $v \mapsto \partial v$ is a derivation, in the sense that $d(av) = a'v + a d(v)$ (where V is identified with its image in $\mathfrak{D} \hat{\otimes} V$.) It is universal for this property: any pair $(i, d) : V \rightarrow W$, where i is linear, and d is a derivation with respect to i factors through it.

Therefore, a fibre functor on $(\mathcal{C}, \mathcal{D})$ is a fibre functor ω in the sense of tensor categories, together with a functorial derivation $d_{X_0} : \omega(X_0) \rightarrow \omega(\partial X_0)$ (where $\partial X_0 = \mathcal{D}(X_0)_1$), satisfying the Leibniz rule with respect to the tensor product (and additional conditions). The condition that the restriction to ω is the identity corresponds to the derivation being relative to the canonical inclusion of $\omega(X_0)$ in $\omega(\partial X_0)$ given by the differential structure.

Similarly, a differential automorphism of ω is an automorphism t of ω as a tensor functor, with the additional condition that for any object X_0 , the diagram

$$\begin{array}{ccc}
 \omega(X_0) & \xrightarrow{d_{X_0}} & \omega(\partial X_0) \\
 \downarrow t_{X_0} & & \downarrow t_{\partial X_0} \\
 \omega(X_0) & \xrightarrow{d_{X_0}} & \omega(\partial X_0)
 \end{array} \tag{4}$$

commutes. Thus the condition (3) really is about preservation of the differentiation.

4.2.10. *Derivations.* More generally, we define a *derivation* on an object \mathbf{X}_0 of \mathcal{C} to be a morphism $d : \mathbf{X}_1 := \partial(\mathbf{X}_0) \rightarrow \mathbf{X}_0$ such that the composition $\mathbf{X}_0 \rightarrow \mathbf{X}_1 \xrightarrow{d} \mathbf{X}_0$ is the identity.

Given two derivations $d_X : \mathbf{X}_1 \rightarrow \mathbf{X}_0$ and $d_Y : \mathbf{Y}_1 \rightarrow \mathbf{Y}_0$, we define the combined derivation $d = d_X \otimes d_Y : (\mathbf{X} \otimes \mathbf{Y})_1 = (\mathbf{X}_0 \otimes \mathbf{Y}_0)_1 \rightarrow \mathbf{X}_0 \otimes \mathbf{Y}_0$ to be the restriction of $d_X \otimes \text{Id} \oplus \text{Id} \otimes d_Y$ to $(\mathbf{X} \otimes \mathbf{Y})_1$ (this makes sense, since both are the identity on $\mathbf{X}_0 \otimes \mathbf{Y}_0$). In $\mathcal{V}ec_{\mathbb{k}}$ this corresponds to a real derivation, in the sense that it gives a map $d_0 : \mathbf{X}_0 \otimes \mathbf{Y}_0 \rightarrow \mathbf{X}_0 \otimes \mathbf{Y}_0$ which is a derivation over \mathbb{k} and also $d_0(x \otimes y) = d_X(x) \otimes y + x \otimes d_Y(y)$.

There is a canonical derivation d_1 on $\mathbf{1}_0$ given by the first projection $\mathbf{1}_1 = \mathbf{1}_0 \oplus \mathbf{1}_0 \rightarrow \mathbf{1}_0$. It has the property that $d_1 \otimes d = d \otimes d_1$ for any derivation d on any \mathbf{X}_0 , under the canonical identification of $\mathbf{1} \otimes \mathbf{X}_0$ and $\mathbf{X}_0 \otimes \mathbf{1}$ with \mathbf{X}_0 .

4.3. Differential algebraic groups. In this sub-section, we recall and summarise some definitions and basic facts from differential algebraic geometry (developed by Kolchin [11]) and linear differential algebraic groups (studied by Cassidy [1]). We show that the category of differential representations of such a group is a differential tensor category in our sense.

4.3.1. Let \mathcal{K} be a differential field (i.e., a field endowed with a derivation). We recall (Kolchin [11]) that a *Kolchin closed* subset (of a affine n -space) is given by a collection of polynomial (ordinary) differential equations in variables x_1, \dots, x_n , i.e., polynomial equations in variables $\delta^i(x_j)$, for $i \geq 0$. Such a collection determines a set of points (solutions) in any differential field extension of \mathcal{K} . As with algebraic varieties, it is possible to study these sets by considering points in a fixed field, provided that it is *differentially closed*. The Kolchin closed sets form a basis of closed sets for a Noetherian topology. Morphisms are also given by differential polynomials. A differential algebraic group is a group object in this category.

More generally, it is possible to consider a differential \mathcal{K} -algebra, i.e., a \mathcal{K} -algebra with a vector field extending the derivation on \mathcal{K} , and develop these notions there.

4.3.2. By a *linear* differential algebraic group, we mean a differential algebraic group which is represented by a differential Hopf algebra. A differential algebraic group which is affine as a differential algebraic variety need not be linear in this sense, since a morphism of affine differential varieties need not correspond to a map of differential algebras. Any linear differential algebraic group has a faithful representation. All these results appear in Cassidy [1], along with an example of an affine non-linear group. In Cassidy [2] it is shown that any representation of a *linear* group (and more generally, any morphism of linear groups) does correspond to a map of differential algebras.

4.3.3. *Differential representations.* Let \mathcal{G} be a linear differential algebraic group over a differential field \mathbb{k} . A representation of \mathcal{G} is given by a finite dimensional vector space V over \mathbb{k} , together with a morphism $\mathcal{G} \rightarrow \text{GL}(V)$. A map of representations is a linear transformation that gives a map of group representations for each differential \mathbb{k} -algebra. The category of all such representations is denoted $\mathcal{R}ep_{\mathcal{G}}$.

We endow $\mathcal{R}ep_{\mathcal{G}}$ with a differential structure in the same way as for vector spaces. If V is a representation of \mathcal{G} , assigning gv to (g, v) , then the action of \mathcal{G} on $\mathcal{D} \otimes V$ is given by $(g, x \otimes v) \mapsto x \otimes gv$. With this differential structure, the forgetful functor ω into $\mathcal{V}ec_{\mathbb{k}}$ has an obvious structure of a differential tensor functor.

A differential automorphism \mathbf{t} of ω is given by a collection of vector space automorphisms \mathbf{t}_V , for any representation V of G . The commutativity condition (4) above translates to the condition that $\mathbf{t}_{\mathcal{D} \otimes V} = \mathbf{1} \otimes \mathbf{t}_V$.

In particular, given a differential \mathbb{k} -algebra \mathbf{A} , and $\mathbf{g} \in \mathcal{G}(\mathbf{A})$, action by \mathbf{g} gives an automorphism of $\mathbf{A} \otimes \omega$ as a differential tensor functor, since the action of \mathbf{g} on $\mathcal{D} \otimes V$ is deduced from its action on V . Thus we get a map $\mathcal{G} \rightarrow \mathcal{G}_\omega$. We shall prove in Theorem 4.5.5 that the map is an isomorphism.

Example 4.3.4. Let \mathcal{G}_m be the (differential) multiplicative group, and let $\overline{\mathcal{G}_m}$ be the multiplicative group of the constants (thus, as differential varieties, \mathcal{G}_m is given by the equation $xy = 1$, and $\overline{\mathcal{G}_m}$ is the subvariety given by $x' = 0$.) There is a differential algebraic group homomorphism dlog from \mathcal{G}_m to \mathcal{G}_a (the additive group), sending x to x'/x , and $x \mapsto x'$ is a differential algebraic group endomorphism of \mathcal{G}_a . Let V be the standard 2-dimensional algebraic representation of \mathcal{G}_a . Using dlog and the derivative, we get for any $i \geq 0$ a 2-dimensional irreducible representation V_i of \mathcal{G}_m , which are all unrelated in terms of the tensor structure (and unrelated with the non-trivial 1-dimensional algebraic representations of \mathcal{G}_m .)

However, if \mathbf{X} is the \mathcal{G}_m representation corresponding to the identity map on \mathcal{G}_m , an easy calculation shows that V_0 is isomorphic to $\partial\mathbf{X} \otimes \check{\mathbf{X}}$. Similarly, V_{i+1} is a quotient of ∂V_i .

The inclusion of $\overline{\mathcal{G}_m}$ in \mathcal{G}_m gives a functor from $\text{Rep}_{\mathcal{G}_m}$ to $\text{Rep}_{\overline{\mathcal{G}_m}}$. But in $\text{Rep}_{\overline{\mathcal{G}_m}}$, V_0 is isomorphic to $\mathbf{1} \oplus \mathbf{1}$ (and $\partial\mathbf{X}$ to $\mathbf{X} \oplus \mathbf{X}$.)

4.4. Differential schemes in \mathcal{C} . We define affine differential schemes in a differential tensor category, and show that any object can be viewed as a “differential affine space”. This is analogous to the notion of \mathcal{C} -schemes for tensor categories that appears in Deligne [3]. The main application is the proof of elimination of imaginaries in Proposition 4.5.6.

4.4.1. We recall that the \otimes operation on \mathcal{C} extends canonically to $\text{Ind}(\mathcal{C})$, making it again an abelian monoidal category. The prolongation $\mathcal{P}(\text{Ind}(\mathcal{C}))$ can be identified with $\text{Ind}(\mathcal{P}(\mathcal{C}))$, and a differential structure on \mathcal{C} thus extends canonically to a differential structure on $\text{Ind}(\mathcal{C})$.

Recall (Deligne [3, § 7.5]) that if \mathcal{C} is a tensor category, a *ring* in $\text{Ind}(\mathcal{C})$ is an object \mathbf{A} of $\text{Ind}(\mathcal{C})$ together with maps $\mathbf{m} : \mathbf{A} \otimes \mathbf{A} \rightarrow \mathbf{A}$ and $\mathbf{u} : \mathbf{1} \rightarrow \mathbf{A}$ satisfying the usual axioms.

Definition 4.4.2. Let $(\mathcal{C}, \mathcal{D})$ be a differential tensor category, and let \mathbf{A} be a commutative ring in $\text{Ind}(\mathcal{C})$. A *vector field* on \mathbf{A} is a derivation on \mathbf{A} in the sense of 4.2.10, which commutes with the product, and which restricts to the canonical derivation \mathbf{d}_1 on $\mathbf{1}$. A *differential algebra* in $\text{Ind}(\mathcal{C})$ is a commutative ring in $\text{Ind}(\mathcal{C})$ together with a vector field. An (affine) *differential scheme* in \mathcal{C} is a differential algebra in $\text{Ind}(\mathcal{C})$, viewed as an object in the opposite category.

4.4.3. *Higher derivations.* Let \mathbf{X}_0 be an object of a differential tensor category $(\mathcal{C}, \mathcal{D})$. As explained above, $\mathcal{D}(\mathbf{X}_0)$ can be viewed as representing a universal derivation on \mathbf{X}_0 . We now construct the analogue of higher derivatives. More precisely, we define, by induction for each $n \geq 0$, the following data:

- (1) An object \mathbf{X}_n (of \mathcal{C})
- (2) A map $\mathbf{q}_n : \mathbf{X}_{n-1} \rightarrow \mathbf{X}_n$
- (3) A map $\mathbf{t}_n : \partial\mathbf{X}_{n-1} \rightarrow \mathbf{X}_n$

such that $q_{n+1} \circ t_n = t_{n+1} \circ \partial(q_n)$. In the context of $\mathcal{V}ec_k$, the data can be thought of as follows: \mathbf{X}_n is the space of expressions $v_0 + \partial v_1 + \cdots + \partial^n v_n$ with $v_j \in \mathbf{X}_0$; q_n the inclusion of elements as above; the map t_n the linear map corresponding to the derivation.

For the base, setting $\mathbf{X}_{-1} = 0$ determines all the data in an obvious way. Given \mathbf{X}_n , $t_{n+1} : \partial(\mathbf{X}_n) \rightarrow \mathbf{X}_{n+1}$ is defined to be the co-equaliser of the following two maps:

$$\begin{array}{ccc}
 & \mathbf{X}_n & \\
 t_n \nearrow & & \searrow i \\
 \partial(\mathbf{X}_{n-1}) & \xrightarrow{\partial(q_n)} & \partial(\mathbf{X}_n)
 \end{array} \tag{5}$$

Where i is part of the structure of $\mathcal{D}(\mathbf{X}_n)$. The map q_{n+1} is the composition $t_{n+1} \circ i$. Clearly the commutativity condition is satisfied. We note that the two objects we denote by \mathbf{X}_1 coincide, and the map q_1 coincides with the map i for $\mathcal{D}(\mathbf{X}_0)$. The map t_1 is the identity.

Definition 4.4.4. Let \mathbf{X}_0 be an object of \mathcal{C} . The *differential scheme associated with \mathbf{X}_0* , denoted $\mathbb{A}(\mathbf{X}_0)$, is a differential scheme in \mathcal{C} defined as follows: Let \mathbf{D} be the ind-object defined by the system $\check{\mathbf{X}}_i$, with maps q_i (as in 4.4.3). The maps t_i there define a derivation t on \mathbf{D} . This derivation induces a derivation on tensor powers of \mathbf{D} (as in 4.2.10), which descends to the symmetric powers. It is easy to see that this determines a differential algebra structure on the symmetric algebra on \mathbf{D} . We let $\mathbb{A}(\mathbf{X}_0)$ be the associated scheme.

A morphism in \mathcal{C} clearly determines a morphism of schemes on the associated schemes, making $\mathbb{A}(-)$ into a functor.

4.5. Model theory of differential fibre functors. We now wish to prove statements analogous to the ones for algebraic Tannakian categories, using the same approach as in Section 3. We work with a fixed differential tensor category $(\mathcal{C}, \mathcal{D})$, with $\mathbb{k} = \text{End}(\mathbf{1})$ a field. We view \mathbb{k} as a differential field, with the differential structure induced from \mathcal{D} , as in 4.2.2.

We will be using the theory DCF of differentially closed fields. We refer the reader to Marker [14] or Marker et al. [15] for more information.

4.5.1. The theory associated with a differential tensor category. The theory $\mathcal{T}_{\mathcal{C}}$ associated with the data above, as well as a differential field extension \mathcal{K} of \mathbb{k} is an expansion of the theory $\mathcal{T}_{\mathcal{C}}$ as defined in 3.1 by the following structure:

- (1) \mathbf{L} has an additional function symbol $'$, and the theory says that $'$ is a derivation, and that \mathbf{L} is a differentially closed field (and with the restriction of $'$ to \mathcal{K} as given).
- (2) For every object \mathbf{X} , there is a function symbol $d_{\mathbf{X}} : V_{\mathbf{X}} \rightarrow V_{\partial(\mathbf{X})}$. This function is a derivation, in the sense that for any $a \in \mathbf{L}$ and $v \in V_{\mathbf{X}}$,

$$d_{\mathbf{X}}(av) = a'V_{i_{\mathbf{X}}}(v) + ad_{\mathbf{X}}(v)$$

The theory furthermore says that $d_{\mathbf{X}}$ identifies $V_{\partial(\mathbf{X})}$ with $\mathcal{D} \hat{\otimes} V_{\mathbf{X}}$ (in any model), in the sense of 4.2.9 (explicitly, it says that $V_{p_{\mathbf{X}}} \circ d_{\mathbf{X}}$ is the identity map.)

- (3) The maps \mathbf{d} and \mathbf{b} (from the tensor structure) are compatible with the structure of tensor functor of \mathcal{D} : given objects \mathbf{X} and \mathbf{Y} of \mathcal{C} , let $\mathbf{c}_{\mathbf{X},\mathbf{Y}} : \partial(\mathbf{X} \otimes \mathbf{Y}) \rightarrow (\partial(\mathbf{X}) \otimes \partial(\mathbf{Y}))_1$ be the isomorphism supplied with \mathcal{D} . Then we require that $V_{\mathbf{c}_{\mathbf{X},\mathbf{Y}}} \circ \mathbf{d}_{\mathbf{X} \otimes \mathbf{Y}} \circ \mathbf{b}_{\mathbf{X},\mathbf{Y}}$ coincides with $\mathbf{b}_{\partial(\mathbf{X}),\partial(\mathbf{Y})} \circ \mathbf{d}_{\mathbf{X}} \times 1 + \mathbf{b}_{\mathbf{X},\partial(\mathbf{Y})} \circ 1 \times \mathbf{d}_{\mathbf{Y}}$.

4.5.2. Let ω be a differential fibre functor on \mathcal{C} , and let $\mathbf{L} = \mathcal{M}_1$ be a differentially closed field containing \mathcal{K} . As in Section 3, we expand \mathcal{M}_1 to a model \mathcal{M} of $\mathbb{T}_{\mathcal{C}}$ by tensoring with \mathbf{L} . The differential structure of ω gives (as in 4.2.9) a universal derivation $\omega(\mathbf{X}) \rightarrow \omega(\partial(\mathbf{X}))$, which extends uniquely to a (universal) derivation $(\mathbf{d}_{\mathbf{X}})_{\mathcal{M}}$ on $\mathcal{M}_{\mathbf{X}}$.

As in the algebraic case, we get:

Proposition 4.5.3. *Assume that \mathcal{C} has a differential fibre functor. Then $\mathbb{T}_{\mathcal{C}}$ is consistent, and (in a model) $\text{dcl}(0) \cap \mathbf{L} = \mathbb{k}$. The theory $\mathbb{T}_{\mathcal{C}}$ is stable, and \mathbf{L} is a pure differentially closed field.*

Proof. Same as in Proposition 3.2 and Proposition 3.4.3. \square

4.5.4. *Internality.* Since the differential $\mathbb{T}_{\mathcal{C}}$ is an expansion of the algebraic one with no new sorts, it is again an internal cover of \mathbf{L} . Furthermore, if \mathbf{B} is a basis for some $V_{\mathbf{X}}$, then $\mathbf{B} \cup \mathbf{d}_{\mathbf{X}}(\mathbf{B})$ is a basis for $V_{\partial(\mathbf{X})}$. Therefore, if \mathcal{C} is generated as a differential tensor category by one object (in the sense that the objects $\partial^i \mathbf{X}$ generate \mathcal{C} as a tensor category), then all of the sorts are internal using the same finite parameter. As usual, the general case is obtained by taking a limit of such.

Theorem 4.5.5. *Let \mathcal{C} be a differential Tannakian category over \mathbb{k} , and let $\omega : \mathcal{C} \rightarrow \text{Vec}_{\mathbb{k}}$ be a differential fibre functor (4.2.8).*

- (1) *The functor $\underline{\text{Aut}}^{\circ}(\omega)$ (restricted to differential fields) is represented by a linear differential group \mathcal{G} over \mathbb{k} .*
- (2) *The fibre functor ω factors through a differential tensor equivalence $\omega : \mathcal{C} \rightarrow \text{Rep}_{\mathcal{G}}$.*
- (3) *If $\mathcal{C} = \text{Rep}_{\mathcal{H}}$ for some linear differential group \mathcal{H} , then the natural map $\mathcal{H} \rightarrow \mathcal{G}$ given in 4.3.3 is an isomorphism.*

Proof. The proof is completely analogous to the proof of the corresponding statement 2.8, which is given, respectively, in 3.7, 3.10 and 3.9. We only mention the differences.

For (1), the main point is again that $\underline{\text{Aut}}^{\circ}(\omega)(\mathcal{K})$ is isomorphic to $\mathbf{G}_{\omega}(\mathcal{K})$ (functorially in \mathcal{K}), where \mathbf{G}_{ω} is a definable copy of the internality group inside \mathbf{L} . The sort \mathbf{L} is now a pure differentially closed field, so the result follows from the fact that any definable group in DCF is differential algebraic (Pillay [21]).

For (2), the argument in 3.10 goes through without a change.

For (3), again the proof in 3.9 applies, once we classify the imaginaries in $\mathbb{T}_{\mathcal{C}}$. This is the content of Proposition 4.5.6. \square

Proposition 4.5.6. *$\mathbb{T}_{\mathcal{C}}$ eliminates imaginaries to the level of projective spaces.*

Proof. Both the statement and the proof are analogous to Hrushovski [9, Proposition 4.2].

We need to show that any definable set S over parameters can be defined with a canonical parameter. Since, by assumption, no new structure is induced on \mathbf{L} , and any set is internal to \mathbf{L} , every such set is Kolchin constructible. By Noetherian induction, it is enough to consider S Kolchin closed.

We claim that the algebra of differential polynomials on a sort \mathbf{U} is ind-definable in $\mathcal{T}_{\mathcal{C}}$. Indeed, it is precisely given by the interpretation of scheme structure associated with \mathbf{U} (Definition 4.4.4). We only mention the definition of the evaluation map (using the notation there): it is enough to the evaluation on (the interpretation of) \mathbf{D} , since the rest is as in the algebraic case. We define the evaluation $e_n : \check{\mathbf{U}}_n \times \mathbf{U} \rightarrow \mathbf{L}$ by induction on n : e_0 is the usual evaluation. If $\mathbf{u} \in \mathbf{U}$, the map $d \mapsto e_n(d, \mathbf{u})'$ is a derivation on $\check{\mathbf{U}}_n$, and so defines a linear map from $\partial(\check{\mathbf{U}}_n)$ to \mathbf{L} . Inspection of the definition of $\check{\mathbf{U}}_{n+1}$ (for vector spaces) reveals that this map descends to a linear map $e_{n+1}(-, \mathbf{u})$ on $\check{\mathbf{U}}_{n+1}$.

The rest of the proof is the same as in Hrushovski [9], namely, the Kolchin closed set S is determined by the finite dimensional linear space spanned by the defining equations, and this space is an elements of some Grassmanian, which is, in turn, a closed subset of some projective space. \square

REFERENCES

- [1] Phyllis Joan Cassidy. “Differential algebraic groups”. In: *Amer. J. Math.* 94 (1972), pp. 891–954. ISSN: 0002-9327 (cit. on p. 19).
- [2] Phyllis Joan Cassidy. “The differential rational representation algebra on a linear differential algebraic group”. In: *J. Algebra* 37.2 (1975), pp. 223–238. ISSN: 0021-8693 (cit. on p. 19).
- [3] Pierre Deligne. “Catégories tannakiennes”. In: *The Grothendieck Festschrift*. Vol. II. Progress in Mathematics 87. Boston, MA: Birkhäuser Boston Inc., 1990, pp. 111–195. ISBN: 0-8176-3428-2 (cit. on pp. 1, 3, 13, 20).
- [4] Pierre Deligne and James S. Milne. “Tannakian categories”. In: Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*. Lecture Notes in Mathematics 900. Berlin: Springer-Verlag, 1982. Chap. II, pp. 101–228. ISBN: 3-540-11174-3. URL: <http://jmilne.org/math> (cit. on pp. 1–3, 7, 13, 14).
- [5] Peter Freyd. *Abelian categories. An introduction to the theory of functors*. Harper’s Series in Modern Mathematics. New York: Harper & Row Publishers, 1964 (cit. on p. 15).
- [6] Alexander Grothendieck et al. *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) 3. Séminaire de géométrie algébrique du Bois Marie 1960–61., Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. Paris: Société Mathématique de France, 2003. ISBN: 2-85629-141-4. arXiv:math/0206203 (cit. on p. 14).
- [7] Bradd Hart and Ziv Shami. “On the type-definability of the binding group in simple theories”. In: *J. Symbolic Logic* 70.2 (2005), pp. 379–388. ISSN: 0022-4812 (cit. on p. 1).
- [8] Ehud Hrushovski. “Computing the Galois group of a linear differential equation”. In: *Differential Galois theory*. Ed. by Teresa Crespo and Zbigniew Hajto. Banach Center Publications 58. May 28–June 1, 2001. Warsaw: Polish Academy of Sciences Institute of Mathematics, 2002, pp. 97–138 (cit. on pp. 2, 5).
- [9] Ehud Hrushovski. *Groupoids, imaginaries and internal covers*. Mar. 2006. arXiv:math/0603413 (cit. on pp. 2, 3, 5, 10, 12, 22, 23).

- [10] Ehud Hrushovski. “Unidimensional theories are superstable”. In: *Ann. Pure Appl. Logic* 50.2 (1990), pp. 117–138. ISSN: 0168-0072 (cit. on p. 1).
- [11] Ellis R. Kolchin. *Differential algebra and algebraic groups*. Pure and Applied Mathematics 54. New York: Academic Press, 1973 (cit. on p. 19).
- [12] Saunders Mac Lane. *Categories for the working mathematician*. 2nd ed. Graduate Texts in Mathematics 5. New York: Springer-Verlag, 1998. ISBN: 0-387-98403-8 (cit. on p. 7).
- [13] David Marker. *Model theory: An introduction*. Graduate Texts in Mathematics 217. New York: Springer-Verlag, 2002. ISBN: 0-387-98760-6 (cit. on p. 3).
- [14] David Marker. “Model theory of differential fields”. In: *Model theory, algebra, and geometry*. Ed. by Deirdre Haskell, Anand Pillay, and Charles Steinhorn. Mathematical Sciences Research Institute Publications 39. Cambridge: Cambridge University Press, 2000, pp. 53–63. ISBN: 0-521-78068-3. URL: <http://www.msri.org/communications/books/Book39/contents.html> (cit. on p. 21).
- [15] David Marker, Margit Messmer, and Anand Pillay. *Model theory of fields*. 2nd ed. Lecture Notes in Logic 5. La Jolla, CA: Association for Symbolic Logic, 2006. ISBN: 978-1-56881-282-3; 1-56881-282-5 (cit. on p. 21).
- [16] James S. Milne. *Étale cohomology*. Princeton Mathematical Series 33. Princeton, N.J.: Princeton University Press, 1980. ISBN: 0-691-08238-3 (cit. on p. 10).
- [17] Alexey Ovchinnikov. “Tannakian approach to linear differential algebraic groups”. In: *Transform. Groups* 13.2 (2008), pp. 413–446. ISSN: 1083-4362 (cit. on p. 2).
- [18] Alexey Ovchinnikov. “Tannakian categories, linear differential algebraic groups, and parametrized linear differential equations”. In: *Transform. Groups* 14.1 (2009), pp. 195–223. ISSN: 1083-4362 (cit. on p. 2).
- [19] Anand Pillay. “Algebraic D-groups and differential Galois theory”. In: *Pacific J. Math.* 216.2 (2004), pp. 343–360. ISSN: 0030-8730 (cit. on p. 2).
- [20] Anand Pillay. *Model theory*. Course lecture notes. 2002. URL: http://www.math.uiuc.edu/People/pillay/lecturenotes_modeltheory.pdf (cit. on p. 3).
- [21] Anand Pillay. “Some foundational questions concerning differential algebraic groups”. In: *Pacific J. Math.* 179.1 (1997), pp. 179–200. ISSN: 0030-8730 (cit. on p. 22).
- [22] Bruno Poizat. *Stable groups*. Mathematical Surveys and Monographs 87. Providence, RI: American Mathematical Society, 2001. ISBN: 0-8218-2685-9 (cit. on p. 11).
- [23] Bruno Poizat. “Une théorie de Galois imaginaire”. In: *J. Symbolic Logic* 48.4 (1983), 1151–1170 (1984). ISSN: 0022-4812 (cit. on pp. 1, 5).
- [24] Neantro Saavedra-Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Berlin: Springer-Verlag, 1972 (cit. on pp. 1, 7).
- [25] Gerald E. Sacks. *Saturated model theory*. Reading, Mass.: W. A. Benjamin, Inc., 1972 (cit. on p. 4).
- [26] William C. Waterhouse. *Introduction to affine group schemes*. Graduate Texts in Mathematics 66. New York: Springer-Verlag, 1979. ISBN: 0-387-90421-2 (cit. on p. 12).

- [27] Boris I. Zil'ber. "Totally categorical theories: structural properties and the nonfinite axiomatizability". In: *Model theory of algebra and arithmetic*. Proceedings of the Conference on Applications of Logic to Algebra and Arithmetic. (Karpacz, September 1–7, 1979). Ed. by L. Pacholski and J. Wierzejewski. Lecture Notes in Mathematics 834. Berlin: Springer, 1980, pp. 381–410. ISBN: 3-540-10269-8 (cit. on p. 1).

DEPARTMENT OF MATH, UNIVERSITY OF NOTRE-DAME, NOTRE-DAME, IN, 46556, USA

E-mail address: <mailto:kamensky.1@nd.edu>

URL: <http://mkamensky.notlong.com>