TANNAKIAN FORMALISM OVER FIELDS WITH OPERATORS

MOSHE KAMENSKY

Abstract. We develop a theory of tensor categories over a field endowed with abstract operators. Our notion of a "field with operators", coming from work of Moosa and Scanlon, includes the familiar cases of differential and difference fields, Hasse–Schmidt derivations, and their combinations. We develop a corresponding Tannakian formalism, describing the category of representations of linear groups defined over such fields. The paper extends the previously know (classical) algebraic and differential algebraic Tannakian formalisms.

Introduction

We study fields with operators (briefly described below, and more thoroughly in §2), and linear groups over such fields. Given such a group **G** (as defined in §3), our goal is to describe the category $\mathcal{R}ep_{\mathbf{G}}$ of finite dimensional representations of **G**, in a manner similar to the classical Tannakian formalism. In addition to generalising the usual Tannakian formalism, this paper forms a natural generalisation and reformulation of the theory of differential Tannakian categories (Ovchinnikov [16]), and especially of the definition of differential tensor categories in Kamensky [10, § 4].

We mention that results this kind are expected to have applications to Galois theory of linear equations with various operators. The classical Galois theories of ordinary differential and difference linear equations (as explained in Put and Singer [20] and Put and Singer [19], respectively) may be approached via the classical Tannakian formalism (also in Deligne [6, § 9]). More recently, there are the Galois theory of (linear) partial differential equations (initiated by Cassidy and Singer [4]) to which the differential Tannakian theory mentioned above was applied in Ovchinnikov [17] (see also Gillet et al. [8]), as well as linear equations involving both derivatives and automorphisms (Hardouin and Singer [9]), and other variants. It is hoped that the present paper will provide the tools to approach all these Galois theories in a uniform manner, from the Tannakian point of view (of course, the classical Tannakian theory has many more applications in different areas, and we hope that similar applications will be found for the generalised theory in this paper).

The main result of the paper, describing the analogue of tensor categories, as well as the statement that shows the notion to be adequate, i.e., that it does axiomatise categories of representations, is in §3 (specifically, Definition 3.2.1 and Theorem 3.2.6). Both the definition and the statement are rather immediate once

---

2010 *Mathematics Subject Classification.* Primary 20G05,12H99,20G42; Secondary 34M15.

Key words and phrases. Tensor categories, Tannakian formalism, Hasse–Schmidt derivations, prolongation spaces.

The author is thankful for support by NSF grant No. 1001741.

the fundamental ideas are developed, so we now turn to a brief overview of the ideas that appear in the first two sections.

Our notion of a "field with operators" comes from (a variant of) the formalism developed by Moosa and Scanlon [15, 14]. This formalism includes at least the cases of differential fields (fields endowed with a derivation, or a vector field), difference fields (fields with an endomorphism), Hasse–Schmidt derivations, and their combinations. To explain the idea, consider a field $\mathbb{k}$ with an endomorphism. One could alternatively describe the situation by saying that we are given an action of the monoid $\mathbb{N}$ of natural numbers on $\mathbb{k}$. More generally, one could consider the action of a monoid M on $\mathbb{k}$. When M is infinite, it cannot be viewed as a scheme. However, as a set, it is the (filtered) union of finite sets, each of which can be viewed as a scheme. Furthermore, the monoid operation maps the product of finite such sets into another. In other words, M is a monoid in the category of ind-finite schemes, and we are given an action of M on $\operatorname{spec}(\mathbb{k})$.

Since any set is the filtered union of its finite subsets, the description above accounts for all discrete monoid actions. However, some finite schemes do not come from finite sets. Recall that the data of a derivation on the field $\mathbb{k}$ over the subfield $\mathbb{k}_0$ is equivalent to that of a $\mathbb{k}_0$-algebra map $\mathbb{k} \to \mathbb{k}[\epsilon] = \mathbb{k} \otimes_{\mathbb{k}_0} \mathbb{k}_0[\epsilon]$ whose composition with the unique map $\mathbb{k}[\epsilon] \to \mathbb{k}$ is the identity. Geometrically, we are given an "action" $\mathfrak{M}_0 \times \operatorname{spec}(\mathbb{k}) \to \operatorname{spec}(\mathbb{k})$ of the scheme $\mathfrak{M}_0 = \operatorname{spec}(\mathbb{k}_0[\epsilon])$, in such a way that the $\mathbb{k}_0$-point of $\mathfrak{M}_0$ acts as the identity. The finite scheme $\mathfrak{M}_0$ is not a monoid, but it is part of a system defining an ind-scheme, the additive formal group $\mathfrak{M}$, and in characteristic 0, each "action" of $\mathfrak{M}_0$ extends uniquely to an action of $\mathfrak{M}$. As with the discrete case, the further components of the system $\mathfrak{M}$ correspond to iterative application of the operator, i.e., to higher derivations (this example, which is classical, is discussed in detail in §2. We note that one could think of the additive formal group as a "limit" of the additive group of the integers, as the generator 1 "tends to 0").

It is therefore reasonable to define a "field with operators" simply as a field with an arbitrary ind-finite scheme monoid action. This is (essentially) the approach taken in Moosa and Scanlon [15, 14], and which we adopt here. We mentioned that similar ideas appear before: for example, Buium [2, § 2.4] discusses the encoding of a certain class of operators by suitable algebra maps (In fact, the approach there is somewhat more general, see §4.6). The case of $\mathbb{k}[\epsilon]$ goes back (at least) to Weil (unpublished), and is, in any case, classical. The case of Hasse–Schmidt derivations is discussed in Matsumura [13, § 27]. It appears that the geometric description we give is new (though Gillet et al. [8] appears to take a geometric approach of the differential case).

We name ind-finite schemes "formal sets". The guiding principle is that whatever can be done with usual sets should also be possible with formal sets. For example, for any set S there is a free monoid generated by S. If S is a set of endomorphisms of $\mathbb{k}$, the resulting free monoid acts on $\mathbb{k}$ (and vice versa). The same is true for formal sets (Proposition 2.3.2). The free monoid generated by $S = \operatorname{spec}(\mathbb{k}_0[\epsilon])$ is the additive formal group precisely if $\mathbb{k}_0$ contains $\mathbb{Q}$. This explains why in characteristic 0 (and only in this case), a derivation is the same as an action of this formal group.

As another example, if S is a set and M is a (discrete) monoid, there is an "induced" M-set S^M of functions from M to S, and this operation is right adjoint to the "restriction", from M-sets to sets, where one forgets the M-action.

A similar construction is available when M is now a formal set, and S is a (nice) scheme. The resulting space is (essentially) what one calls the *prolongation space* of S, which we discuss in §2.2 (and which is defined originally in Moosa and Scanlon [14]). One may then do "M-geometry" (analogous to Kolchin's differential algebraic geometry), where the prolongation spaces allow one to view usual algebraic schemes as M-schemes. In particular, it possible to consider linear groups, and their representations.

Recall that in the algebraic case, the category of representations of a (pro-) linear group scheme (over a field) is described by equipping the pure category structure with a tensor structure, satisfying suitable properties. A theorem of Saavedra (Saavedra-Rivano [21]) then shows that if one also remembers the "forgetful" functor into the category of vector spaces, the description is complete: any tensor category as above is equivalent to the category of representations of a linear group scheme, which can be recovered from the fibre functor (See Deligne and Milne [7] for an exposition).

In the case of differential algebraic groups, the tensor structure is insufficient. For example, the multiplicative group $\mathbf{G}_m$, viewed as a differential algebraic group, admits a non-trivial differential algebraic homomorphism to the additive group $\mathbf{G}_a$, and $\mathbf{G}_a$ itself has the derivative as an endomorphism. To recover the differential algebraic group in this case, Ovchinnikov [16] introduces a new operation on representations, as follows: One considers a 2-dimensional $\Bbbk$-vector space D, which admits an additional vector space structure (on the right), coming from the derivation. Given a vector space (or a representation) V, one obtains a new such object $\tau(V) = D\hat{\otimes}V$, where the tensor product is with respect to the right structure (and the vector space structure on $\tau(V)$ comes from the left one). It is then shown that with this additional structure, the differential algebraic group can be recovered.

In Kamensky [10, § 4], the operation described above is abstracted to apply in an arbitrary tensor (abelian) category. This is done by defining the prolongation of a tensor category $\mathcal{C}$ as a certain tensor category of exact sequences of objects of $\mathcal{C}$. The differential structure is then given by a tensor functor τ from $\mathcal{C}$ to its prolongation. While this formalism does work in the required way, the definition of the prolongation category is somewhat ad hoc. One of the goals of the current paper is to build the prolongation category in a more systematic way. The relation of the current formalism with the original one is explained in Example 1.2.11. We remark that an alternative formulation of the differential theory, including an extension to the case of several derivations, is also suggested in a recent preprint, Gillet et al. [8]. Their approach seems to be similar to the one taken here, and it would be interesting to make a precise comparison.

In this paper, we imitate the differential case, but construct the prolongation category more systematically. Given a field $\Bbbk$ on which a formal monoid M (eventually) acts, we identify the analogue of the space D above: essentially, it is the dual of the pro-algebra corresponding to M. In §1, we define and study the prolongation of an abstract tensor category over $\Bbbk$ with respect to M. In this section, only M itself (or rather, its algebra) is used, the action does not yet appear. In §2 we discuss the notion of a field with operators. This is mostly an exposition of parts of Moosa and Scanlon [14], although there are also new results (and the exposition is somewhat different). Finally, in §3 we give our main result. We remark that, though the statement is completely analogous to the previous cases, the proof in this paper is different: rather than repeating a proof similar to the classical case, our current proof *reduces* to the classical statement, using the description of M-schemes employed in this paper. As a result, the notion of schemes in $\mathcal{C}$, and the scheme structure on objects of $\mathcal{C}$ does not play a role as it played in the differential case (and less explicitly, in the classical case). Nevertheless, we discuss this structure in §3.3.

In the process of writing the paper, I ran into some questions that are not essential for the main results, but they do occur naturally. I list some of these questions in §4.

Remarks on notation: for most of the paper, $\mathbb{k}_0$ is a base ring, consisting of "constants" for the operators. Hence, all maps will be maps over $\mathbb{k}_0$. The (pro-) finite algebra corresponding to the acting monoid is usually denoted by E_0, and the monoid itself by $\mathfrak{M}_0$. $\mathbb{k}$ will be an extension of $\mathbb{k}_0$ (usually a field), on which $\mathfrak{M}_0$ acts. I remove the subscript 0 when changing the base to $\mathbb{k}$: $E = \mathbb{k} \otimes_{\mathbb{k}_0} E_0$, etc. I try to use different font styles for different kind of objects. This should be visible.

1. CATEGORICAL PROLONGATIONS

In this section, our goal is to define the prolongation of a tensor category with respect to an algebra E. This will be another tensor category, as described in the introduction. The goal is achieved in Definition 1.2.8 in the finite case, and extended to the pro-finite case (which is the general case) in Definition 1.3.9. We start by discussing the action of E on objects in an arbitrary $\mathbb{k}$-linear category. Much of the material comes from Deligne [6, § 5].

1.1. Modules in $\mathbb{k}$-linear categories.

Let $\mathbb{k}$ be a field, fixed for the entire section. We denote by $\mathcal{V}ec_{\mathbb{k}}$ the category of finite dimensional vector spaces over $\mathbb{k}$. For any vector space V over $\mathbb{k}$, $\check{V}$ denotes the linear dual.

If E is a finite $\mathbb{k}$-algebra, then $\check{E}$ is an E-module. We note that $\check{E}$ need not be free:

Example 1.1.1. Let $E = \mathbb{k}[x, y]$, with $x^2 = y^2 = xy = 0$. Let $(\delta, \delta_x, \delta_y) \in \check{E}^3$ be the basis dual to the basis $(1, x, y)$ of E. Then $x\delta = x\delta_y = 0$ and similarly for y, so $\check{E}$ cannot be free. □

The actual situation is described by the following.

Proposition 1.1.2. *Let M and N be finite E-modules. Then $(M \otimes_E N)^\vee$ is canonically isomorphic to $Hom_E(M, \check{N})$. In particular, N is flat if and only if $\check{N}$ is injective (as with any commutative ring, this is also equivalent to N being projective and locally free).*

Proof. An element of $Hom_E(M, \check{N})$ corresponds, by adjunction, to a $\mathbb{k}$-linear map $\phi : M \otimes N \to \mathbb{k}$, such that $\phi(em, n) = \phi(m, en)$ for all $e \in E$. These are precisely the elements of $(M \otimes_E N)^\vee$. □

We will be interested in modules in arbitrary $\mathbb{k}$-linear categories.

Definition 1.1.3. Let $\mathbb{k}$ be a field. By a $\mathbb{k}$-*linear category* we mean an additive category $\mathcal{C}$, together with a $\mathbb{k}$-vector space structure on each abelian group $Hom(\mathbf{X}, \mathbf{Y})$, such that

(1) Composition is $\mathbb{k}$-bilinear.

(2) Each $Hom(\mathbf{X}, \mathbf{Y})$ is finite dimensional
(3) Each object has finite length (i.e., the length of a strictly descending chain starting from a given object is bounded).

In particular, each $End(\mathbf{X})$ is a finite (associative) $\Bbbk$-algebra.

1.1.4. *Action of $\mathcal{V}ec_{\Bbbk}$ on $\mathcal{C}$.* For any object $\mathbf{X}$ of a $\Bbbk$-linear category $\mathcal{C}$, the functor $\mathbf{Y} \mapsto Hom(\mathbf{X}, \mathbf{Y})$ (into $\mathcal{V}ec$) has a left adjoint $V \mapsto V \otimes_{\Bbbk} \mathbf{X}$ (Deligne and Milne [7, § 2]). Set $\underline{Hom}_{\Bbbk}(V, \mathbf{X}) = \check{V} \otimes_{\Bbbk} \mathbf{X}$.

Proposition 1.1.5. *Let $\mathcal{C}$ be a $\Bbbk$-linear category, $\mathbf{X}$ an object of $\mathcal{C}$.*

(1) *The functor $\mathbf{Y} \mapsto Hom(\mathbf{Y}, \mathbf{X})^{\vee}$ (from $\mathcal{C}$ to $\mathcal{V}ec_{\Bbbk}$) is left adjoint to $V \mapsto V \otimes \mathbf{X}$.*
(2) *For any vector space V finite dimensional over $\Bbbk$, $\underline{Hom}_{\Bbbk}(V, -)$ is right adjoint to $V \otimes_{\Bbbk} -$*
(3) *The functor $- \otimes_{\Bbbk} \mathbf{X}$ is exact*
(4) *In $\mathcal{C}^{op}$, $(V \otimes_{\Bbbk} \mathbf{X})^{op}$ is canonically isomorphic to $\underline{Hom}_{\Bbbk}(V, \mathbf{X}^{op})$.*

Proof. (1) Given A morphism $f : \mathbf{Y} \to V \otimes_{\Bbbk} \mathbf{X}$, we obtain a map

$$\check{V} \otimes_{\Bbbk} \mathbf{Y} \xrightarrow{id \otimes_{\Bbbk} f} \check{V} \otimes_{\Bbbk} V \otimes_{\Bbbk} \mathbf{X} \xrightarrow{ev \otimes_{\Bbbk} id} \mathbf{X} \tag{1}$$

which induces, by adjunction, a map $\check{V} \to Hom(\mathbf{Y}, \mathbf{X})$, and by duality a map $\check{f} : Hom(\mathbf{Y}, \mathbf{X})^{\vee} \to V$.

In the other direction, a map $g : Hom(\mathbf{Y}, \mathbf{X})^{\vee} \to V$ corresponds by duality to a map $\check{V} \to Hom(\mathbf{Y}, \mathbf{X})$, which corresponds by adjunction to a map $\check{V} \otimes_{\Bbbk} \mathbf{Y} \to \mathbf{X}$. Tensoring with V and combining with co-evaluation, we get

$$\check{g} : \mathbf{Y} \to V \otimes_{\Bbbk} \check{V} \otimes_{\Bbbk} \mathbf{Y} \to V \otimes_{\Bbbk} \mathbf{X} \tag{2}$$

The statement that the two constructions are inverse to each other is precisely the statement that the usual evaluation and co-evaluation determine a rigid monoidal structure on $\mathcal{V}ec_{\Bbbk}$.

(2) Apply the previous statement with $\check{V}$
(3) The functor has both left and right adjoints
(4) A morphism $\mathbf{Y} \to V \otimes_{\Bbbk} \mathbf{X}$ in $\mathcal{C}^{op}$ corresponds to a morphism $V \otimes_{\Bbbk} \mathbf{X} \to \mathbf{Y}$ in $\mathcal{C}$, which correspond to a linear map $V \to Hom_{\mathcal{C}}(\mathbf{X}, \mathbf{Y}) = Hom_{\mathcal{C}^{op}}(\mathbf{Y}, \mathbf{X})$, corresponding by duality to a map $Hom_{\mathcal{C}^{op}}(\mathbf{Y}, \mathbf{X})^{\vee} \to \check{V}$, which corresponds by adjunction to a map $\mathbf{Y} \to \underline{Hom}_{\Bbbk}(V, \mathbf{X})$ (in $\mathcal{C}^{op}$). Hence the two objects represent the same functor.

□

Definition 1.1.6. Let E be an associative $\Bbbk$-algebra. A *(left) E-module in $\mathcal{C}$* is an object $\mathbf{X}$ of $\mathcal{C}$, together with a $\Bbbk$-algebra map $E \to End(\mathbf{X})$. We denote by $E - \mathcal{C}$ the category of left E-modules in $\mathcal{C}$ (with E-action preserving maps).

1.1.7. We fix a finite-dimensional $\Bbbk$-algebra E. Then an E-module structure on $\mathbf{X}$ is the same as a morphism $E \otimes \mathbf{X} \to \mathbf{X}$ satisfying the obvious relations, and by Proposition 1.1.5, it is also the same as a morphism $\mathbf{X} \to \check{E} \otimes_{\Bbbk} \mathbf{X} = \underline{Hom}_{\Bbbk}(E, \mathbf{X})$ that makes $\mathbf{X}$ an $\check{E}$-comodule.

1.1.8. Given an E-module $\mathbf{X}$ in $\mathcal{C}$ and an object $\mathbf{Y}$ in $\mathcal{C}$, the space $Hom(\mathbf{X},\mathbf{Y})$ is a (usual) right E-module. In other words, $\mathbf{X}$ represents a functor $\mathbf{Y}\mapsto Hom(\mathbf{X},\mathbf{Y})$ from $\mathcal{C}$ to the (abelian, $\Bbbk$-linear) category $(E)_{coh}$ of finitely generated right E-modules. This functor has a left adjoint $\mathscr{F}_{\mathbf{X}} : M\mapsto M\otimes_E\mathbf{X}$, where $M\otimes_E\mathbf{X}$ is the co-equaliser

$$M\otimes E\otimes\mathbf{X}\rightrightarrows M\otimes\mathbf{X}\to M\otimes_E\mathbf{X} \tag{3}$$

Definition 1.1.9. An E-module $\mathbf{X}$ in $\mathcal{C}$ is *flat* if the functor $M\mapsto M\otimes_E\mathbf{X}$ from $(E)_{coh}$ to $\mathcal{C}$ is exact. We denote by $\mathcal{C}_{(E)}$ the full sub-category of $E-\mathcal{C}$ consisting of flat E-modules.

Lemma 1.1.10. *If for any right ideal I of E, the map $I\otimes_E\mathbf{X}\to\mathbf{X}$ is monic, then $\mathbf{X}$ is flat.*

Proof. Same as for usual modules □

Example 1.1.11. Let $\mathbb{D}(\mathcal{C})$ be the category of exact sequences $0\to\mathbf{X}\xrightarrow{i}\mathbf{Y}\xrightarrow{\pi}\mathbf{X}\to 0$ in $\mathcal{C}$, where the morphisms are morphisms of exact sequences, in which the two side maps agree. This category can be identified with $\mathcal{C}_{(E)}$, for $E=\Bbbk[\epsilon]$ (with $\epsilon^2=0$). Namely, a sequence as above is identified with $\mathbf{Y}$, with ϵ acting as $i\circ\pi$ (and the sequence is exact precisely if $\mathbf{Y}$ is flat). □

Example 1.1.12. If $E=E_1\times E_2$, then $E-\mathcal{C}$ can be identified with $E_1-\mathcal{C}\times E_2-\mathcal{C}$, and likewise for $\mathcal{C}_{(E)}$. In particular, for $E=\Bbbk\times\Bbbk$, both $E-\mathcal{C}$ and $\mathcal{C}_{(E)}$ are the category of pairs of objects of $\mathcal{C}$. □

Proposition 1.1.13 (Deligne [6, § 5.2])**.** *Given an E-module $\mathbf{X}$ in $\mathcal{C}$, the functor $\mathscr{F}_{\mathbf{X}}$ from 1.1.8 is right-exact. The assignment $\mathbf{X}\mapsto\mathscr{F}_{\mathbf{X}}$ is an equivalence between $E-\mathcal{C}$ and right-exact, $\Bbbk$-linear functors from $(E)_{coh}$ to $\mathcal{C}$. Flat modules correspond to exact functors under this equivalence.*

For convenience, we sketch the proof.

Proof. Given a right-exact functor $\mathscr{F}:(E)_{coh}\to\mathcal{C}$, let $\mathbf{X}=\mathscr{F}(E)$. Since $E=End_E(E)$ (endomorphisms of right E-modules), $\mathbf{X}$ is a left E-module. Given any coherent right E-module M, applying $\mathscr{F}$ to the co-equaliser diagram

$$M\otimes_{\Bbbk}E\otimes_{\Bbbk}E\rightrightarrows M\otimes_{\Bbbk}E\to M\otimes_E E=M \tag{4}$$

and using the fact that $\mathscr{F}$ is right-exact and $\Bbbk$-linear, we get a co-equaliser diagram

$$M\otimes_{\Bbbk}E\otimes_{\Bbbk}\mathbf{X}\rightrightarrows M\otimes_{\Bbbk}\mathbf{X}\to M\otimes_E\mathbf{X}=\mathscr{F}(M) \tag{5}$$

Hence $\mathscr{F}$ is isomorphic to $\mathscr{F}_{\mathbf{X}}$. The other claims are obvious. □

1.1.14. We will require a few more results from Deligne [6, § 5]. Given a left E-module M and an object $\mathbf{X}$ of $\mathcal{C}$, $M\otimes\mathbf{X}$ is naturally an E-module in $\mathcal{C}$. Hence, there is a functor $\otimes:(E)^{coh}\times\mathcal{C}\to E-\mathcal{C}$, $\Bbbk$-linear and exact in each coordinate (where $(E)^{coh}$ is the category of finite left E-modules).

Deligne [6, § 5.1] defines the tensor product of two abelian $\Bbbk$-linear categories $\mathcal{C}_1$ and $\mathcal{C}_2$ to be an abelian $\Bbbk$-linear category $\mathcal{C}=\mathcal{C}_1\otimes_{\Bbbk}\mathcal{C}_2$, together with a "universal" $\Bbbk$-bilinear right-exact (in each coordinate) functor $\otimes:\mathcal{C}_1\times\mathcal{C}_2\to\mathcal{C}$.

Proposition 1.1.15 (Deligne [6, § 5.11])**.** *The functor $\otimes:(E)^{coh}\times\mathcal{C}\to E-\mathcal{C}$ identifies $E-\mathcal{C}$ with the tensor product of $(E)^{coh}$ and $\mathcal{C}$.*

Proposition 1.1.16 (Deligne [6, § 5.13])**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two abelian $\Bbbk$-linear categories.*

1. *$\mathcal{C} = \mathcal{C}_1 \otimes_{\Bbbk} \mathcal{C}_2$ exists, and is again $\Bbbk$-linear*
2. *The "tensor product" $\otimes : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ is exact in each coordinate.*
3. *If $\Bbbk$ is perfect, and $\mathscr{F} : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ is exact in each coordinate, then the induced functor $\mathcal{C} \to \mathcal{D}$ is exact as well. (That this may fail if $\Bbbk$ is not perfect is explained in Deligne [6, § 5.6].)*

1.2. **Tensor structure.**

We now assume that $\Bbbk$ is a perfect field, E is a finite commutative $\Bbbk$-algebra, and $\mathcal{C}$ is abelian and $\Bbbk$-linear. We also assume that we are given a monoidal structure $(\otimes, \phi, \psi)$ on $\mathcal{C}$ (so that $\otimes$ is $\Bbbk$-linear in each coordinate, and has a unit $\mathbf{1}$, but $\mathcal{C}$ is not necessarily rigid). We assume $\otimes$ to be exact in each coordinate (this is automatic if $\mathcal{C}$ is rigid). We would like to define a monoidal structure on $E - \mathcal{C}$. It will be convenient to define and work with two dual such structures.

We fix a unit $\mathbf{1}$ in $(\mathcal{C}, \otimes)$. The functor $V \mapsto V \otimes_{\Bbbk} \mathbf{1}$ has a natural tensor structure, making it a fully faithful exact tensor embedding of $\mathcal{V}ec_{\Bbbk}$ into $\mathcal{C}$. We will therefore view $\mathcal{V}ec_{\Bbbk}$ as a subcategory of $\mathcal{C}$. The meaning of all notions we have defined (and will define) for both vector spaces and objects of $\mathcal{C}$ is easily seen to be compatible with this identification. For example, we have $V \otimes_{\Bbbk} \mathbf{X} = V \otimes \mathbf{X}$ and $\underline{Hom}_{\Bbbk}(V, \mathbf{X}) = \underline{Hom}(V, \mathbf{X})$ (in particular, the latter exists), so we drop the decoration $\Bbbk$ from now on.

1.2.1. Given two E-modules $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{C}$, their usual tensor product $\mathbf{X} \otimes_E \mathbf{Y}$ is defined as the largest quotient of $\mathbf{X} \otimes \mathbf{Y}$ on which the two actions of E agree (cf. Deligne and Milne [7, § 3]). In other words, it is the co-equaliser

$$E \otimes \mathbf{X} \otimes \mathbf{Y} \rightrightarrows \mathbf{X} \otimes \mathbf{Y} \to \mathbf{X} \otimes_E \mathbf{Y} \tag{6}$$

The E-module structure is induced, as usual, by the action on either coordinate.

The dual tensor product $\mathbf{X} \otimes^E \mathbf{Y}$ is defined as $(\mathbf{X}^{op} \otimes_E \mathbf{Y}^{op})^{op}$, where $\mathbf{X}^{op}$ is $\mathbf{X}$ viewed as an object of the opposite category $\mathcal{C}^{op}$ (since E is commutative, $E - \mathcal{C}^{op} = (E - \mathcal{C})^{op}$). In other words, it is the largest sub-object of $\mathbf{X} \otimes \mathbf{Y}$ annihilated by all maps $e \otimes 1 - 1 \otimes e$ with $e \in E$ (this exists since E is finite). Again, the E-module structure comes from the action on either coordinate.

Since the associativity and commutativity constraints are functorial, they commute with the action of E, and therefore induce similar constraints ϕ_E, ψ_E, ϕ^E and ψ^E on the respective tensor structures. We set $\mathcal{C}_E = (E - \mathcal{C}, \otimes_E, \phi_E, \psi_E)$ and $\mathcal{C}^E = (E - \mathcal{C}, \otimes^E, \phi^E, \psi^E)$.

Lemma 1.2.2. *The inclusion of $\mathbf{X} \otimes^E \mathbf{Y}$ in $\mathbf{X} \otimes \mathbf{Y}$ is the equaliser of the two maps $\mathbf{X} \otimes \mathbf{Y} \to \underline{Hom}(E, \mathbf{X} \otimes \mathbf{Y})$.*

Proof. This follows from dualising the diagram (6), using Proposition 1.1.5. □

The following proposition lists the basic properties of these operations.

Proposition 1.2.3. *Let $(\mathcal{C}, \otimes)$ and E be as in 1.2.1.*

1. *$\mathcal{C}_E$ and $\mathcal{C}^E$ are monoidal categories*
2. *If $\mathcal{C}$ is closed, then so is $\mathcal{C}_E$.*
3. *If $\mathcal{C}$ is rigid, then*

$$(\check{\mathbf{X}} \otimes_E \check{\mathbf{Y}})^{\vee} = \mathbf{X} \otimes^E \mathbf{Y} = \underline{Hom}_E(\check{\mathbf{X}}, \mathbf{Y}) \tag{7}$$

Hence, if $\mathcal{C}$ is rigid, $\mathbf{X} \mapsto \check{\mathbf{X}}$ induced a monoidal equivalence $\mathcal{C}^E \to {\mathcal{C}_E}^{op}$.

Proof. (1) This was discussed in 1.2.1. The only additional point is that E is a unit for $\mathcal{C}_E$, and dually, $\check{E}$ is a unit in $\mathcal{C}^E$.

(2) Given two E-modules $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{C}$, E acts on $\underline{Hom}(\mathbf{X}, \mathbf{Y})$ in two ways. Let $\underline{Hom}_E(\mathbf{X}, \mathbf{Y})$ be the equaliser of the two actions, with E structure coming from either.

A map $f : \mathbf{Z} \to \underline{Hom}_E(\mathbf{X}, \mathbf{Y})$ determines a map $\mathbf{Z} \to \underline{Hom}(\mathbf{X}, \mathbf{Y})$, and therefore a map $g : \mathbf{Z} \otimes \mathbf{X} \to \mathbf{Y}$. If $\mathbf{Z}$ is an E-module, and f commutes with the action of E, then the two compositions $E \otimes \mathbf{Z} \otimes \mathbf{X} \to \mathbf{Z} \otimes \mathbf{X} \xrightarrow{g} \mathbf{Y}$ are equal, so g descends to a map $\bar{g} : \mathbf{Z} \otimes_E \mathbf{X} \to \mathbf{Y}$. Furthermore, since f factors through $\underline{Hom}_E(\mathbf{X}, \mathbf{Y})$, $\bar{g}$ is a map of E-modules. The argument in the other direction is similar.

(3) The first equality follows from the fact that $\mathbf{X} \mapsto \check{\mathbf{X}}$ is an exact tensor equivalence of $\mathcal{C}$ with $\mathcal{C}^{op}$, taking $E \otimes \mathbf{X}$ to $\underline{Hom}(E, \check{\mathbf{X}})$ (and using Lemma 1.2.2).

The second equality follows from Lemma 1.2.2 and the construction of $\underline{Hom}_E(\check{\mathbf{X}}, \mathbf{Y})$ as an equaliser (together with the isomorphism $\check{\mathbf{X}} \otimes \mathbf{Y} = \underline{Hom}(\mathbf{X}, \mathbf{Y})$ in any rigid category).

□

Remark 1.2.4. The equivalence mentioned in the Proposition does not imply that $\mathcal{C}^E$ is closed (which is generally false), since the notion of a closed category is not self-dual (However, see 1.2.15). □

1.2.5. *Flatness.* If $\mathbf{X}$ is an E-module in $\mathcal{C}$, the functor $\mathbf{Y} \mapsto \mathbf{Y} \otimes_E \mathbf{X}$ is always right exact (since it is a co-equaliser). We would like to consider those modules for which the functor is exact. Since the usual E-modules are included in $E - \mathcal{C}$, each such module is flat in the sense of 1.1.9. It follows from Deligne's result that flatness is sufficient for the exactness of this functor in general.

Proposition 1.2.6. *Assume that $\mathbb{k}$ is perfect. Then for any flat E-module $\mathbf{X}$, the functor $- \otimes_E \mathbf{X}$ (from $\mathcal{C}_E$ to itself) is exact.*

If $\mathcal{C}$ is rigid, then this is also equivalent to the exactness of $\mathbf{Y} \mapsto \underline{Hom}_E(\mathbf{Y}, \check{\mathbf{X}})$.

The last part is an analogue of Proposition 1.1.2.

Proof. It is enough to prove that $- \otimes_E \mathbf{X}$ is exact as a functor from $E - \mathcal{C}$ to $\mathcal{C}$. According to Proposition 1.1.15, $E - \mathcal{C}$ can be identified with $(E)^{coh} \otimes_{\mathbb{k}} \mathcal{C}$. Since $\mathbb{k}$ is perfect, it is enough, by Proposition 1.1.16, to prove that the induced functor $(E)^{coh} \times \mathcal{C} \to \mathcal{C}$ is exact in each coordinate. This induced functor is given by $(M, \mathbf{Y}) \mapsto (M \otimes_E \mathbf{X}) \otimes \mathbf{Y}$, so precisely equivalent to the flatness of $\mathbf{X}$ (recall that $\otimes$ was assumed to be exact).

The second statement follows from the first together with equation (7). □

From now on, *we assume that $\mathcal{C}$ is rigid.*

Corollary 1.2.7. *The full sub-category $\mathcal{C}_{(E)}$ of $\mathcal{C}_E$ consisting of flat modules is a (non-abelian) tensor sub-category. So is the full sub-category of $\mathcal{C}^E$ consisting of objects $\mathbf{X}$ for which $\check{\mathbf{X}}$ is flat.*

Proof. We need only to prove that if $\mathbf{X}$ and $\mathbf{Y}$ are flat, then so is $\mathbf{X} \otimes_E \mathbf{Y}$. Hence we need to prove that the functor $M \mapsto M \otimes_E (\mathbf{X} \otimes_E \mathbf{Y})$ is exact. Since $\mathcal{C}_E$ is a tensor category, the last object is equal to $(M \otimes_E \mathbf{X}) \otimes_E \mathbf{Y}$, so this is a composition of two exact functors (using Proposition 1.2.6 for $\mathbf{Y}$). ☐

Definition 1.2.8. Let $\Bbbk$ be a perfect field, $\mathcal{C}$ a rigid abelian $\Bbbk$-linear tensor category, and E a finite $\Bbbk$-algebra. An object $\mathbf{X}$ of $E - \mathcal{C}$ will be called *E-injective* if $\check{\mathbf{X}}$ is E-flat.

The *E-prolongation* of $\mathcal{C}$, $\mathcal{C}^{(E)}$, is defined to be the full tensor sub-category of $\mathcal{C}^E$ consisting of E-injective modules.

Remark 1.2.9. If $\mathbf{X}$ is E-injective, it follows that the functor $M \mapsto \underline{Hom}_E(M, \mathbf{X})$ (from $(E)^{coh}$ to $\mathcal{C}$) is exact. The converse is also true, using the same argument as in the proof of Proposition 1.2.6. Hence, the notion of E-injective objects can also be defined on the level of abelian categories, without mentioning the tensor structure. Also, as with flatness, it is enough to check the exactness on inclusions of an ideal of E in E. On the other hand, being E-injective is not the same as being an injective object in $E - \mathcal{C}$.

Similarly, it follows from equation (7) that an object $\mathbf{X}$ is E-flat if and only if it is E-projective, in the sense that $\underline{Hom}_E(\mathbf{X}, -)$ is exact, either on $\mathcal{C}_E$ or on $(E)^{coh}$ (and this is again different from being projective in $E - \mathcal{C}$). ☐

Corollary 1.2.10. *The tensor equivalence* $\mathbf{X} \mapsto \check{\mathbf{X}}$ *from Proposition 1.2.3 induces a tensor equivalence* $\mathcal{C}^{(E)} \to {\mathcal{C}_{(E)}}^{op}$.

Example 1.2.11. Let $E = \Bbbk[\epsilon]$. In Example 1.1.11, we have already identified the flat E-modules in $\mathcal{C}$ with exact sequences $0 \to \mathbf{X} \xrightarrow{i} \mathbf{Y} \xrightarrow{\pi} \mathbf{X} \to 0$ in $\mathcal{C}$. Since the dual module corresponds to the dual exact sequence, an E-module is E-flat if and only if it is injective. Thus, this is also the category of injective E-modules.

Let $0 \to \mathbf{X}_1 \xrightarrow{i} \mathbf{Y}_1 \xrightarrow{\pi} \mathbf{X}_1 \to 0$ and $0 \to \mathbf{X}_2 \xrightarrow{i} \mathbf{Y}_2 \xrightarrow{\pi} \mathbf{X}_2 \to 0$ be two exact sequences. The inclusions induce inclusions $0 \to \mathbf{X}_1 \otimes \mathbf{Y}_2 \xrightarrow{i\otimes 1} \mathbf{Y}_1 \otimes \mathbf{Y}_2$ and $0 \to \mathbf{Y}_1 \otimes \mathbf{X}_2 \xrightarrow{1\otimes i} \mathbf{Y}_1 \otimes \mathbf{Y}_2$, and therefore a map

$$\mathbf{X}_1 \otimes \mathbf{Y}_2 \oplus \mathbf{Y}_1 \otimes \mathbf{X}_2 \xrightarrow{i\otimes 1 - 1\otimes i} \mathbf{Y}_1 \otimes \mathbf{Y}_2$$

whose kernel (by a simple diagram chase) is $\mathbf{X}_1 \otimes \mathbf{X}_2$. Taking the quotient by this kernel, we therefore obtain a sub-object $\mathbf{Z}$ of $\mathbf{Y}_1 \otimes \mathbf{Y}_2$. The equaliser $\mathbf{W}$ of the two maps $\pi \otimes 1$ and $1 \otimes \pi$ is clearly a sub-object of $\mathbf{Z}$, there is an exact sequence $0 \to \mathbf{X}_1 \otimes \mathbf{X}_2 \to \mathbf{W} \to \mathbf{X}_1 \otimes \mathbf{X}_2 \to 0$, which was defined in Kamensky [10] to be the tensor product of the two given sequences.

Viewing the $\mathbf{Y}_i$ as E-injective modules, with $\epsilon = i \circ \pi$ on each $\mathbf{Y}_i$, the equaliser of $\epsilon \otimes 1$ and $1 \otimes \epsilon$ coincides with the equaliser of $\pi \otimes 1$ and $1 \otimes \pi$, so the above definition coincides with (the exact sequence corresponding to) the E-module $\mathbf{Y}_1 \otimes^E \mathbf{Y}_2$. Hence $\mathcal{C}^{(E)}$ coincides, as a tensor category, with what was called the prolongation of $\mathcal{C}$ in Kamensky [10]. ☐

Example 1.2.12. When $E = E_1 \times E_2$, and we identify $E - \mathcal{C}$ with $(E_1 - \mathcal{C}) \times (E_2 - \mathcal{C})$, as in Example 1.1.12, all notions again work component wise. Hence, for $E = \Bbbk \times \Bbbk$, $\mathcal{C}^{(E)}$ is $\mathcal{C} \times \mathcal{C}$, as a tensor category. ☐

Proposition 1.2.13. *Let* $\mathcal{C}$ *be rigid, and let* $\mathbf{X}$ *be a flat* E*-module in* $\mathcal{C}$*. We set* $\mathbf{X}^* = \underline{Hom}_E(\mathbf{X}, E)$.

(1) $\mathbf{X}^*$ *is* E*-flat.*

(2) *For any E-module $\mathbf{Y}$ in $\mathcal{C}$, the canonical map $\mathbf{X}^* \otimes_E \mathbf{Y} \to \underline{Hom}_E(\mathbf{X},\mathbf{Y})$ is an isomorphism.*

Proof. Both parts follow from the following special case of the second part.

Claim 1.2.14. *For any (usual) coherent E-module M, the canonical map $\mathbf{X}^* \otimes_E M \to \underline{Hom}_E(\mathbf{X},M)$ is an isomorphism.*

Proof of claim. M has a finite free resolution,

$$0 \to E^{n_k} \to \dots \to E^{n_2} \to E^{n_1} \to M \to 0$$

Applying the (exact) functor $\underline{Hom}_E(\mathbf{X},-)$ to the sequence, we get an exact sequence

$$0 \to (\mathbf{X}^*)^{n_k} \to \dots \to (\mathbf{X}^*)^{n_2} \to (\mathbf{X}^*)^{n_1} \to \underline{Hom}_E(\mathbf{X},M) \to 0$$

On the other hand, $\mathbf{X}^* \otimes_E M$ is, by definition, the co-kernel of the map $(\mathbf{X}^*)^{n_2} \to (\mathbf{X}^*)^{n_1}$ in that sequence. ☐

We now return to the proof of the Proposition.

(1) We need to prove that the functor $\mathbf{X}^* \otimes_E - : (E)^{coh} \to \mathcal{C}$ is exact. According to the claim, this functor coincides with $\underline{Hom}_E(\mathbf{X},-)$, and since $\mathbf{X}$ is E-flat, the result follows.
(2) The two exact functors $\underline{Hom}_E(\mathbf{X},-)$ and $\mathbf{X}^* \otimes_E -$ from $E-\mathcal{C} \to \mathcal{C}$ restrict, according to Proposition 1.1.15, to functors on $(E)^{coh} \times \mathcal{C}$, and it is enough to show that they coincide on this category. The former restricts to the functor $(M,\mathbf{Y}) \mapsto \underline{Hom}_E(\mathbf{X},M) \otimes \mathbf{Y}$, while the latter to $(M,\mathbf{Y}) \mapsto (\mathbf{X}^* \otimes_E M) \otimes \mathbf{Y}$. Hence the functors are isomorphic by the claim. ☐

Corollary 1.2.15. *Assume that $\mathcal{C}$ is rigid. Then so are $\mathcal{C}_{(E)}$ and $\mathcal{C}^{(E)}$.*

Proof. From Proposition 1.2.13 together with Corollary 1.2.7, we conclude that $\underline{Hom}_E(\mathbf{X},\mathbf{Y})$ is flat whenever $\mathbf{X}$ and $\mathbf{Y}$ are. Since it clearly satisfies the adjunction property, the rigidity of $\mathcal{C}_{(E)}$ follows from the second part Proposition 1.2.13 and Deligne [6, §§ 2.3,2.5].

For $\mathcal{C}^{(E)}$, the statement follows from Corollary 1.2.10, since the opposite of a rigid category is rigid. We mention only that

$$\underline{Hom}^E(\mathbf{X},-) := \check{\mathbf{X}} \otimes_E - \tag{8}$$

is the right adjoint to $\mathbf{X} \otimes^E -$ in $\mathcal{C}^{(E)}$.

☐

1.3. **Passing to the limit.**

We would like now to replace the finite algebra E by a pro-finite one. This is done, essentially, by glueing a matching sequence of flat or injective modules along a filtering system.

1.3.1. *Limits of categories.*

Let $\pi : \mathcal{C} \to \mathcal{I}$ be a fixed functor. We say that an object $\mathbf{X}$ of $\mathcal{C}$ is over an object $\mathbf{J}$ of $\mathcal{I}$ if $\pi(\mathbf{X}) = \mathbf{J}$ (and likewise for morphisms). The *fibre* $\mathcal{C}_{\mathbf{J}}$ of $\mathcal{C}$ (or π) over $\mathbf{J}$ is the sub-category of $\mathcal{C}$ consisting of objects over $\mathbf{J}$ and morphisms over the identity of $\mathbf{J}$.

Recall (say, from Deligne and Milne [7, Appendix]) that $\mathcal{C}$ (or π) is a *fibred category* if for any map $f : \mathbf{I} \to \mathbf{J}$ in $\mathcal{I}$, and any object $\mathbf{X}$ over $\mathbf{J}$, there is a universal map over f in $\mathcal{C}$ from an object $f^*(\mathbf{X})$ over $\mathbf{I}$, and furthermore, $(gf)^*(\mathbf{X}) = f^*(g^*(\mathbf{X}))$

for all $f : \mathbf{I} \to \mathbf{J}$, $g : \mathbf{J} \to \mathbf{K}$ (more precisely, we are given functorial isomorphisms between the two, satisfying pentagon identities).

In particular, f^* is a functor from $\mathcal{C}_{\mathbf{J}}$ to $\mathcal{C}_{\mathbf{I}}$. Conversely, given a collection $\mathcal{C}_{\mathbf{J}}$ of categories, one for each object $\mathbf{J}$ of $\mathcal{I}$, and functors f^* for morphisms f of $\mathcal{I}$, with compatible isomorphisms as above, one constructs a fibred category with the prescribed fibres and pullbacks. In this sense, a fibred category can be thought of as a presheaf of categories.

Given two fibred categories $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{I}$, a *Cartesian functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ over $\mathcal{I}$, together with functorial identifications $g^*\mathscr{F}(\mathbf{X}) = \mathscr{F}(g^*\mathbf{X})$ for all morphisms g in $\mathcal{I}$. A morphism between Cartesian functors is a morphism of functors over the identity on $\mathcal{I}$, which commutes with the identifications.

We view $\mathcal{I}$ as a fibred category over itself, via the identity functor. More generally, for any category $\mathcal{D}$, we have a fibred category $\mathcal{D} \times \mathcal{I}$ over $\mathcal{I}$.

Definition 1.3.2. Let $\pi : \mathcal{C} \to \mathcal{I}$ be a fibred category. The *inverse limit* $\varprojlim_{\mathcal{I}} \mathcal{C}$ is the category of Cartesian functors from $\mathcal{I}$ to $\mathcal{C}$.

Hence, an object of $\varprojlim_{\mathcal{I}} \mathcal{C}$ is given by a collection of objects $\mathbf{X}_{\mathbf{J}}$ of $\mathcal{C}_{\mathbf{J}}$, one for each object $\mathbf{J}$ of $\mathcal{I}$, together with, for each morphism $f : \mathbf{I} \to \mathbf{J}$ in $\mathcal{I}$, an isomorphism $\mathbf{X}_{\mathbf{I}} \to f^*(\mathbf{X}_{\mathbf{J}})$, such that the system of such isomorphisms is compatible with compositions. In particular, if $\mathcal{I}$ has a terminal object $\mathbf{1}$, then the assignment $(\mathbf{X}_{\mathbf{J}}) \mapsto \mathbf{X}_{\mathbf{1}}$ is an equivalence of categories $\varprojlim_{\mathcal{I}} \mathcal{C} \xrightarrow{\sim} \mathcal{C}_{\mathbf{1}}$.

Intuitively, one may think of objects of $\mathcal{I}$ as pieces of some geometric objects, and of the morphisms as glueing instructions. The category $\mathcal{C}$ can be viewed as objects of a particular kind (say, vector bundles) over these pieces. An object of $\varprojlim_{\mathcal{I}} \mathcal{C}$ can then be viewed as an object of the same kind on the (hypothetical) glued space.

We note that $\varprojlim_{\mathcal{I}} \mathcal{C}$ satisfies the expected universal property: The category of functors from $\mathcal{D}$ to $\varprojlim_{\mathcal{I}} \mathcal{C}$ is equivalent to the category whose objects are compatible collections of functors $\mathcal{D} \to \mathcal{C}_{\mathbf{J}}$ (in other words, to the category of Cartesian functors $\mathcal{D} \times \mathcal{I} \to \mathcal{C}$).

1.3.3. As before, when the *Hom* sets are abelian groups or $\Bbbk$-vector spaces, we assume that the pullback functors preserve this structure. We note that the limit of abelian categories need not be abelian in general (see also 4.1). Also, the limit of $\Bbbk$-linear categories need not be $\Bbbk$-linear in our definition, since the finiteness conditions need not hold. However, when $\mathcal{I}$ is filtering (which is the case of interest for us), we may think of $Hom(\mathbf{X}, \mathbf{Y})$ as a pro-finite $\Bbbk$-vector space, and composition is a morphism in this category. Furthermore, each object has pro-finite length. We may call such categories *pro-$\Bbbk$-linear.*

1.3.4. *Tensor structure.* Assume now that each $\mathcal{C}_{\mathbf{J}}$ is a monoidal category, that the functors f^* are given with monoidal structure, and that the monoidal structures are compatible, and compatible with the composition isomorphisms (This is equivalent to saying that we are given a *Cartesian* functor $\otimes : \mathcal{C} \times_{\mathcal{I}} \mathcal{C} \to \mathcal{C}$, etc.). Then the limit category $\varprojlim_{\mathcal{I}} \mathcal{C}$ also has a monoidal structure, given pointwise. If each $\mathcal{C}_{\mathbf{J}}$ admits internal *Hom*s, and each f^* is closed, then the limit category again admits internal *Hom*s. Finally, if each $\mathcal{C}_{\mathbf{J}}$ is rigid, then so is the limit category (note that in this case, pullbacks are automatically closed, Deligne and Milne [7, Prop. 1.9]).

1.3.5. *The fibre-wise opposite.* Given a fibred-category $\pi : \mathcal{C} \to \mathcal{I}$, each pullback functor f^* determines a functor between the opposite categories. It is easy to see that the data of the fibred-category determines a fibred-category data on the collection of opposite categories, which we call the opposite fibred category $\mathcal{C}^{op}$. Hence we are given a functor $\pi : \mathcal{C}^{op} \to \mathcal{I}$, and the underlying category $\mathcal{C}^{op}$ is *not* the opposite category in the usual sense, but we shall never consider the latter in this setting, so there is no room for confusion.

When the fibres are abelian, or monoidal, or rigid, then so are the opposites, and if the original data came (e.g.) from a fibred monoidal category, then the opposite is again fibred monoidal. In particular, if each fibre is rigid, then the assignment $\mathbf{X} \mapsto \check{\mathbf{X}}$ is a Cartesian tensor equivalence between $\mathcal{C}$ and $\mathcal{C}^{op}$.

We also note that the limit of the opposite category is the opposite of the limit category. This is true even including the monoidal structure.

1.3.6. *Pullbacks for modules.* We now assume that we are given a category $\mathcal{C}$ as in 1.2. It will be convenient to think about E-modules in $\mathcal{C}$ geometrically, just like with usual modules. Thus, an E-module in $\mathcal{C}$ is thought of as a family of objects of $\mathcal{C}$, parametrised by $\operatorname{spec}(E)$, a flat module corresponds to a bundle of such objects, etc.

Given a map from a finite $\Bbbk$-algebra E to another such algebra F, corresponding to a map $f : \operatorname{spec}(F) \to \operatorname{spec}(E)$ (over $\Bbbk$), we have functors $f^*, f^! : E-\mathcal{C} \to F-\mathcal{C}$, and a functor $f_* : F-\mathcal{C} \to E-\mathcal{C}$, given by $f^*(\mathbf{X}) = F \otimes_E \mathbf{X}$, $f^!(\mathbf{X}) = \underline{Hom}_E(F, \mathbf{X})$ and $f_*(\mathbf{X}) = \mathbf{X}$. It follows directly from the definitions that f^* is left adjoint to f_*, which is left adjoint to $f^!$. Also, given another map $g : \operatorname{spec}(G) \to \operatorname{spec}(F)$, there are obvious isomorphisms $g^* \circ f^* \to (fg)^*$ and $g^! \circ f^! \to (fg)^!$. Therefore, the determine two fibred categories $\mathcal{C}^*$ and $\mathcal{C}^!$ over the category $\mathcal{S}$ of finite schemes over $\Bbbk$, with pullbacks given by f^* and $f^!$, respectively.

Proposition 1.3.7. *Assume $\mathcal{C}$ is rigid, and let $f : \operatorname{spec}(F) \to \operatorname{spec}(E)$ be a map over $\Bbbk$.*

1. *For any object $\mathbf{X}$ of $E - \mathcal{C}$, there are canonical isomorphisms $f^!(\check{\mathbf{X}}) = (f^*(\mathbf{X}))^\vee$*
2. *If $\mathbf{X}$ is E-flat, then $f^*(\mathbf{X})$ is F-flat.*
3. *If $\mathbf{X}$ is E-injective, then $f^!(\mathbf{X})$ is F-injective.*

Proof.
1. The isomorphism is given by Equation (7) of Proposition 1.2.3. Since all constructions are functorial, it commutes with the F-action.
2. The same as for usual modules
3. By the first two parts □

We note that the tensor structure was not used in any essential way (the duality could be replaced by passing to the opposite category). On the other hand, given the tensor structure, the restriction functors are tensor functor for the corresponding structure, in the obvious way, and the canonical isomorphisms are tensor isomorphisms.

It follows from the proposition that the fibred categories $\mathcal{C}^*$ and $\mathcal{C}^!$ above contain fibred sub-categories $\mathcal{C}^f$ and $\mathcal{C}^i$ of flat and injective modules, respectively (over the same base).

1.3.8. *Modules over pro-finite algebras.* Let E be a pro-finite algebra over $\Bbbk$. Hence E is a co-filtering system of finite algebras, indexed by a category $\mathcal{I}$. Equivalently, it is given by an ind-object $\operatorname{spec}(E)$ of the category $\mathcal{S}$ of finite schemes over $\Bbbk$ (i.e., it is a formal set in the terminology of §2).

Pulling back the categories $\mathcal{C}^f$ and $\mathcal{C}^i$ from above, we obtain fibred categories $\mathcal{C}^f|_{\mathcal{I}}$ and $\mathcal{C}^i|_{\mathcal{I}}$ over $\mathcal{I}$.

Definition 1.3.9. For a pro-finite algebra E, we defined $\mathcal{C}_{(E)}$, the category of *flat E-modules* in $\mathcal{C}$, to be the limit $\varprojlim_{\mathcal{I}} \mathcal{C}^f$ of the fibred category of flat modules along $\mathcal{I}$. Dually, we define the category of E-injective modules (or the *E-prolongation of* $\mathcal{C}$) $\mathcal{C}^{(E)}$ as the limit $\varprojlim_{\mathcal{I}} \mathcal{C}^i$.

We note that if E happens to be a finite algebra, this definition agrees with the previous one by the remarks following the definition of the limit.

Corollary 1.3.10. *For any pro-finite algebra E, the categories $\mathcal{C}_{(E)}$ and $\mathcal{C}^{(E)}$ are rigid (non-abelian) tensor categories, and $\mathbf{X} \mapsto \check{\mathbf{X}}$ determines a tensor equivalence $\mathcal{C}^{(E)} \to {\mathcal{C}_{(E)}}^{op}$.*

Proof. By Corollaries 1.2.7, 1.2.10 and 1.2.15, and the discussions in 1.3.4 and 1.3.5. □

We note that, as in the finite case, a flat coherent E-module M (i.e., an object of $\mathcal{V}ec_{(E)}$) determines, for each object $\mathbf{X}$ of $\mathcal{C}$ an object $M \otimes \mathbf{X}$ of $\mathcal{C}_{(E)}$ and an object $Hom(M, \mathbf{X})$ of $\mathcal{C}^{(E)}$.

1.3.11. The discussion on pullbacks (1.3.6) and Proposition 1.3.7 extend to maps between pro-finite algebras. Let $f : \operatorname{spec}(F) \to \operatorname{spec}(E)$ correspond to a map between two pro-finite algebras E and F. We define $f^! : \mathcal{C}^{(E)} \to \mathcal{C}^{(F)}$ as follows (f^* is analogous).

First, assume that F is finite. Then f is induced by some $f_1 : \operatorname{spec}(F) \to \operatorname{spec}(E_1)$, where E_1 is finite. Hence we have a functor $f_1^! : \mathcal{C}^{(E_1)} \to \mathcal{C}^{(F)}$. Given a Cartesian functor $\mathbf{X} : E \to \mathcal{C}^i$ (where we think of E as the index category), define $f^!(\mathbf{X}) = f_1^!(\mathbf{X}(E_1))$. This is well defined since $\mathbf{X}$ is Cartesian.

A general F is the inverse limit of finite ones, and for each finite piece F_a we obtain from the previous step a functor $f_a^! : \mathcal{C}^{(E)} \to \mathcal{C}^{(F_a)}$. These functors form a matching family, and hence determine a functor to the limit $\mathcal{C}^{(F)}$.

2. Fields with operators

In this section, we recall the formalism introduced in Moosa and Scanlon [15] and Moosa and Scanlon [14] (adapted to our setting). As indicated in the introduction, this formalism provides a unified framework for fields endowed with operators, including differential and difference fields, and (Kolchin-style) algebraic geometry over them.

The use of geometric language is mainly for the purpose of intuition. The case we will eventually be interested in is when $\mathbf{X} = \operatorname{spec}(\Bbbk_0)$ and $\mathbf{Z} = \operatorname{spec}(\Bbbk)$, with $\Bbbk$ a field extending $\Bbbk_0$, and the reader will not lose (or gain) anything by assuming this from the beginning (on the other hand, some of the examples are mainly interesting when $\Bbbk_0$ is not a field).

2.1. **Formal sets.**

Definition 2.1.1. Let $\mathbf{X}$ be a quasi-compact Noetherian scheme. By a *formal set* over $\mathbf{X}$, we mean an ind-object in the category of flat, finite schemes over $\mathbf{X}$ (i.e., schemes over $\mathbf{X}$ whose sheaf of algebras is flat and coherent as an $\mathbf{X}$-module).

The formal set is *strict* if it can be represented by a system of closed embeddings.

A *pointed* formal set is a formal set $\mathfrak{M}$ together with a map $\mathbf{X} \to \mathfrak{M}$ over $\mathbf{X}$.

A *formal (abelian) monoid* (etc.) over $\mathbf{X}$ is an (abelian) monoid object in this category.

Example 2.1.2. Let S be a finite set. The co-product $S \times \mathbf{X} = \coprod_S \mathbf{X}$ is clearly flat and finite over $\mathbf{X}$, and is therefore a (finite) formal set over $\mathbf{X}$. A map $f : S \to T$ to another finite set induces a morphism $f \times 1 : S \times \mathbf{X} \to T \times \mathbf{X}$ over $\mathbf{X}$, and any morphism over $\mathbf{X}$ comes from a map of sets. Hence $S \mapsto S \times \mathbf{X}$ is a fully faithful exact embedding of finite sets in formal sets over $\mathbf{X}$. Since a set is the same as an ind-finite set, this also determines a fully faithful exact embedding of the category of sets into (strict) formal sets.

Since the embedding is exact, it induces an embedding of pointed sets, (abelian) monoids, etc., into the category of formal such objects. For instance, the monoids of natural numbers or integers can be viewed as formal monoids over $\mathbf{X}$. As with usual sets, it is *not* the case that a formal monoid is an ind-object in finite formal monoids.

As explained in the introduction, the guiding principle for all that follows is that any construction or result valid for usual sets should extend to formal sets. □

Example 2.1.3. Let $\mathbf{Y}$ be a scheme over $\mathbf{X}$. If $\mathbf{X}$ is smooth of dimension at most 1 (possibly this condition is redundant), the collection of flat and finite closed subschemes of $\mathbf{Y}$ over $\mathbf{X}$ forms a filtering system, and so determines a strict formal set $\mathfrak{M}$ (which might be empty).

There is a map $\mathfrak{M} \to \mathbf{Y}$ (in the category of ind-schemes over $\mathbf{X}$), and every map from a formal set to $\mathbf{Y}$ (all over $\mathbf{X}$) factors uniquely via $\mathfrak{M}$. For such $\mathbf{X}$, the previous example is obtained from this one as a special case by taking, for an arbitrary set S, $\mathbf{Y} = \coprod_S \mathbf{X}$. □

Example 2.1.4. In the situation of the previous example, let $\mathbf{Y}_0$ by a fixed subscheme of $\mathbf{Y}$, flat and finite over $\mathbf{X}$. The sub-system of $\mathfrak{M}$ consisting of subschemes with the same set-theoretic support as $\mathbf{Y}_0$ is again filtering (even without the restrictions on $\mathbf{X}$), and can be identified with the completion of $\mathbf{Y}$ along $\mathbf{Y}_0$.

For instance, if $\mathbf{X} = \operatorname{spec}(\Bbbk)$ is a point, $\mathbf{Y} = \mathbb{A}^1$ and $\mathbf{Y}_0$ the point at the origin, we obtain the formal scheme with one point, and structure sheaf $\Bbbk[[x]]$. □

Example 2.1.5. As a special case of the previous example, any formal group law over a ring $\Bbbk$ determines a formal abelian group over $\mathbf{X} = \operatorname{spec}(\Bbbk)$ in our sense. □

Example 2.1.6. Any Hasse–Schmidt system $\underline{\mathcal{D}}$ over A, in the sense of Moosa and Scanlon [14, Def. 2.2], determines a pointed strict formal set $(\operatorname{spec}(\mathcal{D}_i(A)))_i$ over $\mathbf{X} = \operatorname{spec}(A)$, indexed by the natural numbers. Conversely, any such pointed strict formal set $(\mathbf{Y}_i)$ can be extended to a Hasse–Schmidt system by choosing a compatible system of A bases, where $\mathcal{D}_i(\mathbf{Z}) = \mathbf{Y}_i \times_{\mathbf{X}} \mathbf{Z}$ (cf. Moosa and Scanlon [14, after Def. 2.1]).

Likewise, an iterative Hasse–Schmidt system over A (Moosa and Scanlon [14, Def. 2.17]) is naturally identified (up to a choice of basis) with a formal abelian monoid over $\mathbf{X}$. $\square$

We work in the category of ind-schemes over $\mathbf{X}$. Given a formal set $\mathfrak{M}$, we get, for any scheme $\mathbf{Z}$ over $\mathbf{X}$, an ind-scheme $\mathfrak{M} \times_{\mathbf{X}} \mathbf{Z}$, and if $\mathfrak{M}$ is pointed, a map $\mathbf{Z} \to \mathfrak{M} \times_{\mathbf{X}} \mathbf{Z}$. This process extends to the case of ind-schemes $\mathbf{Z}$.

Definition 2.1.7. Let $\mathfrak{M}$ be a pointed formal set over $\mathbf{X}$ (as above). A $\mathfrak{M}$-*scheme* is a scheme $\mathbf{Z}$ over $\mathbf{X}$, together with a map $\mathfrak{M} \times_{\mathbf{X}} \mathbf{Z} \to \mathbf{Z}$ over $\mathbf{X}$, such that the induced map $\mathbf{Z} \to \mathbf{Z}$ resulting from the point is the identity.

If $\mathfrak{M}$ is a formal abelian monoid, an *iterative* $\mathfrak{M}$-*scheme* is an $\mathfrak{M}$-scheme in which the structure map is a monoid action (when $\mathfrak{M}$ is such a monoid, all $\mathfrak{M}$-schemes we will consider will be iterative, so we will usually omit the title "iterative").

If $\mathbf{Z}$ is an (iterative) $\mathfrak{M}$-scheme, an *(iterative)* $\mathfrak{M}$-*scheme over* $\mathbf{Z}$ is a map of (iterative) $\mathfrak{M}$-schemes $\mathbf{W} \to \mathbf{Z}$.

Example 2.1.8. Under the identification in Example 2.1.6, an affine $\mathfrak{M}$-scheme is the same as a Hasse–Schmidt ring in the sense of Moosa and Scanlon [14, Def. 2.4] (note that the choice of basis in the definition of a Hasse–Schmidt system does not play any role in the definition of a Hasse–Schmidt ring). Likewise, with the adjective "iterative" added. $\square$

Example 2.1.9. If $\mathbf{X}$ is over $\mathrm{spec}(\Bbbk)$, and $\mathfrak{M} = \mathbf{X} \times_{\mathrm{spec}(\Bbbk)} \mathrm{spec}(\Bbbk[\epsilon])$, where $\Bbbk[\epsilon]$ is the ring of dual numbers ($\epsilon^2 = 0$), pointed in the only possible way, then an $\mathfrak{M}$-structure on $\mathbf{Z}$ is the same as a vector field on $\mathbf{Z}$ over $\mathbf{X}$.

If $\Bbbk$ has characteristic 2, then $\mathfrak{M}$ is a finite flat group sub-scheme of the additive group over $\mathbf{X}$ (and therefore a formal group in our sense), and an iterative $\mathfrak{M}$-structure on $\mathbf{Z}$ corresponds to a vector field ∂ such that $\partial^2 = 0$. $\square$

Example 2.1.10. If $\mathbf{X}$ is over $\mathrm{spec}(\Bbbk)$, where $\Bbbk$ is a field of characteristic 0, and $\mathfrak{M} = \mathbf{X}[[t]]$, the additive formal group over $\mathbf{X}$ (as in Example 2.1.5), then an (iterative) $\mathfrak{M}$-structure on a scheme $\mathbf{Z}$ over $\mathbf{X}$ is again the same as a vector field on $\mathbf{Z}$ over $\mathbf{X}$. In general, such a structure corresponds to a system of Hasse–Schmidt derivations. This is explained in detail in Moosa and Scanlon [14, Prop. 2.20], but we recall the computation for convenience.

Since everything is local, we may assume that $\mathbf{X} = \mathrm{spec}(A)$ and $\mathbf{Z} = \mathrm{spec}(B)$ are affine. Then $\mathfrak{M} = \mathrm{spec}(A[[t]])$, and an $\mathfrak{M}$-structure on $\mathbf{Z}$ is an algebra map $d : B \to B[[t]]$ over A. Hence it is given by $d(b) = \sum_{i \in \omega} \partial_i(b) t^i$ for some maps $\partial_i : B \to B$. The statement that d is an algebra map means that each ∂_i is an A-module map, and that $\partial_n(ab) = \sum_{i \leqslant n} \partial_i(a)\partial_{n-i}(b)$ for $a, b \in B$. The condition on the base point $A[[t]] \to A$ means that ∂_0 is the identity. Finally, iterativity means that d makes B an $A[[t]]$-comodule (for the additive group law $c : t \mapsto t \otimes 1 + 1 \otimes t$), so that for all $b \in B$

$$\sum_{i,j \in \omega} \binom{i+j}{i} \partial_{i+j}(b) t^i \otimes t^j = (1 \otimes c)(d(b)) = d(d(b)) = \sum_{i,j \in \omega} \partial_i(\partial_j(b)) t^i \otimes t^j \qquad (9)$$

Hence, $\binom{i+j}{i}\partial_{i+j} = \partial_i \circ \partial_j$, which is precisely the definition of an iterative Hasse–Schmidt derivation. See also Example 2.3.4, where we make a similar computation for usual derivations. $\square$

Example 2.1.11. If $\mathfrak{M}$ is a pointed set (over $\mathbf{X}$), then an $\mathfrak{M}$-structure on $\mathbf{Z}$ is simply a collection of endomorphisms of $\mathbf{Z}$, indexed by $\mathfrak{M}$, such that the point corresponds to the identity. Likewise, if $\mathfrak{M}$ is a (discrete) monoid (or group), an (iterative) $\mathfrak{M}$-structure is an action of $\mathfrak{M}$ on $\mathbf{Z}$. In particular, for $\mathfrak{M} = (\mathbb{Z}, +)$, an $\mathfrak{M}$-structure on $\mathbf{Z}$ is the same as an automorphism of $\mathbf{Z}$ (cf. Moosa and Scanlon [14, § 5.1]). □

2.1.12. *Free formal monoids.* We will see in Prop. 2.3.2 below that to any pointed formal set $\mathfrak{M}$ we may associate a free formal monoid $\mathcal{F}(\mathfrak{M})$, such that $\mathfrak{M}$-schemes are identified with iterative $\mathcal{F}(\mathfrak{M})$-schemes. For this reason, we are free to restrict our attention to the iterative case from now on.

2.2. **Prolongations.** From now on we fix a base scheme $\mathbf{X}$, and take all schemes, formal sets, etc., to be over $\mathbf{X}$, without mentioning it. We also fix a pointed formal set $\mathfrak{M}_0$.

2.2.1. For a scheme $\mathbf{Y}$, we write $\mathcal{S}ch/\mathbf{Y}$ for the category of quasi-projective schemes over $\mathbf{Y}$.

We say that a map $q : \mathbf{W} \to \mathbf{Y}$ of ind-objects is *compact* if for any map $r : \mathbf{Y}_0 \to \mathbf{Y}$ where $\mathbf{Y}_0$ is compact (i.e., an object in the original category), the pullback $r^*(\mathbf{W})$ is compact as well (the term "proper" would be better, but this becomes confusing in the context of schemes). For $\mathbf{Y}$ an ind-scheme, we denote by $\mathcal{S}ch/\mathbf{Y}$ the category of ind-quasi-projective schemes compact over $\mathbf{Y}$ (note that a compact ind-scheme over a scheme is a scheme, so there is no contradiction).

2.2.2. *Weil restriction.* A map $p : \mathbf{Y} \to \mathbf{Z}$ of schemes determines a base change functor $p^* : \mathcal{S}ch/\mathbf{Z} \to \mathcal{S}ch/\mathbf{Y}$. When p is flat and finite, this functor has a right adjoint, p_*, called the *Weil restriction* (cf, e.g., Conrad et al. [5, A.5]). Hence, if $q : \mathbf{W} \to \mathbf{Y}$ is a scheme over $\mathbf{Y}$, and $\mathbf{T}$ is a scheme over $\mathbf{Z}$, a $\mathbf{T}$ point of $p_*(\mathbf{W})$ corresponds to a family of sections of q, parametrised by $\mathbf{T}$.

Given a diagram of schemes over $\mathbf{Z}$

$$\begin{tikzcd}
\mathbf{W}_1 := r^*(\mathbf{W}_2) \arrow[r] \arrow[d, "q_1"'] & \mathbf{W}_2 \arrow[d, "q_2"] \\
\mathbf{Y}_1 \arrow[r, "r"'] \arrow[dr, "p_1"'] & \mathbf{Y}_2 \arrow[d, "p_2"] \\
& \mathbf{Z}
\end{tikzcd} \tag{10}$$

a section of q_2 restricts to a section of q_1, so we obtain a map $p_{2*}(\mathbf{W}_2) \to p_{1*}(\mathbf{W}_1)$ over $\mathbf{Z}$. Hence, if $p : \mathfrak{M} \to \mathbf{Z}$ is a formal set over $\mathbf{Z}$, and $\mathbf{W}$ is a compact (quasi-projective) ind-scheme over $\mathfrak{M}$, we obtain a pro-scheme $p_*(\mathbf{W})$ over $\mathbf{Z}$. We note that if $\mathbf{T}$ is a scheme over $\mathbf{Z}$, the pullback $p^*(\mathbf{T})$ is compact over $\mathfrak{M}$, and we still have the adjunction property $Hom_{\mathcal{S}ch/\mathfrak{M}}(p^*(\mathbf{T}), \mathbf{W}) = Hom_{\mathrm{Pro}(\mathcal{S}ch/\mathbf{Z})}(\mathbf{T}, p_*(\mathbf{W}))$. This fact further extends formally to the case when $\mathbf{T}$ is an ind-scheme.

We note also that when the map r above is a closed embedding, the resulting map $p_{2*}\mathbf{W}_2 \to p_{1*}\mathbf{W}_1$ is dominant (since an open subset of $\mathbf{W}_1$ comes from an open subset of $\mathbf{W}_2$), so when $\mathfrak{M}$ is strict, $p_*\mathbf{W}$ is strict as well.

Example 2.2.3. If $\Bbbk$ is a field, $\mathbf{Z} = \mathrm{spec}(\Bbbk)$, $\mathfrak{M} = \mathrm{spec}(\Bbbk[[x]])$, and $\mathbf{W}$ is a scheme over $\Bbbk$, then $p_*(p^*(\mathbf{W}))$ is the arc space of $\mathbf{W}$. The adjunction map $\mathbf{W} \to p_*(p^*(\mathbf{W}))$ is the 0-section. □

Remark 2.2.4. The notation is chosen so that it is compatible when the scheme $\mathbf{W}$ over $\mathbf{Z}$ is identified with the presheaf (on $\mathcal{S}ch/\mathbf{Z}$) it represents. When $\mathbf{W}$ is affine over $\mathbf{Z}$, it corresponds to an $\mathcal{O}_{\mathbf{Z}}$-algebra (in particular, $\mathcal{O}_{\mathbf{Z}}$-module) $\mathcal{O}_{\mathbf{W}}$, but the operations above *do not* correspond to the similarly denoted operations on modules. ☐

Definition 2.2.5. Let $\mu : \mathfrak{M}_0 \times \mathbf{Z} \to \mathbf{Z}$ be an $\mathfrak{M}_0$-scheme, and let $\mathbf{W}$ be a quasi-projective scheme over $\mathbf{Z}$. The $\mathbf{Z}$*-prolongation* of $\mathbf{W}$ is the pro-scheme $\tau(\mathbf{W}) = p_*(\mu^*(\mathbf{W}))$ over $\mathbf{Z}$, where p is the projection $p : \mathfrak{M} := \mathfrak{M}_0 \times \mathbf{Z} \to \mathbf{Z}$. The base point of $\mathfrak{M}_0$ determines a map $\pi^{\tau}_{\mathbf{W}} : \tau(\mathbf{W}) \to \mathbf{W}$.

2.2.6. Thus, $\tau(\mathbf{W})$ represents the functor $\mathbf{T} \mapsto \mathbf{W}(\mu_!(\mathfrak{M}_0 \times \mathbf{T}))$ on $\mathcal{S}ch/\mathbf{Z}$, where $\mu_!(\mathfrak{M}_0 \times \mathbf{T})$ is the ind-scheme $\mathfrak{M}_0 \times \mathbf{T}$, with $\mathbf{Z}$ structure given by composition with μ (in other words, $\mu_!$ is the left adjoint of μ^*).

A map $\mathbf{T} \to \mathbf{Z}$ of $\mathfrak{M}_0$-schemes induces a map $\mu_!(\mathfrak{M}_0 \times \mathbf{T}) \to \mathbf{T}$, hence induces by the previous paragraph a function of sets $\nabla^{\mathbf{T}} : \mathbf{W}(\mathbf{T}) \to \tau_{\mathbf{Z}}(\mathbf{W})(\mathbf{T})$ (cf. Moosa and Scanlon [14, Def. 2.10]). In particular, an $\mathfrak{M}_0$-structure on $\mathbf{W}$ is the same as a section of $\pi : \tau(\mathbf{W}) \to \mathbf{W}$.

2.2.7. The functor τ extends in an obvious way to a functor on (quasi-projective) pro-schemes over $\mathbf{Z}$. In particular, $\tau^2\mathbf{W}$ makes sense. Assume now that we are given a monoid structure $m : \mathfrak{M}_0 \times \mathfrak{M}_0 \to \mathfrak{M}_0$, and that μ is a monoid action. Then for any scheme $\mathbf{T}$ over $\mathbf{Z}$ we obtain a map $m \times 1_{\mathbf{T}} : \mu_!(\mathfrak{M}_0 \times \mu_!(\mathfrak{M}_0 \times \mathbf{T})) \to \mu_!(\mathfrak{M}_0 \times \mathbf{T})$. By taking $\mathbf{T}$-points, this map induces a map $m^{\tau} : \tau \to \tau^2$, and the monoid axioms imply that $(\tau, \pi^{\tau}, m^{\tau})$ is a co-monad (on $\mathrm{Pro}(\mathcal{S}ch/\mathbf{Z})$). An $\mathfrak{M}_0$-scheme over $\mathbf{Z}$ is the same as a co-action of this co-monad (See Mac Lane [12, § VI] for co-monads; there is some more discussion on this in §4.6).

In particular, each $\tau\mathbf{W}$ is an $\mathfrak{M}_0$-scheme over $\mathbf{Z}$, which is universal among $\mathfrak{M}_0$-schemes over $\mathbf{W}$. In other words, the functor τ from $\mathrm{Pro}(\mathcal{S}ch/\mathbf{Z})$ to $\mathfrak{M}_0$-pro-schemes over $\mathbf{Z}$ is right adjoint to the forgetful functor (as promised in the introduction).

Example 2.2.8. Let $\Bbbk$ be a field, $\mathbf{Z} = \mathrm{spec}(\Bbbk)$ and $\mathfrak{M}_0 = \mathrm{spec}(\mathbb{Z}[\epsilon])$, as in Example 2.1.9, so that an $\mathfrak{M}_0$-structure on $\mathbf{Z}$ corresponds to a derivation $'$ on $\Bbbk$. If $\mathbf{W}$ is a scheme over $\Bbbk$, $\tau(\mathbf{W})$ is then a scheme whose A points (for a $\Bbbk$-algebra A) are $\mathbf{W}(A[\epsilon])$, where the $\Bbbk$ algebra structure on $A[\epsilon]$ is given by $x \mapsto x + x'\epsilon$. Hence $\tau(\mathbf{W})$ is the twisted tangent bundle of $\mathbf{W}$ over $\Bbbk$.

If A itself is endowed with a vector field $'$ compatible with the one on $\Bbbk$ (i.e., with an $\mathfrak{M}_0$-structure over $\mathbf{Z}$), then the map ∇^A above is induced by pre-composing with $\mathrm{spec}(A[\epsilon]) \to \mathrm{spec}(A)$, $a \mapsto a + a'\epsilon$, i.e., by differentiating the A-points of $\mathbf{W}$. In particular, a vector field on $\mathbf{W}$ (extending that on $\Bbbk$) is the same as a section of the twisted tangent bundle. ☐

Example 2.2.9. Generalising Example 2.2.3, we may consider the special case $\mathbf{X} = \mathbf{Z} = \mathrm{spec}(\Bbbk)$, with the trivial action of (any) $\mathfrak{M}_0$. Then $\tau\mathbf{W}$ is the analogue of the arc space (or the tangent bundle) for $\mathfrak{M} = \mathfrak{M}_0$. In particular, an $\mathfrak{M}$-structure on $\mathbf{W}$ is the same as a section of $\tau\mathbf{W} \to \mathbf{W}$ (this is analogous to the statement that a derivation on $\mathbf{W}$ is the same as a morphism $\Omega^1\mathbf{W} \to \mathbf{W}$ of $\mathbf{W}$ modules).

The functor $\tau\mathbf{W}$ is the internal-Hom $\underline{Hom}(\mathfrak{M}, \mathbf{W})$, in the sense that $\tau\mathbf{W}(\mathbf{T}) = Hom(\mathfrak{M} \times \mathbf{T}, \mathbf{W})$ (a projective limit of morphisms of schemes over $\Bbbk$). This all remains true for any affine $\mathbf{W}$ (not necessarily finitely generated), since any affine scheme can be viewed as an inverse system of finite generated ones.

We note that this construction may be difficult to glue if $\mathfrak{M}$ is not local. □

2.2.10. We call a map of pro-schemes $f : \mathbf{U} \to \mathbf{V}$ a closed embedding if for any map $p : \mathbf{U} \to \mathbf{U}_0$, with $\mathbf{U}_0$ a scheme, there is a map $q : \mathbf{V} \to \mathbf{V}_0$ with $\mathbf{V}_0$ a scheme, such that $q \circ f$ factors through $f_0 \circ p$, with $f_0 : \mathbf{U}_0 \to \mathbf{V}_0$ a closed embedding.

Definition 2.2.11. Let $\mathfrak{M}$ be a formal monoid acting on a scheme $\mathbf{Z}$, and let $\mathbf{W}$ be a quasi-projective scheme over $\mathbf{Z}$. A $\mathfrak{M}$-*subscheme* of $\mathbf{W}$ is a closed subscheme of $\tau\mathbf{W}$ that is closed under the action. In other words, it is an $\mathfrak{M}$ pro-scheme $\mathbf{W}_1$ over $\mathbf{W}$, such that the induced map $\mathbf{W}_1 \to \tau\mathbf{W}$ is a closed embedding.

2.3. **The affine picture.** Through most of this section, we only talk about formal sets, and not their actions, so we will use E and $\mathfrak{M}$ in place of E_0 and $\mathfrak{M}_0$.

2.3.1. Assume that $\mathbf{X} = \operatorname{spec}(A)$. Then a formal set $\mathfrak{M}$ over $\mathbf{X}$ corresponds to a projective system $E = (E_i)$ of finite flat A-algebras, and a base point corresponds to an A-algebra map $E \to A$. A monoid structure m on $\mathfrak{M}$ determines a bi-algebra structure $m^* : E \to E \otimes_A E$ (i.e., m^* is a map of pro-A-algebras. It is not, in general, induced from the finite levels).

Likewise, an affine $\mathfrak{M}$-scheme corresponds to an A-algebra B, together with a pro-algebra map $B \to E \otimes_A B$ (inducing the identity when composed with the base point $E \to A$). The map is iterative if it makes B a co-module over E.

Proposition 2.3.2. *Let $\mathfrak{M}$ be a pointed formal set. Then there is a universal map $\mathfrak{M} \to \mathcal{F}(\mathfrak{M})$ of point formal sets, where $\mathcal{F}(\mathfrak{M})$ is a formal monoid. This map identifies iterative $\mathcal{F}(\mathfrak{M})$-schemes with $\mathfrak{M}$-schemes. Likewise, there is a free formal abelian monoid $\mathcal{A}(\mathfrak{M})$.*

Proof. It is enough to give an affine construction that localises, since the universal property will ensure the glueing. With notation as above, we first ignore the algebra structure, and view E as a pro-finite flat A-module, together with an A-module map $p : E \to A$. We produce a co-algebra TE, and a universal map (from a co-algebra) $\pi : TE \to E$ over A. The construction is dual to that of the tensor algebra.

Let $E^{\otimes n}$ be the n-fold tensor power of E over A, and let $E_n \subseteq E^{\otimes n}$ be the equaliser of all the maps $E^{\otimes n} \to E^{\otimes n-1}$ obtained by tensoring p with identity maps. E_n is finite (since A is Noetherian) and flat over A (for example, if E is free, then so is E_n, and the construction localises). The unique map $E_n \to E^{\otimes n-1}$ determined by these maps clearly factors through E_{n-1}, and we set $TE = (E_i)$. We let π be the projection on $E_1 = E$. The co-multiplication is given by the map $E_{i+j} \to E_i \otimes_A E_j$ which is the restriction of the identity map. It is clear that this is a co-algebra. To get the (co-) commutative version, simply symmetrise the tensors.

Given another co-algebra H and a map $t : H \to E$ over A, we lift it to a map $t_n : H \to E_n$ via $t^{\otimes n} \circ c^{\circ n-1} : H \to E^{\otimes n}$, where $c^{\circ n-1} : H \to H^{\otimes n}$ is the application of the co-multiplication c of H $n-1$ times. The co-algebra axioms imply that this map factors through E_n, and it is clearly a unique co-algebra map over E. We note also that T commutes with filtered inverse limits (in pro-finite flat A-co-algebras). In particular, if E is given by a system (E^α), then TE is the inverse limit of the TE^α.

Finally, assume that E is a system of algebras. By the remark above, we may assume that E itself is a finite flat A-algebra. The multiplication map m determines a map $m \circ \pi \otimes \pi : TE \otimes_A TE \to E$ over A, hence a co-algebra map $TE \otimes_A TE \to TE$, which is easily seen to be an algebra map. □

Example 2.3.3. If $\mathfrak{M}$ is a discrete set, then the free monoid generated by it coincides with the usual free monoid in the category of sets. $\square$

Example 2.3.4. If $\mathfrak{M} = \operatorname{spec}(A[\epsilon])$, as in Example 2.1.9, the bi-algebra of the free monoid can be described as follows. Let $A[[\epsilon_1, \epsilon_2, \dots]]$ be the formal power series algebra in countably many variables ϵ_i, each satisfying $\epsilon_i^2 = 0$. The symmetric group S_ω of the natural numbers acts on this algebra, and TE is the sub-algebra of invariant elements (i.e., symmetric power series). Each element of TE can be written as $\sum_{i\in\omega} a_i e_i$, where e_i is the i-th elementary symmetric power series $e_i = \sum_{j_1<\dots<j_i} \epsilon_{j_1}\dots\epsilon_{j_i}$ in the variables ϵ_k. In this presentation, the algebra structure is given by $e_i e_j = \binom{i+j}{i} e_{i+j}$, and the co-algebra structure is given by $e_k \mapsto \sum_{i\leqslant k} e_i \otimes e_{k-i}$.

We show, by explicit calculation, that TE satisfies the required property (this can be contrasted with the calculation in Example 2.1.10). A map $d : A \to TE$ may thus be written as $d(a) = \partial_0(a) + \partial_1(a) + \dots$, where ∂_i are some maps $A \to A$. The requirement that the unit (corresponding to the map $e_i \mapsto 0$ for $i > 0$) acts as the identity means that $\partial_0(a) = a$. The requirement that d makes A a comodule (i.e., that we have a monoid action) means the following: applying d again to the coefficients of $d(a)$, we obtain

$$d(d(a)) = \sum_{i,j\in\omega} \partial_i(\partial_j(a)) e_i \otimes e_j \tag{11}$$

(recall that we view TE as a pro-algebra, so the tensor product consists of "power series" in the $e_i \otimes e_j$). Comparing this with the co-algebra structure, we see that $\partial_{i+j}(a) = \partial_i \circ \partial_j$. Hence d is determined by ∂_1. Finally, the statement that d is an algebra map means that ∂_1 is a derivation (and the product formula makes it consistent for higher i). In other words, an action of $\mathfrak{M}$ is precisely the same as a derivation (in any characteristic!), so $\operatorname{spec}(TE)$ is indeed the free monoid on $\operatorname{spec}(A[\epsilon])$.

The algebra map $A[[x]] \to A[\epsilon]$ from the additive formal group induces the bi-algebra map $f : A[[x]] \to TE$, $f(x) = e_1$, and when A contains $\mathbb{Q}$, this map is an isomorphism. On the other hand, if A contains $\mathbb{F}_p$, then TE is generated (as a power series algebra) by the e_{p^k}, with $e_i^p = 0$ for $i > 0$.

We note that e_k^d is divisible by $d!$, and the assignment $e_k^{(d)} = e_k^d/d!$ determines a divided power structure on TE (with respect to the ideal generated by all e_i), which could be called the universal complete divided power A-algebra in one variable. $\square$

2.3.5. *Cartier duality.* The system $E = (E_i)$ determines a direct system $\breve{E} = (\breve{E_i})$ of finite dimensional co-algebras over A ($\breve{E_i}$ is the dual with respect to A). Since the category of co-algebras over A is equivalent (by taking limits) to the category of ind-finite co-algebras (Deligne and Milne [7, Prop. 2.3]), this is just a co-algebra over A. A base point $E \to A$ corresponds to an element $1 \in \breve{E}$. A monoid structure then corresponds to an algebra structure $m_* : \breve{E} \otimes_A \breve{E} \to \breve{E}$, which commutes with the co-algebra structure, and which is commutative if the original monoid was commutative.

Hence, to a commutative formal monoid $\mathfrak{M}$ corresponds a commutative affine monoid scheme $\breve{\mathfrak{M}}$, which we call the *Cartier dual* of $\mathfrak{M}$ (this is precisely the usual Cartier duality when $\mathfrak{M}$ is finite, cf Waterhouse [22, § 2.4] or Pink [18]). Reversing the arguments above, we see that conversely, to a commutative affine monoid scheme over A corresponds a commutative formal monoid, and that the two operations are inverse to each other. We also note that the A-points of $\mathfrak{M}$ can be viewed as elements of $\check{E}$.

Given an $\mathfrak{M}$-scheme corresponding to an A-algebra B, the co-module structure on B corresponds to a module structure for $\check{E}$. The fact that the co-module structure is an algebra map means that we have the following commutative diagram:

$$\begin{tikzcd}
\check{E}\otimes B\otimes B \arrow[rr, "c\otimes 1\otimes 1"] \arrow[d, "1\otimes m"'] & & \check{E}\otimes\check{E}\otimes B\otimes B \arrow[d] \\
\check{E}\otimes B \arrow[r] & B & B\otimes B \arrow[l, "m"]
\end{tikzcd} \tag{12}$$

where c is the co-multiplication of $\check{E}$, and m is the multiplication on B.

Example 2.3.6. Assume that $\mathfrak{M}$ is a discrete commutative monoid Y. Then $\check{E}$ is the group algebra $A[Y]$, with co-algebra structure given by $y \mapsto y\otimes y$ for $y \in Y$. An $A[Y]$ module is then the same as an action of Y by A-linear map. The diagram (12) then means that Y acts by A-algebra endomorphisms, i.e., Y acts on $\mathrm{spec}(B)$. For □

Example 2.3.7. Let $\mathfrak{M}$ be the additive formal group ($E = A[[x]]$). Then $\check{E}$ is the A-algebra generated by elements u_i, $i > 0$, with relations $u_i u_j = \binom{i+j}{i} u_{i+j}$, and co-multiplication $c(u_n) = \sum_{i\leqslant n} u_i \otimes u_{n-i}$. In other words, it is the sub-bi-algebra of the algebra of the free monoid on the dual numbers consisting of finite sums (in yet other words, it is the free divided powers algebra in one variable over A).

If A has characteristic 0, we get $\check{\mathfrak{M}} = \mathbf{G_a}$, the additive group. In particular, a module over $\check{E} = A[x]$ is simply an A-linear action of x. Diagram (12) then reflects that x is a derivation. Similarly, in characteristic $p > 0$, $\check{E}$ is generated by u_{p^k} for $k > 0$ (with some relations), and an action satisfying (12) corresponds to a sequence of Hasse–Schmidt derivations. □

Remark 2.3.8. The procedure described in 2.3.5 is valid also when the monoid is not commutative, but the resulting algebra $\check{E}$ is not commutative, so the geometric interpretation as a scheme is no longer available. □

Remark 2.3.9. If $\mathfrak{N}$ is a formal monoid acting on $\mathfrak{M}$ by monoid endomorphisms, then it also acts on $\check{\mathfrak{M}}$, making $\check{\mathfrak{M}}$ a $\mathfrak{N}$-scheme, on which $\mathfrak{N}$ acts by monoid endomorphisms.

This happens for example if $\mathfrak{M}$ is (the additive monoid of) a formal semi-ring, and $\mathfrak{N}$ is the multiplicative monoid. For instance, if $\mathfrak{M}$ is the (discrete) ring of integers, then $\check{\mathfrak{M}}$ is the multiplicative group, and $\mathbb{Z}$ acts by endomorphisms in the usual way. Likewise, the dual of $\mathbb{Z}[i]$ is ${\mathbf{G_m}}^2$, with $i(a,b) = (-b,a)$.

It also happens with the additive formal group, on which the usual derivation acts by group endomorphisms. We therefore get a derivation on the dual, the divided powers algebra, given by $u'_i = u_{i-1}$. □

2.3.10. *The prolongations of affine spaces.* Assume again, as in 2.3.1, that we are given a formal monoid $\mathfrak{M}_0$ over an affine scheme $\mathrm{spec}(A)$, acting on $\mathbf{Z} = \mathrm{spec}(\Bbbk)$, where $\Bbbk$ is a field. Then $\mathfrak{M} = \mathfrak{M}_0 \otimes_A \Bbbk$ is given by a projective system $E = (E_i)$ of finite algebras over $\Bbbk$. We denote the projection and the action maps $\mathfrak{M} \to \mathrm{spec}(\Bbbk)$ by p and μ, respectively. The correspond to pro-algebra maps $\Bbbk \to E$ (over A).

Given a finite dimensional vector space V over $\Bbbk$, we let $\underline{V} = \operatorname{spec}(\operatorname{Sym}(\check{V}))$ be the associated affine space. Hence, for any $\Bbbk$-algebra B, the B-points of $\underline{V}$ correspond to $\Bbbk$-linear maps $\check{V} \to B$, i.e., to elements of $V \otimes_{\Bbbk} B$.

More generally, for a projective system $V = (V_i)$ of such spaces, $\underline{V} = (\underline{V_i})$ is the corresponding pro-scheme. We would like to compute the prolongation $\tau\underline{V}$ with respect to the given action. We denote by $E \otimes_{\mu} V$ the tensor product over $\Bbbk$, where E is given a $\Bbbk$-structure via μ. We view it as a vector space over $\Bbbk$ via the map p.

Proposition 2.3.11. *For a (pro-) finite-dimensional vector space V over $\Bbbk$, $\tau\underline{V} = \underline{E \otimes_{\mu} V}$.*

Proof. It is enough to prove that for any $\Bbbk$-algebra B, the two pro-schemes have the same B-points. Also, it suffices to prove the statement when E and V are finite.

By 2.2.6, the B-points of $\tau\underline{V}$ correspond to the $B_E = B \otimes_{\Bbbk} E$ points of $\underline{V}$, where the tensor product is taken with respect to the $\Bbbk$-vector space structure on E given by p, but the $\Bbbk$-structure on B_E is given by μ. Hence, by the above discussion, they correspond to elements of $(B \otimes_{\Bbbk} E) \otimes_{\mu} V = B \otimes_{\Bbbk} (E \otimes_{\mu} V)$. Again by the same discussion, these elements correspond to the B-points of $\underline{E \otimes_{\mu} V}$. ☐

Remark 2.3.12. It is easy to describe the action $\mathfrak{M} \times \tau\underline{V} \to \tau\underline{V}$ in these terms: it suffices to give an (ind-pro-) vector space map $\check{E} \otimes_{\Bbbk} E \otimes_{\mu} V \to E \otimes_{\mu} V$. The map is given by the "transpose" $m^t : \check{E} \otimes_{\Bbbk} E \to E$ of the co-algebra map $m : E \to E \otimes_{\Bbbk} E$. ☐

3. TANNAKIAN CATEGORIES

We now arrive at the main point, the description of the category of representations of a linear group. The description is completely analogous to the one given in Kamensky [10, § 4] in the special case of differential fields. However, the proof is simpler, since we reduce to the algebraic case, instead of mimicking its proof.

3.1. **Linear groups.** We fix a base action $\mathfrak{M}_0 \times \mathbf{Z} \to \mathbf{Z}$ with $\mathbf{Z} = \operatorname{spec}(\Bbbk)$, $\Bbbk$ a perfect field, and work in the category of $\mathfrak{M}_0$-pro-schemes over $\mathbf{Z}$ (as before, $\mathfrak{M}_0$, $\mathbf{Z}$ and all maps, products, etc. are over some base ring $\Bbbk_0$, which we generally omit from the notation). We set $\mathfrak{M} = \mathfrak{M}_0 \times \mathbf{Z}$. As explained in 2.2.7, each scheme $\mathbf{X}$ over $\mathbf{Z}$ (in the usual sense) determines an $\mathfrak{M}$-pro-scheme $\tau\mathbf{X}$. Since τ has a left-adjoint, it preserves products. In particular, a group pro-scheme $\mathbf{G}$ over $\mathbf{Z}$ determines a group object $\tau\mathbf{G}$ in the category of $\mathfrak{M}$-pro-schemes over $\mathbf{Z}$ (we call these $\mathfrak{M}$-*groups* from now on).

Definition 3.1.1. Let $\mathbf{G}$ be an $\mathfrak{M}$-group. A *representation* of $\mathbf{G}$ is a map (of $\mathfrak{M}$-groups) $\mathbf{G} \to \tau\mathbf{GL}(V)$ for some finite dimensional $\Bbbk$-vector space V. As customary, we sometimes write V for the whole representation. A representation is *faithful* if it is a closed embedding. The group $\mathbf{G}$ is *linear* if it admit a faithful representation.

3.1.2. We note that already in the differential case, there are affine groups that are not linear (Cassidy [3]), so the definition is reasonable. We also note that we have a slight discrepancy with the terminology of Deligne and Milne [7, Cor. 2.5].

Given an $\mathfrak{M}$-group scheme $\mathbf{G}$, we denote by $\overline{\mathbf{G}}$ the underlying group-pro-scheme. If $\mathbf{G}$ is a linear $\mathfrak{M}$-group, the category $\mathcal{R}ep_{\mathbf{G}}$ of representations of $\mathbf{G}$ is abelian and $\Bbbk$-linear in the usual way. With the usual tensor structure, it is a rigid tensor category.

The forgetful functor shows it is neutral Tannakian. We have the following simple observation.

Proposition 3.1.3. *Let* $\mathbf{G}$ *be a linear* $\mathfrak{M}$*-group. The algebraic group associated to the Tannakian category* $\mathcal{R}ep_{\mathbf{G}}$ *is* $\overline{\mathbf{G}}$.

Proof. By 2.2.7, τ is right adjoint to the forgetful functor. Since all functors involved are left exact, we get a similar result for groups. Applying this to the map $\mathbf{G} \to \tau\mathbf{GL}(V)$, we get the result. $\square$

3.1.4. Our goal is thus to describe an additional structure on $\mathcal{C} = \mathcal{R}ep_{\mathbf{G}}$ that will allow us to recover the action of $\mathfrak{M}$. We pass back to algebra: let E be the pro-algebra corresponding to $\mathfrak{M} \times \mathbf{Z}$. We ignore, at first, the monoid structure on $\mathfrak{M}$, and so deal with each piece separately. Thus, we assume that E is a finite algebra. The projection and action maps are denoted by $p, \mu : \mathrm{spec}(E) \to \mathrm{spec}(\Bbbk)$, respectively.

3.1.5. Recall from 1.3.6, that given a map $f : \mathrm{spec}(E) \to \mathrm{spec}(\Bbbk)$, there is a pullback functor $f^* : \mathcal{C} \to E - \mathcal{C}$, given by $f^*(\mathbf{X}) = E \otimes_{\Bbbk} \mathbf{X}$. We note that in the present situation, the functor is defined even if f is not finite. When f is finite, f^* has a right adjoint, f_* given by viewing an E-module as a $\Bbbk$-vector space via f (in general f_* is defined as a functor into $\mathrm{Ind}(\mathcal{C})$, but we will not need it). f^* is a tensor functor, and we have an internal version of the adjunction:

$$f_*(\underline{Hom}_E(f^*(\mathbf{X}), \mathbf{Y})) = \underline{Hom}(\mathbf{X}, f_*(\mathbf{Y})) \tag{13}$$

for any object $\mathbf{X}$ and E-module $\mathbf{Y}$. We also have an isomorphism

$$f_*(f^*(\mathbf{X}) \otimes_E \mathbf{Y}) = \mathbf{X} \otimes_{\Bbbk} f_*(\mathbf{Y}) \tag{14}$$

Assume that f is finite. Then f_* also has a right adjoint $f^! : \mathcal{C} \to E - \mathcal{C}$, given by $f^!(\mathbf{X}) = \underline{Hom}_{\Bbbk}(E, \mathbf{X})$. We note that $f^!(\mathbf{1}) = \check{E}$, where duality is with respect to f. More generally, $M^\vee = \underline{Hom}_E(M, f^!(\mathbf{1}))$ for a finite E-module M, and we may prefer the second notation to stress the dependence on f. We thus have, for any object $\mathbf{X}$, an isomorphism (in $E - \mathcal{C}$), as in Proposition 1.3.7

$$f^!(\check{\mathbf{X}}) = \underline{Hom}_E(f^*(\mathbf{X}), f^!(\mathbf{1})) = (f^*(\mathbf{X}))^\vee \tag{15}$$

In particular, $f^*(\mathbf{X})$ is E-flat and $f^!(\mathbf{X})$ is E-injective (as E-modules, disregarding the action of $\mathbf{G}$). Using the identities above, we again have an internal version of the adjunction:

$$\underline{Hom}_{\Bbbk}(f_*(\mathbf{X}), \mathbf{Y}) = f_*(\underline{Hom}_E(\mathbf{X}, f^!(\mathbf{Y}))) \tag{16}$$

Viewing f^* as a functor to the category $\mathcal{C}_{(E)}$ of flat E-modules in $\mathcal{C}$, f^* also has a left adjoint, $f_! : \mathcal{C}_{(E)} \to \mathcal{C}$, given by

$$f_!(\mathbf{X}) = f_*(f^!(\mathbf{1}) \otimes_E \mathbf{X}) = \check{E} \otimes_E \mathbf{X} \tag{17}$$

(this is obviously true when $\mathbf{X}$ is free, hence when $\mathbf{X}$ is flat by localisation). We have, by definition,

$$f_! f^* = f_* f^! \tag{18}$$

We note $f_!(\mathbf{X})$ has, in fact, the structure of an E-injective module. We also note that dually, the functor $f^! : \mathcal{C} \to \mathcal{C}^{(E)}$ has a right adjoint $f_\# : \mathcal{C}^{(E)} \to \mathcal{C}$, defined by $f_\#(\mathbf{X}) = \underline{Hom}_E(f^!(\mathbf{1}), \mathbf{X})$, but we will not use it.

3.1.6. We would like to apply the discussion above to the maps p and μ. Note that p, but not necessarily μ, is finite. We will be interested in the functor

$$\tau(\mathbf{X}) = p_!(\mu^*(\mathbf{X})) = \check{E} \otimes_\mu \mathbf{X} \tag{19}$$

which we view as a functor into either $\mathcal{C}$ or $\mathcal{C}^{(E)}$. Our interest in this functor is explained by Proposition 2.3.11, and the following fact.

Lemma 3.1.7. *Let* $\mathbf{X}$ *be a representation, and let* M *be an* E*-module.*

(1) *There is a canonical isomorphism* $\tau(\mathbf{X})^\vee = p_*\mu^*(\check{\mathbf{X}})$ *as* E*-modules.*
(2) *If* M *is a flat (or injective)* E*-module, then* $M \otimes_E \mu^*(\mathbf{X})$ *is* E*-flat (*E*-injective) in* $\mathcal{R}ep_{\mathbf{G}}$*.*

Proof. (1) We claim that both sides are isomorphic (as E-modules) to the space $\underline{Hom}_E(\mu^*(\mathbf{X}), E)$. For the left side, this follows directly from the adjunction. For the right side, let $\phi \in \check{\mathbf{X}}$. Then $\mu^{\#} \circ \phi$ is a map from $\mathbf{X}$ to E, linear with the respect to the vector-space structure on E given by $\mu^{\#}$ (the algebra map corresponding to μ). This is the same as a map $\mu^*(\mathbf{X}) \to E$ of E-modules, so we have a map $\check{\mathbf{X}} \to \underline{Hom}_E(\mu^*(\mathbf{X}), E)$, which is again $\mu^{\#}$-linear. We obtain an E-module map $\mu^*(\check{\mathbf{X}}) \to \underline{Hom}_E(\mu^*(\mathbf{X}), E)$, which is an isomorphism by dimension.

(2) This is clear, since $M \otimes_E \mu^*(\mathbf{X})$ is a (finite) direct sum of copies of M. ☐

Remark 3.1.8. As a result of this Lemma, we could, instead, work with the functor $\mathbf{X} \mapsto p_*\mu^*\mathbf{X}$, which is a tensor functor into $\mathcal{C}_{(E)}$, and is perhaps more familiar. We choose to use the current setting mostly since it is compatible with the original setup of Kamensky [10, § 4], and also because in our current setting, $\tau(\check{\mathbf{X}})$ has a simple interpretation as consisting of functions on $\mathbf{X}$ (as in Proposition 2.3.11). We discuss this again in the abstract setting of the next section, in Remark 3.2.2. ☐

We now describe the properties of the functor τ. Eventually, we will use these properties to characterise the situation.

Proposition 3.1.9. *The functor* τ *is naturally a tensor functor from* $\mathcal{R}ep_{\mathbf{G}}$ *to* $\mathcal{R}ep_{\mathbf{G}}^{(E)}$*.*

We note that τ is not $\Bbbk$-linear.

Proof. The fact that τ takes values in $\mathcal{R}ep_{\mathbf{G}}^{(E)}$ is explained above. To give τ a tensor structure, we need to provide functorial (E-module) isomorphisms $\tau(U \otimes_\Bbbk V) = \underline{Hom}_E(\tau(U)^\vee, \tau(V))$ (Proposition 1.2.3). The left hand side is isomorphic to $\mu^*(U) \otimes_E \tau(V)$ (directly from definitions), while by Lemma 3.1.7, the right hand side is isomorphic to

$$\underline{Hom}_E(\mu^*(\check{U}), \tau(V)) = \underline{Hom}_E(\mu^*(\check{U}), E) \otimes_E \tau(V) = \mu^*(U) \otimes_E \tau(V) \tag{20}$$

The verification that this is a tensor structure is straightforward. ☐

We now wish to change the algebra.

Proposition 3.1.10. *Assume that* E_1 *and* E_2 *are two rings with maps* p_1, μ_1 *and* p_2, μ_2 *as above, and corresponding functors* τ_1 *and* τ_2*. Assume, further, that we are given a ring map* $f : E_1 \to E_2$ *that preserves both* $\Bbbk$*-algebra structures. Then for any representation* V *we have an isomorphism of* E_2*-modules* $\underline{Hom}_{E_1}(E_2, \tau_1(V)) =$

$\tau_2(V)$, and together these isomorphisms determine an isomorphism of tensor functors.

Proof. Using $\tau_i(V) = \tau_i(\mathbf{1}) \otimes_{E_i} \mu_i^*(V)$ (as in the previous proof), and $\mu_2^*(V) = E_2 \otimes_{E_1} \mu_1^*(V)$, we reduce to the case $V = \mathbf{1}$. Hence, we need to prove that $\underline{Hom}_{E_1}(E_2, E_1{}^\vee) = E_2{}^\vee$ (as E_2-modules). But this is obvious, by taking duals. □

3.1.11. We recall that $\mathfrak{M}$ was assumed to have a base point, which acts as the identity. In other words, we are also given a map $i : \operatorname{spec}(\Bbbk) \to \operatorname{spec}(E)$, such that $\mu \circ i = p \circ i$. The map i induces, as before, a functor $i^! : \mathcal{C}^{(E)} \to \mathcal{C}$, $i^!(\mathbf{X}) = Hom_E(\Bbbk, \mathbf{X})$ (see also 1.3.6; geometrically, $i^!(\mathbf{X})$ consists of sections of $\mathbf{X}$ supported at the base point).

The functor $i^!$ extends, as in 1.3.11, to $\mathcal{C}^{(E)}$ when E is a pro-finite algebra. Applying the previous proposition we obtain, upon passing to inverse systems, the following result.

Corollary 3.1.12. *Let $\mathfrak{M} = \mathfrak{M}_0 \times \mathbf{Z}$ be a formal set (with $\mathbf{Z} = \operatorname{spec}(\Bbbk)$), let $p : \mathfrak{M} \to \mathbf{Z}$ be the projection, $i : \mathbf{Z} \to \mathfrak{M}$ a base point (section of p), and let $\mu : \mathfrak{M} \to \mathbf{Z}$ be a map (action), such that $\mu \circ i$ is the identity. The definitions above determine a tensor functor $\tau : \mathcal{C} = \mathcal{R}ep_{\mathbf{G}} \to \mathcal{C}^{(E)}$, and a (tensor) isomorphism $i^*(\tau(V)) = V$.*

Proof. Apply the proposition above to the maps $i : E_\alpha \to E_2 = \Bbbk$ and the transition maps $E_\beta \to E_\alpha$ of the system, using the definition of $\mathcal{C}^{(E)}$ in 1.3.6. □

Finally, we bring back the monoid structure. Let $m : \operatorname{spec}(E \otimes_\Bbbk E) \to \operatorname{spec}(E)$ be the product map. As in 1.3.11, m determines the functor $m^! : \mathcal{C}^{(E)} \to \mathcal{C}^{(E \otimes_\Bbbk E)}$. On the other hand, τ extends to a functor on ind-objects of $\mathcal{C}$ (which we again denote τ). Proposition 3.1.10 directly generalises to the case where the E_i are pro-finite algebras, and we obtain:

Proposition 3.1.13. *There is a $\Bbbk$-linear tensor isomorphism $\tau \circ \tau \to (p \times p)_* m^! \circ \tau$.*

Proof. As in the proof of Proposition 3.1.10, the statement reduces to $\mathbf{1}$, where we use duality to obtain the isomorphism. The statement that this isomorphism is $\Bbbk$-linear corresponds to the statement that the map $\mu : \Bbbk \to E$ is an action, i.e., $(1 \otimes \mu^\#) \circ \mu^\# = m^\# \circ \mu^\#$. □

3.2. **$\mathfrak{M}$-Tensor categories.** We now introduce the abstract axiomatisation of the situation described for representations. As usual, we fix a base ring $\Bbbk_0$, a perfect field $\Bbbk$ over $\Bbbk_0$, and a formal monoid $(\mathfrak{M}_0, i_0, m_0)$ over $\Bbbk_0$. We denote by E_0 the $\Bbbk_0$-pro-finite algebra corresponding to $\mathfrak{M}_0$, and set $E = E_0 \otimes_{\Bbbk_0} \Bbbk$ and $\mathfrak{M} = \operatorname{spec}(E) = \mathfrak{M}_0 \times \mathbf{Z}$. As before, we denote by $p : \mathfrak{M} \to \mathbf{Z}$ the projection.

The base point $i_0 : \operatorname{spec}(\Bbbk_0) \to \mathfrak{M}_0$ and the product $m_0 : \mathfrak{M}_0 \times \mathfrak{M}_0 \to \mathfrak{M}_0$ induce, by base change, maps $i : \mathbf{Z} \to \mathfrak{M}$ and $m : \mathfrak{M} \times_{\mathbf{Z}} \mathfrak{M} \to \mathfrak{M}$ over $\Bbbk$. Given an abelian $\Bbbk$-linear tensor category $\mathcal{C}$, these maps determine functors $i^! : \mathcal{C}^{(E)} \to \mathcal{C}$ and $m^! : \mathcal{C}^{(E)} \to \mathcal{C}^{(E \otimes_\Bbbk E)}$, as in 1.3.11. We note that $\mathcal{C}^{(E \otimes_\Bbbk E)}$ can be viewed as a full subcategory of $E \otimes_\Bbbk E - \mathcal{C} = E - \mathcal{C} \otimes_\Bbbk E - \mathcal{C}$. The fact that i_0 is the unit for the action translates into isomorphisms of $i^! \otimes_\Bbbk \mathbf{1} \circ m^!$ and $\mathbf{1} \otimes_\Bbbk i^! \circ m^!$ with the identity on $\mathcal{C}^{(E)}$, and similarly for the associativity of m.

We note that if $\mathcal{C}$ and $\mathcal{D}$ are two categories as above, and $\mathscr{F} : \mathcal{C} \to \mathcal{D}$ is an exact tensor functor, then $\mathscr{F}$ induces a functor from $\mathcal{C}^{(E)}$ to $\mathcal{D}^{(E)}$, which we denote by

$\mathscr{F}^{(E)}$ (and sometimes again by $\mathscr{F}$). We have a natural (tensor, $\mathbb{k}$-linear) isomorphism $i^! \circ \mathscr{F}^{(E)} = \mathscr{F} \circ i^!$, which we indeed denote by "=". Note that $i^!$ denotes the corresponding functor in both categories. Similar remarks apply to the other canonically determined functors, $m^!$, p_*, etc. As in the previous section, we will sometimes omit p_*.

Definition 3.2.1. With notation as above.

(1) An E_0-*structure* (or $\mathfrak{M}_0$-structure) on a $\mathbb{k}$-linear tensor category $\mathcal{C}$ consists of the following data:
   (a) A $\mathbb{k}_0$-linear tensor functor τ from $\mathcal{C}$ to $\mathcal{C}^{(E)}$, which is exact when viewed as a functor into $E - \mathcal{C}$.
   (b) $\mathbb{k}$-linear tensor isomorphisms

$$a : i^! \circ \tau \to Id_{\mathcal{C}} \tag{21}$$

   and

$$b : p_* \circ \tau \circ p_* \circ \tau \to (p \times p)_* \circ m^! \circ \tau \tag{22}$$

   An E_0-*tensor category* is a $\mathbb{k}$-linear tensor category together with an E_0-structure.

(2) If $(\mathcal{C}, \tau, a, b)$ and $(\mathcal{D}, \sigma, c, d)$ are E_0-tensor categories, an E_0-*functor* from the first to the second consists of an exact $\mathbb{k}$-linear tensor functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$, together with an E-linear tensor isomorphism u:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathscr{F}"] \arrow[d, "\tau"'] & \mathcal{D} \arrow[d, "\sigma"] \arrow[dl, Rightarrow, "u"'] \\ \mathcal{C}^{(E)} \arrow[r, "\mathscr{F}^{(E)}"'] & \mathcal{D}^{(E)} \end{tikzcd} \tag{23}$$

   This data is required to satisfy the obvious commutation relation with the structure isomorphism: The diagrams

$$\begin{tikzcd} i^!\sigma\mathscr{F}(\mathbf{X}) \arrow[r, "c_{\mathscr{F}(\mathbf{X})}"] \arrow[d, "i^!(u_{\mathbf{X}})"'] & \mathscr{F}(\mathbf{X}) \\ i^!(\mathscr{F}^{(E)}(\tau\mathbf{X})) \arrow[r, equal] & \mathscr{F}(i^!\tau\mathbf{X}) \arrow[u, "\mathscr{F}(a_{\mathbf{X}})"'] \end{tikzcd} \tag{24}$$

   and

$$\begin{tikzcd} \sigma\sigma\mathscr{F}(\mathbf{X}) \arrow[r, "d_{\mathscr{F}(\mathbf{X})}"] \arrow[d, "\sigma(u_{\mathbf{X}})"'] & m^!\sigma(\mathscr{F}(\mathbf{X})) \arrow[d, "m^!(u_{\mathbf{X}})"] \\ \sigma(\mathscr{F}(\tau\mathbf{X})) \arrow[d, "u_{\tau\mathbf{X}}"'] & m^!\mathscr{F}(\tau\mathbf{X}) \arrow[d, equal] \\ \mathscr{F}(\tau\tau\mathbf{X}) \arrow[r, "\mathscr{F}(b_{\mathbf{X}})"'] & \mathscr{F}(m^!\tau\mathbf{X}) \end{tikzcd} \tag{25}$$

   commute.

(3) If $(\mathscr{F}, u)$ and $(\mathscr{G}, v)$ are E_0-functors from $(\mathcal{C}, \dots)$ to $(\mathcal{D}, \dots)$, an E_0-*map* from $(\mathscr{F}, u)$ to $(\mathscr{G}, v)$ is a map $r : \mathscr{F} \to \mathscr{G}$ of tensor functors, such that the diagram

$$\begin{tikzcd}
\sigma\mathscr{F}(\mathbf{X}) \arrow[r, "u_{\mathbf{X}}"] \arrow[d, "\sigma(r_{\mathbf{X}})"'] & \mathscr{F}(\tau\mathbf{X}) \arrow[d, "r_{\tau\mathbf{X}}"] \\
\sigma\mathscr{G}(\mathbf{X}) \arrow[r, "v_{\mathbf{X}}"'] & \mathscr{G}(\sigma\mathbf{X})
\end{tikzcd} \tag{26}$$

commutes.

Remark 3.2.2. A functor τ as in the definition determines a functor $\check{\tau} : \mathbf{X} \mapsto (\tau(\check{\mathbf{X}}))^{\vee}$, which is a tensor functor into $\mathcal{C}_{(E)}$ (by Corollary 1.3.10), and this process determines an equivalence between the two kinds of tensor functors. Further, an E_0-tensor functor determines, in an obvious manner, an isomorphism $\check{u} : \check{\sigma} \circ \mathscr{F} \to \mathscr{F} \circ \check{\tau}$ (in the terminology of the definition).

Hence, as discussed earlier in Remark 3.1.8, we may instead work with tensor functors from $\mathcal{C}$ to $\mathcal{C}_{(E)}$. Indeed, the dual $\check{\tau}$ is used, for convenience, in the proof of Theorem 3.2.6 below, but everything can be translated back and forth, by dualising. As in the concrete setup of the previous section, objects of the form $\tau(\mathbf{X})$ can be interpreted as "functions" on $\check{\mathbf{X}}$. See §3.3 for more details. □

3.2.3. Corollary 3.1.12 and Proposition 3.1.13 show how, given an action of $\mathfrak{M}_0$ on $\mathbf{Z}$ and an $\mathfrak{M}_0$-group $\mathbf{G}$, $\mathcal{R}ep_{\mathbf{G}}$ acquires an E_0-structure. We note that in this case, $\check{\tau}(\mathbf{X}) = E \otimes_{\mu} \mathbf{X}$ (by Lemma 3.1.7).

Conversely, given an E_0-tensor category $\mathcal{C}$ over $\Bbbk$, the functor τ determines a map $\tau_{\mathbf{1}} : \Bbbk = End(\mathbf{1}) \to End(\check{E}) = E$. The two isomorphisms given with the E_0-structure on $\mathcal{C}$ show that this map corresponds to an action $\mu : \operatorname{spec}(E) = \mathfrak{M}_0 \times \mathbf{Z} \to \mathbf{Z}$.

Proposition 3.2.4. *The process described in 3.2.3 determines a bijection between actions of* $\mathfrak{M}_0$ *on* $\mathbf{Z} = \operatorname{spec}(\Bbbk)$ *and isomorphism classes of* E_0*-structures on* $\mathcal{V}ec_{\Bbbk}$ *(all over* $\Bbbk_0$*)*

Proof. This is a direct computation. Starting with an action $\mu : \mathfrak{M} \to \mathbf{Z}$, corresponding to a pro-algebra map $f : \Bbbk \to E$, we have $\mu^*(\mathbf{1}) = \check{E}$, and given an endomorphism $a \in \Bbbk$ of $\mathbf{1}$, $\mu^*(a)$ is given by the "right" vector space structure on $\check{E}$, via μ. Hence $\tau_{\mathbf{1}} = f$.

Conversely, since the functor τ is exact, it is determined by its value on $\mathbf{1}$ (and $End(\mathbf{1})$), so by the map $f = \tau_1 : \Bbbk \to E$. □

Definition 3.2.5. Let $\mathcal{C}$ be an E_0-tensor category. An E_0*-fibre functor* on $\mathcal{C}$ is an E_0-tensor functor from $\mathcal{C}$ to $\mathcal{V}ec_{\Bbbk}$, where the latter has the E_0-structure corresponding to the action recovered from $\mathcal{C}$.

An E_0*-Tannakian category* is an E_0-tensor category that admits an E_0-fibre functor.

We may now formulate and prove the main Theorem: E_0-Tannakian categories are precisely categories of representations of (pro-) linear $\mathfrak{M}_0$-groups.

Theorem 3.2.6. *Let* ω *be an* E_0*-fibre functor on an* E_0*-tensor category* $\mathcal{C}$*. Then there is a pro-linear* E_0*-group scheme* $\mathbf{G}$ *over* $\Bbbk$*, and an action of* $\mathbf{G}$ *on each* ω*,* *making* ω *an* E_0*-tensor equivalence between* $\mathcal{C}$ *and* $\mathcal{R}ep_{\mathbf{G}}$*. If* $\mathcal{C} = \mathcal{R}ep_{\mathbf{G}}$ *for some pro-linear* E_0*-group scheme, then* $\mathbf{G}$ *is canonically isomorphic to* $\mathbf{G}$*.*

Proof. Let $\mathbf{G}$ be the usual pro-linear group scheme $\underline{Aut}^{\otimes}(\omega)$ over $\Bbbk$ associated to the fibre functor ω. As indicated by Proposition 3.1.3, this should be the underlying pro-linear group scheme, so our task is to give $\mathbf{G}$ the structure of a $\mathfrak{M}_0$-scheme, over the $\mathfrak{M}_0$-structure on $\mathbf{Z} = \operatorname{spec}(\Bbbk)$. Thus, we should define an action map $\mu : \mathfrak{M}_0 \times \mathbf{G} \to \mathbf{G}$ over $\Bbbk$, where the domain is given the $\Bbbk$-structure coming from the action map $\mu_0 : \mathfrak{M}_0 \times \mathbf{Z} \to \mathbf{Z}$.

In other words, we should provide a compatible system μ_A of monoid actions $\mu_A : \mathfrak{M}_0(A) \times \mathbf{G}(A) \to \mathbf{G}(A)$, one for each $\Bbbk_0$-algebra A. Here, $\mathbf{G}(A)$ is the set of maps $\operatorname{spec}(A) \to \mathbf{G}$ over $\Bbbk_0$, and similarly for $\mathfrak{M}_0$. Furthermore, these maps should respect the $\Bbbk$-structure in the following sense: Given an element $y \in \mathfrak{M}_0(A)$, and an element $g \in \mathbf{G}(A)$ mapping to an element $p \in \mathbf{Z}(A)$, $\mu_A(y, g)$ is a map of schemes $h : \operatorname{spec}(A) \to \mathbf{G}$ such that the diagram

$$\begin{tikzcd}
\operatorname{spec}(A) \arrow[r, "h"] \arrow[d, "{(y,p)}"'] & \mathbf{G} \arrow[d] \\
\mathfrak{M}_0 \times \mathbf{Z} \arrow[r, "\mu_0"'] & \mathbf{Z}
\end{tikzcd} \tag{27}$$

commutes. We denote by $A^{(y)}$ the ring A with the $\Bbbk$-algebra structure coming from diagram (27) (in the language of 2.2.6, $\operatorname{spec}(A^{(y)}) = \mu_!(\operatorname{spec}(A))$ as schemes over $\mathbf{Z}$).

We thus fix a $\Bbbk$-algebra A. By the definition of $\mathbf{G}$, we should produce, for each A-point y of $\mathfrak{M}_0$, and each tensor automorphism g of $A \otimes_{\Bbbk} \omega$ over A, an automorphism $\mu(y, g)$ of $A^{(y)} \otimes_{\Bbbk} \omega$, again over A.

An A-point of $\mathfrak{M}_0$ factors, by definition, through a finite sub-scheme $\mathfrak{M}'_0$. Also by definition, the structure τ restricts to a tensor functor $\mathcal{C}^{(E')}$, where $\mathfrak{M}'_0 = \operatorname{spec}(E'_0)$, and $E' = E_0 \otimes \Bbbk$. So as long as the A-point y is fixed, we may assume that $E = E_0 \otimes \Bbbk$ is finite (the same is true for any finite number of points). We are thus given a map of $\Bbbk_0$-algebras $y : E_0 \to A$ (in fact, we may at this point assume $A = E$ and y the identity, but this will be inconvenient when comparing several points).

Consider now an object $\mathbf{X}$. The E-structure on the fibre functor ω determines an E-module isomorphism $u_{\mathbf{X}} : \omega(\check{\tau}(\mathbf{X})) \to E \otimes_{\mu} \omega(\mathbf{X})$ (the right hand side is what we denoted $E^{(z)} \otimes_{\Bbbk} \omega(\mathbf{X})$, where $z : E \to E$ is the identity). Using y we thus obtain an A-module isomorphism

$$A \otimes_E u_{\mathbf{X}} : A \otimes_E \omega(\check{\tau}(\mathbf{X})) \to A^{(y)} \otimes \omega(\mathbf{X}) \tag{28}$$

If $g \in \mathbf{G}(A)$ is an automorphism of $A \otimes_{\Bbbk} \omega$, we note that $g_{\check{\tau}(\mathbf{X})} : A \otimes_{\Bbbk} \omega(\check{\tau}(\mathbf{X})) \to A \otimes_{\Bbbk} \omega(\check{\tau}(\mathbf{X}))$ is an E-module automorphism (since E acts on objects in $\mathcal{C}$), so it descends to an automorphism of $A \otimes_E \omega(\check{\tau}(\mathbf{X}))$. Hence we obtain an induced map

$$\begin{tikzcd}
A \otimes_E \omega(\check{\tau}(\mathbf{X})) \arrow[r, "A \otimes_E u_{\mathbf{X}}"] \arrow[d, "g_{\check{\tau}(\mathbf{X})}"'] & A^{(y)} \otimes_{\Bbbk} \omega(\mathbf{X}) \arrow[d, "\mu(y,g)_{\mathbf{X}}"] \\
A \otimes_E \omega(\check{\tau}(\mathbf{X})) \arrow[r, "A \otimes_E u_{\mathbf{X}}"] & A^{(y)} \otimes_{\Bbbk} \omega(\mathbf{X})
\end{tikzcd} \tag{29}$$

with $\mu(y, g)$ the map as indicated. This concludes the definition of μ.

To verify that the map is a monoid action, if, in the above definition, $y : E_0 \to \mathbb{k}$ corresponds to the identity of $\mathfrak{M}_0$, then $A^{(y)} \otimes_{\mathbb{k}} \omega(\mathbf{X}) = \omega(\mathbf{X})$, and the induced map in (29) is just $g_{\mathbf{X}}$ (using the existence of the isomorphism a from (21) of Definition 3.2.1).

Next, assume that we are given two points $y_1, y_2 : E_0 \to A$, and let $y = (y_1, y_2) \circ m^{\#}$ be their product in $\mathfrak{M}_0$. We first note that from the fact that we have an action on $\mathbf{Z}$ we get an isomorphism (over $\mathbb{k}$) $(A^{(y_2)})^{(y_1)} \to A^{(y)}$. Now, applying the definition (29) twice, we get a diagram

$$\begin{tikzcd}
A \otimes_E \omega(\check{\tau}\check{\tau}(\mathbf{X})) \arrow[r] \arrow[d, "g_{\check{\tau}\check{\tau}(\mathbf{X})}"'] & {A^{(y_2)}}^{(y_1)} \otimes_{\mathbb{k}} \omega(\mathbf{X}) \arrow[d, "\mu(y_1, \mu(y_2, g))_{\mathbf{X}}"] \\
A \otimes_E \omega(\check{\tau}\check{\tau}(\mathbf{X})) \arrow[r] & {A^{(y_2)}}^{(y_1)} \otimes_{\mathbb{k}} \omega(\mathbf{X})
\end{tikzcd} \tag{30}$$

Applying the isomorphism b from Definition 3.2.1, we may replace $\omega(\check{\tau}\check{\tau}(\mathbf{X}))$ with $E \otimes_{\mathbb{k}} E \otimes_m \omega(\check{\tau}(\mathbf{X}))$, so the left part of the diagram becomes $A \otimes_E \omega(\check{\tau}(\mathbf{X}))$, with the E-structure on A given by y. This concludes the proof that μ is an action. The fact that ω induces an equivalence of $\mathcal{C}$ with $\mathcal{R}ep_{\mathbf{G}}$ follows from the adjunction, as in Proposition 3.1.3.

For the last part, assume that $\mathcal{C}$ is the E_0-tensor category associated with a pro-linear $\mathfrak{M}$-group scheme. By the usual Tannakian formalism and the construction, the underlying group scheme is $\mathbf{G}$. Thus, we need only verify that the action of $\mathfrak{M}$ is the same. This is clear, since the functor τ was defined in the same way for representations of $\mathbf{G}$ and for vector spaces. □

Remark 3.2.7. Analogously to the algebraic case, if $\mathbf{T} = \operatorname{spec}(B)$ is an $\mathfrak{M}$-scheme over $\mathbf{Z}$, and $g_0 : \mathbf{T} \to \mathbf{G}$ is a $\mathbf{T}$-point of $\mathbf{G}$ (so it commutes with the action), then g determines an automorphism of $B \otimes_{\mathbb{k}} \omega$ as an *E_0-functor*. Indeed, such a point determines an automorphism g_0 of $B \otimes_{\mathbb{k}} \omega$ as a tensor functor. Let $A = B \otimes_{\mathbb{k}} E$, let $y : E \to A$ be the obvious map, and let g be the induced automorphism of $A \otimes \omega$. Then, by diagram (29) we have a diagram

$$\begin{tikzcd}
B \otimes_{\mathbb{k}} \omega(\check{\tau}(\mathbf{X})) \arrow[r, "A \otimes_E u_{\mathbf{X}}"] \arrow[d, "g_{\check{\tau}(\mathbf{X})}"'] & A^{(y)} \otimes_{\mathbb{k}} \omega(\mathbf{X}) \arrow[d, "\mu(y, g)_{\mathbf{X}}"] \\
B \otimes_{\mathbb{k}} \omega(\check{\tau}(\mathbf{X})) \arrow[r, "A \otimes_E u_{\mathbf{X}}"] & A^{(y)} \otimes_{\mathbb{k}} \omega(\mathbf{X})
\end{tikzcd} \tag{31}$$

Now, since the action of y on $\mathbf{G}$ commutes with that on $\mathbf{T}$, the map on the right is $(B \otimes E_0) \otimes_B g_{0\mathbf{X}}$ (tensor product with respect to the action $B \to B \otimes E_0$). Composing with the isomorphism $B \otimes E_0 \otimes_B B \otimes_{\mathbb{k}} \omega(\mathbf{X}) \to B \otimes_{\mathbb{k}} E \otimes_{\mu} \omega(\mathbf{X})$ (coming from the fact that that action on $\mathbf{T}$ is over $\mathbf{Z}$), we get the compatibility required in Definition 3.2.1. □

3.3. **$\mathfrak{M}$-schemes in $\mathcal{C}$.** As with usual tensor categories, it is possible to define the category of (affine) $\mathfrak{M}$-schemes. This is the opposite category to the category of $\mathfrak{M}$-algebras in $\mathcal{C}$, defined as follows. Recall, first, that an algebra in $\mathcal{C}$ is an ind-object $\mathbf{X}$ of $\mathcal{C}$, together with (suitable) maps $m : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ and $u : \mathbf{1} \to \mathcal{C}$. If $\mathcal{C}$ is given with an E_0-structure (τ, a, b), the tensor structure on τ makes $\tau(\mathbf{X})$ an algebra as well. We note that the isomorphism a induces a map $a_{\mathbf{X}} : \mathbf{X} \to \tau\mathbf{X}$, which is an algebra map (since a is a tensor isomorphism).

We recall that in the case of usual algebras, $\tau(\mathbf{X})$ was the analogue of the algebra of functions on the arc space of the scheme associated to $\mathbf{X}$. Hence the following definition is natural.

Definition 3.3.1. Let $(\mathcal{C}, \tau, a, b)$ be an E-tensor category, $\mathbf{X}$ an algebra in $\mathcal{C}$. An *E-structure* on $\mathbf{X}$ consists of an algebra map $d : \tau(\mathbf{X}) \to \mathbf{X}$, such that $d \circ a_{\mathbf{X}}$ is the identity, and the following diagram commutes

$$\begin{tikzcd}
\tau\tau\mathbf{X} \arrow[rr, "\tau(d)"] \arrow[d, "b_{\mathbf{X}}"'] & & \tau\mathbf{X} \arrow[d, "d"] \\
m^{!}\tau\mathbf{X} \arrow[r, "m"'] & \tau\mathbf{X} \arrow[r, "d"'] & \mathbf{X}
\end{tikzcd} \tag{32}$$

The category of (affine) $\mathfrak{M}_0$*-schemes in* $\mathcal{C}$ is the opposite of the category of algebras with E-structure (and maps of between them that preserve this structure)

As in previously known cases, any object $\mathbf{X}$ has an associated "affine space" $\mathcal{C}$-scheme $A(\mathbf{X})$. These affine spaces were important in the differential case to achieve elimination of imaginaries in the corresponding theory (cf Kamensky [10, §§ 4.4–4.5]), and played an important role in Deligne [6].

The construction of $A(\mathbf{X})$ is a special case of the following result, which says that prolongation spaces also exist in $\mathcal{C}$. The defining property is taken to be analogous to the adjunction in 2.2.7 (the affine case should also be compared to Proposition 2.3.11).

Proposition 3.3.2. *Let $\mathcal{C}$ by an E-tensor category. The forgetful functor from $\mathfrak{M}$-schemes to schemes in $\mathcal{C}$ has a right adjoint.*

Proof. Given a (usual) scheme with algebra $\mathbf{W}$ in $\mathcal{C}$, the induced $\mathfrak{M}$-scheme corresponds to $\tau\mathbf{W}$, with $d : \tau\tau\mathbf{W} \to \tau\mathbf{W}$ induced by the product on $\mathfrak{M}$. The fact that this is an $\mathfrak{M}$-scheme structure comes from the monoid axioms on $\mathfrak{M}$.

Given some other $\mathfrak{M}$-scheme $\mathbf{X}$, $d : \tau\mathbf{X} \to \mathbf{X}$, an algebra map $f : \mathbf{W} \to \mathbf{X}$ determines a algebra maps $\tau(f) : \tau\mathbf{W} \to \tau\mathbf{X}$ and $\tau^2(f) : \tau^2\mathbf{W} \to \tau^2\mathbf{X}$, resulting, upon composition with d and $\tau(d)$, in maps $\tilde{f} : \tau\mathbf{W} \to \mathbf{X}$ and $\tilde{f}_\tau : \tau\tau\mathbf{W} \to \tau\mathbf{X}$. The fact that these maps determine a map of $\mathfrak{M}$-schemes comes from diagram (32).

In the other direction, a map $\tau\mathbf{W} \to \mathbf{X}$ restricts to a map $\mathbf{W} \to \mathbf{X}$. It is standard to check that these are inverse to each other. □

Remark 3.3.3. If $\mathfrak{M}$ is the free formal monoid generated by some formal set $\mathfrak{M}_1$, the data in Definition 3.3.1 can be given in terms of $\mathfrak{M}_1$, as a map $d_1 : \tau_1(\mathbf{X}) \to \mathbf{X}$, without the condition (32). The full $d : \tau\mathbf{X} \to \mathbf{X}$ can then be reconstructed as the unique map extending d_1 and satisfying (32). In the case where $\mathfrak{M}_1 = \mathrm{spec}(\Bbbk_0[\epsilon])$ (where $\Bbbk_0$ is a field of characteristic 0 and $\epsilon^2 = 0$), the corresponding $\mathfrak{M}$ is the additive formal group (Example 2.3.4), and the explicit construction of $\tau\mathbf{X}$ and of d was (essentially) carried out in Kamensky [10, § 4.4.3] ($\tau\mathbf{X}$ is the ind-object with maps q_n there; the map d is essentially given by the t_n). □

4. Questions and speculations

In this section I point out some questions and other issues I would like to clarify. At least some of them should be easy to answer, but I do not see the answer immediately, and they are not directly relevant to the main point of the paper, so I leave them unanswered. Nevertheless, I think they are interesting.

4.1. **Sheaves on formal sets.** As explained in §1.3, if $\mathfrak{M}$ is a formal set (viewed as a filtering system), and $\mathcal{C} \to \mathfrak{M}$ is a fibred category over $\mathfrak{M}$, whose fibres are (say) sheaves of some kind over the corresponding finite piece, then $\varprojlim_{\mathfrak{M}} \mathcal{C}$ can be viewed as the category of sheaves of the same kind on $\mathfrak{M}$. However, it does not seem to be straightforward to deduce properties of $\varprojlim_{\mathfrak{M}} \mathcal{C}$ from properties of the fibres.

For instance, assume that each fibre is abelian. May we conclude that the limit is abelian? The answer is "no" in general, and "yes" if each pullback functor is exact. However, in our situation this assumption does not hold. In the context of 1.3.6, we know that the pullbacks are either left or right-exact (and, indeed, admit a left or right adjoint), but not both. Can anything be said in this case? In special cases such as completions of Noetherian local rings and finitely generated modules, one ends up with an abelian category, but this does depend on the Artin–Rees Lemma or similar results.

More generally, this appears like it should be a classical construction, but I don't know what would be a good reference.

4.2. **Cartier duality.** The Cartier dual of $\mathbf{G} = \mathbf{G}_m$ (and of any group of multiplicative type) can also be obtained as follows: it is the group of isomorphism classes of invertible (rank 1) representations of $\mathbf{G}$, with group structure induced by $\otimes$. Since such a group is reductive and all its irreducible representations are invertible, the information about $\mathbf{G}$ is captured by the dual.

This is clearly false for other groups (for instance, $\mathbf{G}_a$ has only the trivial invertible representation), but Example 2.3.7 suggests that (in characteristic 0), the additive formal group could be viewed as the Picard group for the category of representations of $\mathbf{G}_a$. The question is what is the precise meaning of this formal group in terms of the category, and whether one can compute a meaningful (formal) Picard group like that for tensor categories in general.

Another question related to the duality: In characteristic 0, the group schemes $\mathbf{G}_m$ and $\mathbf{G}_a$ correspond, respectively, to the cases of an automorphisms and a derivation, and they are the only affine groups of dimension 1. So it seems that we have shown that the only "rank 1" operators in characteristic 0 are automorphisms and derivations. The question is how to explain what "rank 1" means, without going through Cartier duality (this might be related to the classification mentioned in Buium [2, § 2.4]).

4.3. **Quotients.** In the usual treatment of differential and difference fields, an important role is played by the "field of constants". It played no role in this paper, but it is still interesting to define it in the general context of §2.

We have a categorical description. Given a scheme $\mathbf{X}$ (over $\Bbbk_0$), one may view $\mathbf{X}$ as an $\mathfrak{M}$-scheme $\underline{\mathbf{X}}$ via the trivial action. We may then define quotient by $\mathfrak{M}$ to be the "left-adjoint" to this functor: Given an $\mathfrak{M}$-scheme $\mathbf{Z}$, $\mathbf{Z}/\mathfrak{M}$ is defined as a *covariant* functor on schemes $\mathbf{X}$ by $(\mathbf{Z}/\mathfrak{M})(\mathbf{X}) = Hom_{\mathfrak{M}}(\mathbf{Z}, \underline{\mathbf{X}})$. The problem is that there is no reason that this functor should be representable (unless, of course, $\mathfrak{M}$ is finite), and furthermore, it seems impossible to describe maps from a scheme to $\mathbf{Z}/\mathfrak{M}$.

In the affine case, we do have an algebra associated to $\mathbf{Z}/\mathfrak{M}$: if $\mathbf{Z} = \operatorname{spec}(\Bbbk)$, with action of $\mathfrak{M}$ given by m and projection given by p (both on the level of algebras), then $\mathbf{Z}/\mathfrak{M}$ corresponds to the sub-algebra given by $p(a) = m(a)$. If $\Bbbk$ is a field, then this sub-algebra is field as well, and this definition coincides with the usual one in the difference and differential case.

4.4. **Relation to crystals.** Some of the notions presented here appear to be closely related to notions that appear in the context of the infinitesimal and the crystalline sites, as presented, for example in Berthelot and Ogus [1] or Lurie [11] (in the case when $\mathfrak{M}$ is local). Essentially, the structure of a crystal on an object $\mathbf{X}$ is what we called an E-structure in Definition 3.3.1, so that an $\mathfrak{M}_0$-scheme (in $\mathcal{C}$) corresponds to a crystal of $\mathcal{C}$-schemes. A similar resemblance exists for crystals in positive characteristic. It would be interesting to understand the precise relation, and whether it is useful.

4.5. **Changing the monoid.** Throughout, we work with a fixed formal monoid $\mathfrak{M}$. It makes sense, of course, to ask what happens when we let $\mathfrak{M}$ vary. For example, in the context of several derivations, it could be desirable to pass to a subset of the derivations, or to a more convenient choice of them.

In particular, in the context of the Tannakian formalism, $\mathfrak{M}$ is recovered from $\mathcal{C}^{(E)}$ (as $End(\mathbf{1}_E)$), so one could ask to replace $\mathcal{C}^{(E)}$ by an abstract category of prolongations $\mathcal{D}$. This would entail finding conditions under which $\mathcal{D}$ is canonically isomorphic to $\mathcal{C}^{(E)}$ for $E = End(\mathbf{1}_{\mathcal{D}})$ (over a given tensor functor $\mathcal{C} \to \mathcal{D}$). Such a formalism would treat all formal monoid actions at once. I leave it to some other time.

4.6. **More general monads.** Instead of working with with a formal monoid $\mathfrak{M}$, as we did, we could work more generally with the corresponding monad $\mathscr{W}_{\mathfrak{M}}$, given by $\mathscr{W}_{\mathfrak{M}}(\mathbf{Z}) = \mathfrak{M} \times \mathbf{Z}$. The advantage is that we may then forget about $\mathfrak{M}$ and work just with a monad $\mathscr{W}$ on the category of ind-schemes (over a given base $\Bbbk_0$; we would probably be assuming that $\mathscr{W}$ is "continuous", i.e., determined by its restriction to schemes). Given such a monad, an $\mathfrak{M}$-scheme is replaced by a $\mathscr{W}$-algebra $\mathbf{Z}$, and likewise $\mathfrak{M}$-schemes over $\mathbf{Z}$ are replaced by $\mathscr{W}$-algebras over $\mathbf{Z}$. The main difference with our approach is that we are no longer assuming to have a functorial map $p : \mathscr{W}(\mathbf{X}) \to \mathbf{X}$ (the projection). An important example covered by this more general approach is the case of (say) the p-adic Witt scheme of length 2, $\mathscr{W} = \mathscr{W}_2$, corresponding to the arithmetic differential equations of Buium [2] (See especially Buium [2, § 2.4]).

If $\mathscr{W}$ happens to have a right adjoint τ_0 (possibly going from schemes to pro-schemes), which is then automatically a co-monad, then an algebra $\mathbf{Z}$ for $\mathscr{W}$ determines a co-algebra $t : \mathbf{Z} \to \tau_0\mathbf{Z}$ for τ_0. Given a scheme $\mathbf{X}$ over $\mathbf{Z}$, we set $\tau(\mathbf{X}) = \tau_0(\mathbf{X}) \times_t \mathbf{Z}$, and call it the prolongation of $\mathbf{X}$ (viewed as a pro-scheme over $\mathbf{Z}$). As in 2.2.7, τ is a co-monad on pro-schemes over $\mathbf{Z}$, and is, by construction, right adjoint to the forgetful functor from $\mathscr{W}$-schemes to schemes.

The main issue with extending the results of the paper is now to find the analogue of tensoring with E to this setting, i.e., we need a canonical way to extend $\mathscr{W}$ (or τ) to a tensor category over $\mathbf{Z}$. This should be possible.

References

[1] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Mathematical Notes 21. Princeton, N.J.: Princeton University Press, 1978, vi+243. ISBN: 0-691-08218-9 (cit. on p. 31).

[2] Alexandru Buium. *Arithmetic differential equations.* Mathematical Surveys and Monographs 118. Providence, RI: American Mathematical Society, 2005, xxxii+310. ISBN: 0-8218-3862-8 (cit. on pp. 2, 30, 31).

[3] Phyllis Joan Cassidy. "The differential rational representation algebra on a linear differential algebraic group". In: *J. Algebra* 37.2 (1975), 223–238. ISSN: 0021-8693 (cit. on p. 21).

[4] Phyllis J. Cassidy and Michael F. Singer. "Galois theory of parameterized differential equations and linear differential algebraic groups". In: *Differential equations and quantum groups.* Ed. by Daniel Bertrand, Benjamin Enriquez, Claude Mitschi, Claude Sabbah, and Reinhard Schäfke. IRMA Lectures in Mathematics and Theoretical Physics 9. Andrey A. Bolibrukh memorial volume. European Mathematical Society (EMS), Zürich, 2007, 113–155. ISBN: 978-3-03719-020-3. arXiv:`math/0502396` (cit. on p. 1).

[5] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups.* New Mathematical Monographs 17. Cambridge: Cambridge University Press, 2010, xx+533. ISBN: 978-0-521-19560-7 (cit. on p. 16).

[6] Pierre Deligne. "Catégories tannakiennes". In: *The Grothendieck Festschrift.* Vol. II. Progress in Mathematics 87. Boston, MA: Birkhäuser Boston Inc., 1990, 111–195. ISBN: 0-8176-3428-2 (cit. on pp. 1, 4, 6, 7, 10, 29).

[7] Pierre Deligne and James S. Milne. "Tannakian categories". In: Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties.* Lecture Notes in Mathematics 900. Berlin: Springer-Verlag, 1982. Chap. II, 101–228. ISBN: 3-540-11174-3. URL: `http://jmilne.org/math/xnotes/tc.html` (cit. on pp. 3, 5, 7, 10, 12, 19, 21).

[8] Henri Gillet, Sergey Gorchinskiy, and Alexey Ovchinnikov. *Parameterized Picard–Vessiot extensions and Atiyah extensions.* Oct. 2011. arXiv:`1110.3526v1` (cit. on pp. 1–3).

[9] Charlotte Hardouin and Michael F. Singer. "Differential Galois theory of linear difference equations". In: *Math. Ann.* 342.2 (2008), 333–377. ISSN: 0025-5831. arXiv:`0801.1493` (cit. on p. 1).

[10] Moshe Kamensky. *Model theory and the Tannakian formalism.* submitted. Aug. 2009. arXiv:`0908.0604` (cit. on pp. 1, 3, 9, 21, 23, 29).

[11] Jacob Lurie. *D-modules and D-schemes via crystals.* Notes from Gaitsgory's "Geometric Representation theory" seminar. 2009. URL: `http://www.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov17-19(Crystals).pdf` (cit. on p. 31).

[12] Saunders Mac Lane. *Categories for the working mathematician.* 2nd ed. Graduate Texts in Mathematics 5. New York: Springer-Verlag, 1998. ISBN: 0-387-98403-8 (cit. on p. 17).

[13] Hideyuki Matsumura. *Commutative ring theory.* 2nd ed. Cambridge Studies in Advanced Mathematics 8. Translated from the Japanese by M. Reid. Cambridge: Cambridge University Press, 1989, xiv+320. ISBN: 0-521-36764-6 (cit. on p. 2).

[14] Rahim Moosa and Thomas Scanlon. "Generalised Hasse-Schmidt varieties and their jet spaces". In: *Proceedings of the London Mathematical Society* (). arXiv:`0908.4230` (cit. on pp. 2, 3, 13–17).

[15] Rahim Moosa and Thomas Scanlon. "Jet and prolongation spaces". In: *J. Inst. Math. Jussieu* 9.2 (2010), 391–430. ISSN: 1474-7480. DOI: `10.1017/S1474748010000010`. arXiv:`0806.4196` (cit. on pp. 2, 13).

[16] Alexey Ovchinnikov. "Tannakian approach to linear differential algebraic groups". In: *Transform. Groups* 13.2 (2008), 413–446. ISSN: 1083-4362. arXiv:`math/0702846` (cit. on pp. 1, 3).

[17] Alexey Ovchinnikov. "Tannakian categories, linear differential algebraic groups, and parametrized linear differential equations". In: *Transform. Groups* 14.1 (2009), 195–223. ISSN: 1083-4362. arXiv:`math/0703422` (cit. on p. 1).

[18] Richard Pink. *Finite Group Schemes.* Course lecture notes. 2004. URL: `http://www.math.ethz.ch/~pink/FiniteGroupSchemes.html` (cit. on p. 19).

[19] Marius van der Put and Michael F. Singer. *Galois theory of difference equations.* Lecture Notes in Mathematics 1666. Berlin: Springer-Verlag, 1997. ISBN: 3-540-63243-3 (cit. on p. 1).

[20] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 328. Berlin: Springer-Verlag, 2003. ISBN: 3-540-44228-6. URL: `http://www4.ncsu.edu/~singer/papers/dbook2.ps` (cit. on p. 1).

[21] Neantro Saavedra-Rivano. *Catgories Tannakiennes.* Lecture Notes in Mathematics, Vol. 265. Berlin: Springer-Verlag, 1972 (cit. on p. 3).

[22] William C. Waterhouse. *Introduction to affine group schemes.* Graduate Texts in Mathematics 66. New York: Springer-Verlag, 1979. ISBN: 0-387-90421-2 (cit. on p. 19).

DEPARTMENT OF MATH, UNIVERSITY OF NOTRE-DAME, NOTRE-DAME, IN, USA
E-mail address: `mailto:mkamensky@nd.edu`
URL: `http://mkamensky.notlong.com`