

# Ind- and Pro- definable sets

MOSHE KAMENSKY

ABSTRACT. We describe the ind- and pro- categories of the category of definable sets, in some first order theory, in terms of points in a sufficiently saturated model.

## 1. INTRODUCTION

Given the direct limit  $Y$  of some system  $Y_i$  in a given category, the morphisms from  $Y$  to another object  $X$  are described, by definition, as certain collections of morphisms from each  $Y_i$  to  $X$ . In contrast, there is, in general, no simple description of morphisms in the other direction, from  $X$  to  $Y$ . However, if the category in question is, for example, a category of topological spaces, and  $X$  is compact, then any morphism from  $X$  to  $Y$  will factor via some  $Y_i$ .

The category  $\text{Ind}(\mathcal{C})$  of ind-objects of a category  $\mathcal{C}$  is a category containing the original category  $\mathcal{C}$ , in which any filtering system has a limit, and the objects of the original category are “compact” (or, more formally, finitely presented) in the above sense. This construction, which appears in [1], can be applied to any category, and is described below. The dual construction, of the category of pro-objects, is described as well.

In the context of first order logic, and definable sets, there is a natural notion of compactness, and given a system of definable sets, one may compute limits of their points in a given model. The purpose of this note is to describe how the categorical notions of ind- and pro- objects apply to definable sets, and in particular to describe the categories of ind- and pro- definable sets in terms of points in a model. The main result is corollary 9, which describes morphisms in terms of such points. We also describe, in section 3.1, a categorical construction (the Grothendieck construction) that identifies models with pro-definable sets, allowing one to evaluate the  $M$  points of any functor from definable sets to sets. The final statement of the results is in corollary 10.

**Acknowledgement.** This work is part of my PhD research, performed in the Hebrew university under the supervision of Ehud Hrushovski. I would like to thank him for his guidance.

I would also like to thank Tibor Beke for explaining to me the correct categorical context of notions considered here. In particular, the formulation of section 3 follows closely his suggestions.

---

2000 *Mathematics Subject Classification.* Primary 03C07; Secondary 18A35.  
*Key words and phrases.* compactness,limits,inddefinable,prodefinable.

## 2. CATEGORICAL NOTIONS

We begin by recalling some general notions from category theory. The reference to all this is [1]. Let  $\mathcal{C}$  be a category (which we assume to be small),  $\widehat{\mathcal{C}}$  the category of presheaves on  $\mathcal{C}$  (i.e., contra-variant functors from  $\mathcal{C}$  to the category of sets), and  $\mathbf{y} : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$  the Yoneda embedding, given by  $\mathbf{y}(X)(Z) = \text{Hom}(Z, X)$ . We shall describe a construction of  $\text{Ind}(\mathcal{C})$  as a sub-category of  $\widehat{\mathcal{C}}$ .

A *filtering category* is a small category  $I$  such that:

- For any two objects  $i, j$  of  $I$ , there are morphisms  $i \rightarrow k$  and  $j \rightarrow k$  for some object  $k$ .
- For any two morphism  $t_1, t_2 : i \rightarrow j$  there is a morphism  $s : j \rightarrow k$  with  $s \circ t_1 = s \circ t_2$ .

A *filtering system* in  $\mathcal{C}$  is a functor from a filtering category  $I$  to  $\mathcal{C}$ . Such a system will be denoted  $(X_i)$ , where  $X_i$  is the object of  $\mathcal{C}$  associated with  $i$ . The category  $I$  is called the *index category*.

Any directed partially ordered set can be viewed as a filtered category, and conceptually a filtering system can be thought of as a partially ordered one. In fact, it can be shown that any filtering system is isomorphic to a partially ordered one. However, in some cases (like in section 3.1 below), the natural index category has the more general form.

We now define  $\text{Ind}((X_i))$ , the *ind-object* of  $\mathcal{C}$  associated with the system  $(X_i)$ , to be  $\varinjlim \mathbf{y}(X_i)$  (an object of  $\widehat{\mathcal{C}}$ .) Recall that direct limits in  $\widehat{\mathcal{C}}$  can be computed “pointwise”. Thus, we have for any object  $Y$  of  $\mathcal{C}$ ,

$$\text{Hom}(Y, \text{Ind}((X_i))) = \text{Ind}((X_i))(Y) = \varinjlim \text{Hom}(Y, X_i)$$

The category  $\text{Ind}(\mathcal{C})$  is defined to be the full subcategory of  $\widehat{\mathcal{C}}$  of presheaves isomorphic to  $\text{Ind}((X_i))$  for some filtering system  $(X_i)$ .

As observed in remark 1 below, this category can be characterised by the following universal property: it is a category that has all filtering direct limits, with a functor  $\mathbf{y} : \mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$ , which is universal, in the sense that any functor from  $\mathcal{C}$  to another category that has all filtering direct limits extends to a functor on  $\text{Ind}(\mathcal{C})$  that preserves all such limits, and the extension is unique up to an isomorphism. This universal property bestows  $\text{Ind}(\mathcal{C})$  with the title “free filtered cocompletion” of  $\mathcal{C}$ . Each object  $Y$  of  $\mathcal{C}$  is *finitely presented* in  $\text{Ind}(\mathcal{C})$ , in the sense describe in the introduction: giving a map from  $Y$  to a filtering limit of objects  $X_i$  (in  $\text{Ind}(\mathcal{C})$ ) amounts to giving a map into some  $X_i$ .

The category of pro-objects  $\text{Pro}(\mathcal{C})$  is obtained by dualising the universal property and the construction: it is constructed as  $\text{Ind}(\mathcal{C}^\circ)^\circ$ , where  $\mathcal{C}^\circ$  denotes the opposite category to  $\mathcal{C}$ . We describe the construction explicitly in terms of  $\mathcal{C}$  itself: let  $\check{\mathcal{C}} = \widehat{\mathcal{C}^\circ}$  be the category of co-variant functors from  $\mathcal{C}$  to sets,  $\check{\mathbf{y}} : \mathcal{C} \rightarrow \check{\mathcal{C}}$  the (contra-variant) Yoneda embedding. Given a co-filtering system  $(X_i)$  in  $\mathcal{C}$  (i.e., a contra-variant functor from a filtering category to  $\mathcal{C}$ ), the associated pro-object is defined to be the functor  $\text{Pro}((X_i)) = \varprojlim \check{\mathbf{y}}(X_i)$ . For any object  $Y$  of  $\mathcal{C}$  we get

$$\begin{aligned} \text{Hom}_{\text{Pro}(\mathcal{C})}(\text{Pro}((X_i)), Y) &= \text{Hom}_{\check{\mathcal{C}}}(Y, \text{Pro}((X_i))) = \\ &= \text{Pro}((X_i))(Y) = \varprojlim \text{Hom}_{\mathcal{C}}(X_i, Y) \end{aligned}$$

More generally, we have the following formulas for the morphism sets in the *Pro* and *Ind* categories:

$$(1a) \quad \text{Hom}(\text{Ind}((X_i)), \text{Ind}((Y_j))) = \varprojlim_i \varinjlim_j \text{Hom}(X_i, Y_j)$$

$$(1b) \quad \text{Hom}(\text{Pro}((X_i)), \text{Pro}((Y_j))) = \varinjlim_j \varprojlim_i \text{Hom}(X_i, Y_j)$$

*Remark 1.* It follows that any presheaf  $P$  on  $\mathcal{C}$  extends canonically to  $\text{Pro}(\mathcal{C})$  by setting  $P(\text{Pro}((X_i))) = \varinjlim P(X_i)$ : a map of pro-objects

$$f : \text{Pro}((X_i)) \rightarrow \text{Pro}((Y_j))$$

is represented by a sequence of maps  $f_j : X_{i_j} \rightarrow Y_j$ , hence we get maps  $P(f_j) : P(Y_j) \rightarrow P(X_{i_j})$  that represent a map from  $\varinjlim P(Y_j)$  to  $\varinjlim P(X_i)$ . This extension tautologically preserves all filtering direct limits.

We note that, as an ind object is a special kind of presheaf,  $P$  extends canonically to  $\text{Ind}(\mathcal{C})$  as well, via the formula

$$P(\text{Ind}((X_i))) = \text{Hom}_{\widehat{\mathcal{C}}}(\text{Ind}((X_i)), P) = \varprojlim P(X_i)$$

Likewise, any functor from  $\mathcal{C}$  to sets can be extended to a functor on  $\text{Ind}(\mathcal{C})$  or  $\text{Pro}(\mathcal{C})$ . It is clear that the same observation applies when these functors to the category of sets are replaced by functors into an arbitrary category, provided that the corresponding limits in the category exist. This illustrates the universal property of these categories, mentioned earlier.  $\square$

Given an object  $X$  of  $\mathcal{C}$ , the category  $\mathcal{C}/X$  is defined to have  $\mathcal{C}$ -morphisms  $Y \rightarrow X$  as objects, and  $\mathcal{C}$ -morphisms over  $X$  as morphisms. Then  $\text{Ind}(\mathcal{C}/X) = \text{Ind}(\mathcal{C})/X$  and  $\text{Pro}(\mathcal{C}/X) = \text{Pro}(\mathcal{C})/X$ . The first assertion follows by definition (and is true when  $X$  is replaced by any presheaf), while the second uses the fact that the systems are filtered.

We are going to use the following lemma, which describes a sufficient condition for a morphism with a section to be an isomorphism:

**Lemma 2.** *Let  $f : \text{Ind}((X_i)) \rightarrow Y$ ,  $g : Y \rightarrow X_0$  be two morphisms, such that  $f_0 \circ g$  is the identity on  $Y$ . Assume that for any  $i$ , there is a morphism  $t_i : X_i \rightarrow X_j$  in the system, such that for any two morphisms  $h_1, h_2 : V \rightarrow X_i$ , if  $f_i \circ h_1 = f_i \circ h_2$ , then  $t_i \circ h_1 = t_i \circ h_2$  (this is the formal analogue of saying that  $f_j$  is injective on the image of  $t_i$ .)*

*Then  $f$  is an isomorphism with inverse  $g$ .*

*Proof.* First note that for any filtering system  $(X_i)$  and an object  $X$  in the system, the (full) subsystem consisting of all objects that have a system morphism from  $X$  is isomorphic (in the *Ind* category) to the original one. Thus we may assume that there is a system morphism from  $X_0$  to any other object in the system.

To show that  $g$  is the inverse of  $f$ , we need to show that  $g \circ f$  is the identity on  $\text{Ind}((X_i))$  (the other composition is the identity by assumption.) This amounts to showing that for any  $i$ ,  $g \circ f_i$  is identified with some morphism in the system  $(X_i)$ . In other words, we need to show that there are morphisms  $t : X_i \rightarrow X_k$ ,



**Proposition 5.** *For any model  $M$ ,  $q(M)$  is exact, and  $q$  is fully faithful.*

*Proof.* The fact that  $q(M)$  is exact follows from the definition of definable sets. To show that  $q$  is fully faithful, let  $N$  and  $M$  be two models, and  $t : q(M) \rightarrow q(N)$  a morphism. It is obvious that the map  $f = t_{\mathbb{A}} : q(M)(\mathbb{A}) \rightarrow q(N)(\mathbb{A})$  is elementary, and that this is an inverse to  $q$ .  $\square$

It follows from remark 1 that any model can be extended to a functor on the categories  $\text{Ind}(\mathcal{D})$  and  $\text{Pro}(\mathcal{D})$ . However, the truth is that any model can in fact be viewed as a pro-definable set, and can be extended to the category of all functors (or presheaves) on  $\mathcal{D}$ . This fact, which will not be needed in the sequel, is explained in subsection 3.1 below. It follows from a general categorical notion, the Grothendieck construction.

**3.1. Models as pro-definable sets.** The Grothendieck construction is a canonical presentation of any (set valued) functor on a category  $\mathcal{C}$ , as a direct limit (not necessarily filtered) of representable functors. The details of this construction appear in [4]. Given a functor  $F$  on  $\mathcal{C}$ , the index category, called the category of elements, is the category whose objects are pairs  $(X, a)$ , where  $X$  is an object of  $\mathcal{C}$ , and  $a \in F(X)$ , where the morphisms from  $(X, a)$  to  $(Y, b)$  are those morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$  for which  $F(f)(a) = b$ . The functor from this category to  $\mathcal{C}$  over which the limit is taken is simply forgetting the point  $a$ .

Now, if finite inverse limits exist in  $\mathcal{C}$ , and  $F$  is left exact, then finite inverse limits exist also in the category of elements of  $F$  (for example,  $(X \times Y, (a, b))$  is the product of  $(X, a)$  and  $(Y, b)$ .) In particular, this category is co-filtering, and so  $F$  belongs to  $\text{Pro}(\mathcal{C})$ . Conversely, filtered direct limits commute with finite inverse limits, so functors in  $\text{Pro}(\mathcal{C})$  are always left exact. Hence, if  $\mathcal{C}$  has finite inverse limits,  $\text{Pro}(\mathcal{C})$  is exactly (the opposite of) the category of left exact functors on  $\mathcal{C}$ . Similarly, if  $\mathcal{C}$  has all finite direct limits, then  $\text{Ind}(\mathcal{C})$  consists of all presheaves that are left exact (as functors on  $\mathcal{C}^{\circ}$ .)

In our case, the category  $\mathcal{D}$  of definable sets has finite inverse limits, and, as mentioned above, models are exact. Therefore, the Grothendieck construction gives a presentation of each model as a pro-definable set. This presentation can be interpreted as follows. If  $M$  is a model, and  $X$  is a definable subset of  $\mathbb{A}^n$ , then  $X(M)$  contains together with each element, its type (over 0.) In other words, we may write,

$$\begin{aligned} X(M) &= \coprod_{a \in M^n} \text{Hom}(tp(a), X) = \coprod_{a \in M^n} \varinjlim_{\substack{Y \text{ with} \\ a \in Y(M)}} \text{Hom}(Y, X) = \\ &= \varinjlim_{(Y, a \in Y(M))} \text{Hom}(Y, X) \end{aligned}$$

where  $\text{Hom}(X, Y)$  here stands for inclusions (so  $\text{Hom}(X, Y)$  contains one element if  $X \subseteq Y$ , and is empty otherwise.) The Grothendieck construction is obtained by replacing inclusions with arbitrary definable maps in this description.

The observation that any model can be identified with an object of  $\text{Pro}(\mathcal{D})$  allows us to find the  $M$  points of any functor or presheaf on  $\mathcal{D}$ , using remark 1. Explicitly,

for  $P$  a presheaf on  $\mathcal{D}$ , and  $F$  a functor from  $\mathcal{D}$  to sets, we have for any model  $M$ :

$$(2a) \quad F(M) = \text{Hom}_{\mathcal{D}}(F, q(M)) = \text{Hom}_{\mathcal{D}}(F, \varinjlim \check{y}(X_a))$$

$$(2b) \quad P(M) = \varinjlim P(X_a)$$

*Remark 6.* The Grothendieck construction can also be applied to  $F$  (or  $P$ ) above, and one may use it to extend  $q(M)$  to the category of functors (or presheaves.) The result will be the same, though in this case the limit is not necessarily filtered. In this form, this is a special case of a more general construction, the left Kan extension (of  $q(M)$  along the Yoneda embedding of  $\mathcal{D}$ .) See [3] for details.

#### 4. THE CATEGORIES OF IND- AND PRO-DEFINABLE SETS

We now come to the main matter of the paper, a description of the categories of ind- and pro-definable sets in terms of (sufficiently saturated) models. Using remark 1, we find, for (co-) filtering systems  $(Z_i)$  (resp.  $(Y_i)$ ) of definable sets,

$$\text{Ind}((Z_i))(M) = \varinjlim_i Z_i(M)$$

$$\text{Pro}((Y_i))(M) = \varprojlim_i Y_i(M)$$

Thus, to compute the points of a pro-definable set in a model  $M$ , we need to choose a presentation of it as system, and compute the inverse limit of the associated system of sets (and similarly for ind-definable sets.)

We may now identify these sets of points with some familiar model theoretic objects. Let  $\mathfrak{p}$  be any partial type. The definable sets comprising it form a co-filtering system, with all maps the inclusions. The last equations says that computing the  $M$  points of  $\mathfrak{p}$ , viewed as pro-definable set, coincides with computing its  $M$  points as a type, i.e., taking the intersection of the  $M$  points of the definable sets in  $\mathfrak{p}$ . The fact that two such system that give the same pro-definable set also give the same set of points means that this set of points is determined by the set of definable sets containing  $\mathfrak{p}$ .

A partial type such as above is always contained in some definable set. There is a more general construction, called a  $*$ -type, that consists of the intersection of formulas in an arbitrary set of variables. Such types are similarly examples of pro-definable sets.

Analogously, an increasing union of definable set is an example of an ind-definable set. A more complicated example can be formulated as follows: let  $E_i$  be definable equivalence relations on a definable set  $X$ , indexed by natural numbers  $i$ , such that for  $i > j$ ,  $E_i$  is coarser than  $E_j$ . Let  $E$  be the equivalence relation saying that  $xEy$  if  $xE_i y$  for some  $i$ . Then  $E$  is the union of the  $E_i$  and thus an example of an ind-definable equivalence relation. The quotient of  $X$  by  $E$  is another example of an ind-definable set (this situation occurs, for instance, when  $E$  is given by an ascending chain of subgroups; for example,  $E$  may be given by the union of balls of finite radius around 0 in a theory of valued fields.)

Our next purpose is to describe the morphisms between the new objects in terms of their points in models. Considering equations (1) again, we see in particular that any morphism from  $\text{Ind}((X_i))$  to  $\text{Ind}((Y_j))$  gives rise to a filtering system  $\Gamma_i$  of the corresponding graphs of functions from  $X_i$  to  $Y_{j_i}$ . Similarly, a morphism of pro-definable sets gives rise to a cofiltering system. Each such system is isomorphic

to its domain  $X_i$ , and therefore induces a function on the level of points from  $\text{Ind}((X_i))(M)$  to  $\text{Ind}((Y_i))(M)$  (and similarly for pro-definable sets.) We would like to show that conversely, any ind-definable set that gives rise to a function on the points of every model (equivalently, saturated enough model) induces a morphism.

We first restate the compactness theorem in this language:

**Proposition 7.** *Let  $\kappa$  be a cardinal bigger than the cardinality of the index category (i.e., the cardinality of the disjoint union of the morphism sets.)*

- (1) *Let  $f : \text{Ind}(X_i) \rightarrow Y$  be a morphism such that for some  $\kappa$ -saturated model  $M$ ,  $f_M : \text{Ind}(X_i)(M) \rightarrow Y(M)$  is a bijection. Then  $f$  is an isomorphism.*
- (2) *Let  $f : Y \rightarrow \text{Pro}(X_i)$  be a morphism such that for some  $\kappa$ -saturated model  $M$ ,  $f_M : Y(M) \rightarrow \text{Pro}(X_i)(M)$  is a bijection. Then  $f$  is an isomorphism.*

*Proof.* In each case, let  $f_i$  be the maps corresponding to the morphism  $f$ . Note that for definable sets and maps, the claims are true by definition ( $f$  is an isomorphism in this case.) We shall use the criterion of remark 3 (and remark 4.)

- (1) We will find  $g$  and  $t_i$  as required by remark 3. We first show that for some  $k$ ,  $f_k$  is onto. In fact, the collection of sets  $f_i(X_i)(M)$  is a small covering of  $Y(M)$ , hence it has a finite sub-cover. Since the system is filtering, there is an  $X_k$  above all the sets in the sub-cover.

We next note that the  $t_i$  condition requires, in this case, for each  $i$ , a definable map  $t_i : X_i \rightarrow X_j$  in the system such that  $f_i(x) = f_i(y)$  defines the same set as  $t_i(x) = t_i(y)$ . This again holds by compactness: consider the set of formulas consisting of the formula  $f_i(x) = f_i(y)$ , and for each morphism  $t : X_i \rightarrow X_j$  in  $(X_i)$ ,  $t(x) \neq t(y)$ . This set expresses the fact that the elements  $x, y \in X_i$  determine distinct elements of  $\text{Ind}((X_i))(M)$ , that have the same image under  $f$ . Therefore it is not satisfied in  $M$ . Since this collection is small, a finite subset is not satisfied. Therefore, there is some  $t_i : X_i \rightarrow X_j$  such that  $f_i(x) = f_i(y)$  implies  $t_i(x) = t_i(y)$ .

In particular, this means that  $f_j$  is injective on the image of  $t_i$ . Let  $X_0$  be the codomain of  $t_k$  (for the  $X_k$  found above.) Then  $f_0$  restricted to the image of  $t_k$  is a bijection. We take  $g$  to be the inverse of this restriction.

- (2) The proof is dual, using remark 4. The only complication here is that the category of definable sets does not, in general, have finite direct limits. The assumption that such limits exist is called *elimination of imaginaries (EI)*. However, for the specific purpose of the condition in remark 4, we do not actually need these limits. In our case, the condition simply translates to saying that  $f_i$  and  $t_i$  have the same image. Such  $t_i$  can be obtained by compactness, using the surjectivity of the limit map, as in the dual case.

Further, by considering the formulas  $f_i(x) \neq f_i(y)$ , we see that there is an  $X_k$  such that  $f_k$  is injective. In particular, we have  $t_k : X_0 \rightarrow X_k$ , such that  $f_k$  is a bijection between  $Y$  and the image of  $t_k$ . Taking  $g = f_k^{-1} \circ t_k$ , all the conditions of the lemma are satisfied.  $\square$

The promised description of morphisms is just the extension of this criterion to the entire category:

**Proposition 8.** *Let  $\kappa$  be a cardinal bigger than the cardinality of the index category,  $M$  a  $\kappa$  saturated model. Let  $X$  and  $Y$  be ind- (or pro-) definable sets,  $f : X \rightarrow Y$  a morphism that induces a bijection on  $M$  points. Then  $f$  is an isomorphism.*

*Proof.* We prove for the *Ind* category, the *Pro* case is dual. We have  $f : \text{Ind}(X_i) \rightarrow \text{Ind}(Y_i)$ . We first note that for any map  $f : P \rightarrow \text{Ind}(Y_i)$  where  $P$  is a presheaf,  $f$  is an isomorphism if and only if for all  $j$ , the pullback  $f_j : P \times_{\text{Ind}(Y_i)} Y_j \rightarrow Y_j$  is an isomorphism. Indeed, given inverses  $g_j$  to the  $f_j$ , their composition with the projection to  $P$  forms a matching family of maps from the  $Y_i$  to  $P$ , and therefore yields a map from  $\text{Ind}(Y_i)$  to  $P$ , inverse to  $f$ .

Furthermore, if  $P$  itself is ind-definable,  $P = \text{Ind}(X_i)$ , we have

$$P \times_{\text{Ind}(Y_i)} Y_j = \text{Ind}(X_i \times_{\text{Ind}(Y_i)} Y_j) = \text{Ind}(X_i \times_{Y_{\kappa_i}} Y_j)$$

On the other hand, since taking  $M$  points is represented by a pro-definable set, it preserves pullbacks. Therefore, if  $f_M$  is a bijection of  $M$  points, so is  $f_{jM}$ , for any  $j$ . By proposition 7,  $f_j$  is an isomorphism.

The description of the morphism sets is the interpretation of this statement for the projection map from a subset  $R$  of  $X \times Y$  to  $X$ .  $\square$

**Corollary 9.** *Let  $X$  and  $Y$  be ind- (pro-) definable sets. There is a natural bijection between  $\text{Hom}(X, Y)$  and ind- (pro-) definable subsets of  $X \times Y$  whose set of  $M$ -points is a function from  $X(M)$  to  $Y(M)$ .*

*Proof.* Apply proposition 8 to the projection map from a subset  $R$  of  $X \times Y$  to  $X$ .  $\square$

We may summarise the results of this section as follows:

**Corollary 10.** *Let  $M$  be a  $\kappa$ -saturated model.*

*The functor of “taking  $M$  points” is an equivalence of categories between the category  $\text{Pro}_\kappa(\mathcal{D})$  of pro-definable sets representable by systems of length less than  $\kappa$ , and the sub-category of the category of sets whose objects and morphisms are inverse co-filtered limits of  $M$  points of definable sets, of length less than  $\kappa$ .*

*Similarly, the same functor is an equivalence between the category  $\text{Ind}_\kappa(\mathcal{D})$  of ind-definable sets representable by systems of length less than  $\kappa$ , and the sub-category of the category of sets whose objects and morphisms are direct filtered limits of  $M$  points of definable sets, of length less than  $\kappa$ .*

Finally, we note that definable sets are given with canonical inclusions (in the “universe”.) For example, in our terminology, any two points are identified. If we wish to remember the inclusion of the definable sets in some definable set  $X$ , we work in the category  $\mathcal{D}/X$ , and all results continue to hold. This way we get pro-definable subsets of  $X$ . These sets are called also  $\omega$ -definable.

A pro-object in the category of definable groups contained in a definable set  $X$  is called an  $\omega$ -group. In other words, an  $\omega$ -group is the intersection of a language sized system of definable groups. We note that, as in the general case of ind- (and pro-) group objects, not every  $\omega$ -definable group is an  $\omega$ -group. For instance, in the theory of real-closed fields, the intersection of all intervals  $(-\frac{1}{n}, \frac{1}{n})$  (for all natural  $n$ ) is an  $\omega$ -definable set that has a group structure, inherited from the addition. However, by  $o$ -minimality, it is not an intersection of definable subgroups. On the other hand, if the theory is stable, then these two notions coincide (cf. [2].)

## REFERENCES

1. Alexander Grothendieck et al., *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec

- la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269. MR MR0354652 (50 #7130)
2. Ehud Hrushovski, *Unidimensional theories are superstable*, Ann. Pure Appl. Logic **50** (1990), no. 2, 117–138. MR MR1081816 (92g:03052)
  3. Saunders Mac Lane, *Categories for the working mathematician*, 2 ed., Graduate Texts in Mathematics, no. 5, Springer-Verlag, New York, 1998. MR MR1712872 (2001j:18001)
  4. Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994. MR MR1300636 (96c:03119)
  5. Gerald E. Sacks, *Saturated model theory*, W. A. Benjamin, Inc., 1972. MR MR0398817 (53 #2668)

DEPARTMENT OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL

*Current address:* Department of Maths, University of East Anglia, Norwich, NR4 7TJ, England

*E-mail address:* <mailto:m.kamensky@uea.ac.uk>

*URL:* <http://mkamensky.notlong.com>