# THE MODEL COMPLETION OF THE THEORY OF MODULES OVER FINITELY GENERATED COMMUTATIVE ALGEBRAS

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Abstract. We find the model completion of the theory modules over $\mathbb{A}$, where $\mathbb{A}$ is a finitely generated commutative algebra over a field $K$. This is done in a context where the field $K$ and the module are represented by sorts in the theory, so that constructible sets associated with a module can be interpreted in this language. The language is expanded by additional sorts for the Grassmanians of all powers of $K^n$, which are necessary to achieve quantifier elimination.

The result turns out to be that the model completion is the theory of a certain class of "big" injective modules. In particular, it is shown that the class of injective modules is itself elementary. We also obtain an explicit description of the types in this theory.

## 1. Introduction

We begin with some motivation. An algebra $\mathbb{A}$ finitely generated over an algebraically closed field $K$ corresponds to an affine variety $V$ over $K$, and a module over $\mathbb{A}$ corresponds to a (quasi-coherent) sheaf over $V$. Whereas varieties can be reasonably considered within the framework of model theory (for example, as definable sets in the theory $ACF$ of algebraically closed fields), modules (or sheaves) do not appear so naturally. For example, basic results about definability of the fibre dimension are proved, using algebraic methods, for algebras and modules alike (see [2, Theorem 14.4], for example.) On the model theoretic side, the fibre dimension for a map of varieties (or more generally, for definable sets) is well understood, in a much more general framework (e.g., the Morley rank of the fibre is definable; see, e.g., [1, Lemma 5.1.3].) However, the analogous statements for modules can not even be phrased. This work represents, we hope, a first step in approaching these questions.

The purpose of this paper is to find the model completion for the theory of modules over a finitely generated commutative $K$-algebra ($K$ a field), and describe the types in that theory. Our initial approach in formulating this theory is to use a two-sorted language, with a sort $K$ for the field, and another sort $M$ for the module. In addition to the field structure on $K$ and the $K$-vector space structure on $M$, we introduce symbols for $n$ commuting linear operators $T_i$ on $M$, that represent generators of the algebra.

Our goal is to find the model completion. To estimate the feasibility of our task, we consider the case $n = 0$. In this case we simply have a vector space $M$ over $K$. We immediately observe that the most basic relation on $M$, that of linear dependence, cannot be expressed in this theory without quantifiers (this follows by inspection: the only quantifier free relations on pairs of vectors are linear equations

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over the base field, but the field elements giving a linear dependence might not be in the base.) Since this is true independently of the theory of $K$, we conclude that the theory does not have a model completion.

To remedy this situation, we introduce operators on the vector space that provide information about linear dependence. Since the set of tuples witnessing linear dependence of $m$ vectors is a linear subspace of $K^m$, these operators take values in various Grassmanians, which we introduce as additional sorts. With this addition to the language, the above problem is resolved, and it turns out that this is the only obstacle for the existence of a model completion, even for the case when $n > 0$.

Below we give a precise definition of the language and the theory we work with. The rest is divided into the cases $n = 0$ and $n > 0$. Although the first case is not really different, the kind of problems the cases deal with are different and independent: in the first case, we deal with the vector space structure, as well as the new sorts introduced. In the second case, the main interest comes from the action of the operators $T_i$.

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1.1. **Linear spaces.** Before presenting the language and the theory we recall certain facts about linear spaces. Let $V$ be a vector space over a field $\Bbbk$. A *line* in $V$ is a linear subspace of dimension 1. If $V$ is finite dimensional, the set of lines in $V$, denoted $\mathbb{P}(V)$, has a structure of an algebraic variety over $\Bbbk$ (c.f., e.g., [6, Ch. 6] for all facts in this sub-section.) If $\phi \in V^*$ (the dual of $V$) is non-zero, then the set of lines on which $\phi$ is not (identically) zero is an open subset, and the map $[v] \mapsto v/\phi(v)$ is an isomorphism of this set with the affine space given by $\phi(v) = 1$. Here, we denote by $v \mapsto [v]$ the canonical map from $V \setminus 0$ to $\mathbb{P}(V)$.

More generally, the set of $l$-dimensional subspaces of $V$ is called the $l$-th Grassmanian of $V$, $Gr_l(V)$. If $V$ is finite dimensional, it is again an algebraic variety. The map $U \mapsto \bigwedge^l U \subseteq \bigwedge^l V$, for $U$ an $l$-dimensional subspace of $V$, is a closed embedding of $Gr_l(V)$ in $\mathbb{P}(\bigwedge^l V)$. Again there is a canonical map from the set $F_l(V)$ of $l$-tuples of linearly independent vectors in $V$ to $Gr_l(V)$. We shall denote by $Gr(V)$ the disjoint union of all $Gr_l(V)$. It is therefore the set of all (finite-dimensional) sub-spaces of $V$. If $V$ has dimension $n$, we denote by $\pi$ (or $\pi_V$) the map from $V^n$ to $Gr(V)$ assigning to each tuple the subspace it spans.

If $V = \Bbbk^n$, the above structures are interpreted in the theory of fields. The set $F_l(V)$ is then given within $V^l$ by the non-vanishing of some minor (where an element of $V^l$ is viewed as an $n \times l$ matrix), and is therefore defined without quantifiers. In particular, the relation of linear dependence is quantifier free. The projective space $\mathbb{P}(\Bbbk^n) = \mathbb{P}^n$ and the Grassmanians $Gr(\Bbbk^n) = Gr^n$ are imaginary sorts. The standard cover is given by taking for the functional $\phi$ above elements of the dual standard basis. Over the affine open set determined by the $i$-th projection, the map $v \mapsto [v]$ is given by $(x_1, \dots, x_n) \mapsto (x_1/x_i, \dots, x_n/x_i)$ (this is an instance of the elimination of imaginaries for algebraically closed fields.)

1.2. **Homogeneous definable sets.** We shall be interested in definable subsets of (Cartesian products of) Grassmanians. Such definable sets will be considered quantifier free if their inverse image under the natural projection is quantifier free.

A formula in the theory of fields $\phi(\overline{x}, Y)$, where $Y$ is a tuple of size $m^2$ is called homogeneous in $Y$ if for any $\overline{x}$, its truth value depends only on the subspace of $\Bbbk^m$ spanned by the rows of $Y$ (where $Y$ is viewed as a matrix.) We note that the set of formulas homogeneous in $Y$ is closed under boolean combinations.

We now introduce some terminology that will allow us to define precisely the atomic structure in the language we will use. Briefly, any formula that is homogeneous in some subsets of its variables will give rise to an atomic subset relation on the Grassmanians and the field. The precise details will become implicit once the definition is given. See also example 1.

For a natural number $n$, let $[n]$ be the set $\{1, \dots, n\}$. If $\overline{n} = (n_1, \dots, n_k)$ is a sequence of number, and $X$ is a set, we shall call an injective function $f : [n_1]^2 \amalg \dots \amalg [n_k]^2 \to X$ a *matrix form (of shape $\overline{n}$) of $X$*. A matrix form of a formula $\phi$ (in the language of fields) is a matrix form of the set of its free variables. A *homogeneous formula* is a formula $\phi$ together with a matrix form $f$ of shape $\overline{n}$, which is homogeneous in $f([n_i]^2)$, for each $i$. Thus, the subset it defines in $\Bbbk^X$ (where $X$ is the set of free variables of $\phi$) is the inverse image of a subset of $Gr^{n_1} \times \dots \times Gr^{n_k} \times \Bbbk^l$ under the product of the natural maps $\pi_i$ and the identity map. We note, however, that the shape $\overline{n}$ and the matrix form $f$ are part of the data of the homogeneous formula. For convenience, we will also allow the case $k = 0$, in which case there is a unique matrix form, and an homogeneous formula with this form is simply any formula in the language of fields.

Finally, A *homogeneous definable set* is a pair $(X, f)$ where $f$ is a matrix form, and $X$ is a definable set that can be represented by a homogeneous formula with matrix form $f$. It is quantifier free if the formula can be taken to be quantifier free (all with respect to the theory of fields.)

1.3. **The theory of modules over a commutative $K$-algebra.** We work with a multi-sorted language, as described, e.g., in [4, 1.3]. If $S$ is (the symbol for) a sort, and $\mathbf{N}$ is a structure, we denote by $S(\mathbf{N})$ the part of $\mathbf{N}$ corresponding to that sort. Given $n \geq 0$, we use the following language $\mathcal{L} = \mathcal{L}_n$:

$$\begin{aligned}
\mathcal{L}_n = (&K, +, -, \cdot, 0, 1, && (1)\\
&M, +, -, 0, \cdot, && (2)\\
&\{G^i\}_{i>0}, \{\pi_i\}_{i>0}, \{P_{\widetilde{\varphi}}\}_{\widetilde{\varphi}}, \{D_i\}_{i>0}, && (3)\\
&T_1, \dots, T_n) && (4)
\end{aligned}$$

Where

- $K$ is a symbol for a field sort and (1) is the language of fields on it.
- $M$ is a symbol for a vector space sort, and (2) is the language of abelian groups together with a function symbol $\cdot \colon K \times M \to M$
- each $G^i$ is a sort, and $\pi_i \colon K^{i^2} \to G^i$ is a function symbol
- $\widetilde{\varphi}$ is a quantifier free homogeneous definable set, of shape $k_1, \dots, k_m$. Given such a $\widetilde{\varphi}$, $P_{\widetilde{\varphi}}$ is a predicate symbol on $G^{k_1} \times \dots \times G^{k_m} \times K^l$ (where $l$ is the number of variables not in any homogeneous part.)
- Each $D_i : M^i \to G^i$ is a function symbol
- Each $T_i$ is a function symbol $T_i : M \to M$

Let $I \subset \mathbb{Z}[X_1, \dots, X_n]$ (the polynomial algebra in $n$ variables) be an ideal, and let $\mathbb{A} := \mathbb{Z}[X_1, \dots, X_n]/I$. The theory $\mathfrak{T} = \mathfrak{T}_I$ says the following:

- (1) is a field, and (2) is a vector space over it
- For each $i$, $(G^i, \pi_i)$ is the Grassmanian of $K^i$, as described in 1.1. Thus, we view $K^{i^2}$ as an $i$-tuple of row vectors in $K^i$, and the theory says that $\pi_i : K^{i^2} \to G^i$ is surjective, and two elements belong to the same fibre if and only if the corresponding row vectors span the same linear subspace. As noted in 1.1, this last axiom is universal.
- Let $\widetilde{\varphi}$ be a homogeneous definable set, of shape $k_1, \dots, k_m$, and with matrix form $f$. Denote by $\overline{x_i}$ the set of free variables $f([k_i]^2)$, and by $\overline{x}$ the set of free variables not in the image of $f$. Then $P_{\widetilde{\varphi}}$ is the set induced on $G^{k_1} \times \dots \times G^{k_m} \times K^l$ by $\varphi$, i.e., we have the axiom

$$\begin{aligned}
&\forall p_1 \in G^{k_1}, \dots, p_m \in G^{k_m}, \overline{x} \in K^l \\
&\qquad (P_{\widetilde{\varphi}}(p_1, \dots, p_m, \overline{x}) \iff \\
&\qquad\quad \exists \overline{x_1} \in K^{k_1^2}, \dots, \overline{x_m} \in K^{k_m^2} (\varphi(\overline{x_1}, \dots, \overline{x_m}, \overline{x}) \wedge \\
&\qquad\qquad\qquad\qquad \bigwedge_i \pi(\overline{x_i}) = p_i))
\end{aligned}$$

  We note that this is not an additional structure, but part of $K^{eq}$. However, as explained above, this addition (together with the operators $D_i$) is required in our approach in order to achieve quantifier elimination. Also note, that since $\varphi$ is homogeneous, the above axiom is universal.
- For any $m$, $D_m$ is a function from $M^m$ to $G^m$, assigning to any $\overline{v} = v_1, \dots, v_m \in M$, the subspace of all $\overline{x} \in K^m$ such that $\sum x_i v_i = 0$.
- The $T_i$ encode the $K \otimes \mathbb{A}$ module structure on $M$. Thus, they are commuting $K$-linear operators on $M$. Since they commute, for any $p \in \mathbb{Z}[X_1, \dots, X_n]$, $p(T_1, \dots, T_n)$ is a well defined operator. We require that $p(T_1, \dots, T_n)$ is the 0 operator for any $p \in I$ (See remark 21 about the non-commutative case.)

Models of this theory are (determined by) pairs $(K, M)$, where $K$ is a field, and $M$ is a module over $K[X_1, \dots, X_n]/\hat{I}$. Here, $\hat{I}$ is the ideal generated by $I$ in $K[X_1, \dots, X_n]$.

**Example 1.** To illustrate the structure in (3), we give some examples of formulas $P_{\widetilde{\phi}}$.

1. As mentioned in section 1.1, the condition $rank(X) = k$ on an $n \times n$ matrix $X$ is definable by a quantifier free formula $\phi$ (together with matrix form $f : [n]^2 \to X$.) It follows that the subset of $G^n$ consisting of $k$-dimensional subspaces is defined by a basic relation $P_\phi$, which we denote by $G^n_k$.
2. Let $\phi(X, y, z)$ be the formula $X \cdot \binom{y}{z} = 0$, where $X$ is a 4-tuple of variables, viewed as a $2 \times 2$ matrix, and $\cdot$ is the usual matrix multiplication (formally, we consider $\widetilde{\phi}$ to consist of a formula $\phi$ in six variables, together with a matrix form $f : [2]^2 \to X$.) This is a quantifier free homogeneous formula, since passing to a matrix $X'$ with the same row space as $X$ amounts to multiplying by an invertible matrix on the left. It thus defines a basic relation $P_{\widetilde{\phi}}(p, y, z)$ on $G^2 \times K^2$. The relations $R_1(p, z)$ and $R_2(p, y)$ given by $= P_{\widetilde{\phi}} \wedge (y = 1)$ and $P_{\widetilde{\phi}} \wedge (z = 1)$, respectively, each defines a partial function from $\mathbb{P}^1$ to $K$, which together illustrate the fact that every element $p$ in $\mathbb{P}^1$ contains a $p$ definable point. This works in the same way for higher

dimensional projective spaces $\mathbb{P}^n$, by replacing $X$ with an $(n+1)\times(n+1)$ matrix, and $(y,z)$ with an $n+1$-tuple.

(3) The field multiplication extends to a function $m : \mathbb{P}^1 \times \mathbb{P} \to \mathbb{P}^3$ (this is an instance of the Segre embedding, c.f. [3, ex. I.2.14].) The map is given in homogeneous coordinates by $m([x,y],[z,w]) = [xw, yz, xz, yw]$. Its image is given by the homogeneous equation $st = uv$ on $\mathbb{P}^3$, and $m$ gives an (algebraic) isomorphism with its image.

The map $m$ can further be extended to a map from $G^2 \times G^2$ to $G^4$: on the level of matrices, the extension is given by sending the $2\times 2$ matrices $X$ and $Y$ to $X \otimes Y$. Though the formula $X \otimes Y = Z$ is not homogeneous in $X$, $Y$ and $Z$, it can be homogenised: the formula $\phi(X,Y,Z)$ given by $\exists B \in GL_4(X \otimes Y = BZ)$ is homogeneous, since multiplication of $X$ and $Y$ by invertible matrices $B_X$ and $B_Y$ on the left corresponds to multiplying $Z$ by $B_X \otimes B_Y$, which can be absorbed in $B$. Contrary to its appearance, the definable set given by $\phi$ is quantifier free, as explained in section 1.1. It thus defines a relation $P_\phi(p,q,r)$ on $G^2 \times G^2 \times G^4$, that gives rise to a function from $G^2 \times G^2$ to $G^4$, which is an extension of the field multiplication.

Since, according to the previous examples, the projective spaces $\mathbb{P}^1$ and $\mathbb{P}^3$, respectively, and their affine covers are basic relations, we get an interpretation of the fields structure in the sorts $G^i$ via the basic relations. For example, an element $p \in G^2$ satisfies the formula $G^2_1(p) \wedge m(p,p) = \pi(R_1(p), R_1(p), 1, 2)$ if and only if $p$ is the one-dimensional space spanned by $(1, \sqrt{2})$. Here we wrote $R_1$ for the function in the previous example, and used a vector as an argument of $\pi$ instead of the matrix consisting of the same vector four times.

(4) For natural numbers $k \leq n$, let $X$ and $Y$ by matrices of variables of sizes $n$ and $\binom{n}{k}$, respectively, and let $E(X,Y)$ be the (quantifier free) formula saying that $Y = \bigwedge^k X$, the matrix of $k$-minors of $X$. If $A$ and $B$ satisfy $E$, then so do $TA$ and $(\bigwedge^k T)B$ for any invertible $n\times n$ matrix $T$. Hence this formula can be homogenised as in the previous example, to give a relation $P_E(X,Y)$ on $G^n \times G^{\binom{n}{k}}$. This relation restricts to the embedding of $G^n_k$ into $\mathbb{P}^{\binom{n}{k}}$ mentioned in section 1.1.

*Notation* 2. For the sake of readability, we use the following conventions: the letter $G$ is used to denote a $G^i$ with an unspecified $i$. Unless otherwise mentioned, $x, y, z$ are field variables, $p, q, r$ are $G$ variables, and $u, v, w$ are module variables. $X, Y, Z$ are used for tuples of field variables when considered as matrices. We will generally use **bold** letters for structures and their subsets.

Given a model $(\mathbf{K}, \mathbf{M})$ of $\mathfrak{T}$, we identify an element of $\mathbf{K}[X_1, \dots, X_n]$ or $\mathbf{K} \otimes \mathbb{A}$ with the operator it defines on $\mathbf{M}$. Similarly, we identify an element of $\mathbb{Z}[X_1, \dots, X_n]$ or $\mathbb{A}$ with the terms or definable function it defines on the module sort. Also, since the number of operators $T_i$ is fixed in every situation, $n$ is released for other uses.

## 2. The case $n = 0$

In this section we deal with the case of a vector space over a field. We show that, with the dependence functions introduced above, the theory admits a model completion, and describe the types explicitly. A related idea appears in [7].

We are looking for a model completion of the theory above, so in particular the field part $K$ should eliminate quantifiers. Since in $\mathcal{L}$, the quantifier free subsets of the field are only those defined in the field language, this leads us to the requirement that $K$ is algebraically closed. This requirement makes the theory complete, up to the characteristic of $K$ and the dimension of $M$ as a vector space over $K$ (in fact, fixing the characteristic of $K$ and the dimension of $M$, any two models where the fields have the same uncountable cardinality are isomorphic.) Let $\widetilde{\mathfrak{T}^\infty}$ be $\mathfrak{T}_0$, together with the axioms saying that $K$ is algebraically closed, and $M$ is of infinite dimension over $K$. Furthermore, let $\mathfrak{T}_0^n$ be the theory $\mathfrak{T}_0$ together with the (universal) axiom saying that $M$ has dimension at most $n$, and let $\widetilde{\mathfrak{T}^n}$ be the extension of this theory by the axioms that $K$ is algebraically closed, and $M$ is of dimension $n$. Our first goal is:

**Proposition 3.** *The theories $\widetilde{\mathfrak{T}^\infty}$ and $\widetilde{\mathfrak{T}^n}$ eliminate quantifiers. They are the model completions of $\mathfrak{T}_0$ and $\mathfrak{T}_0^n$, respectively.*

Till the end of this section, we let $\mathfrak{T}$ stand for one of the theories $\mathfrak{T}_0^n$ (or $\mathfrak{T}_0$), and $\widetilde{\mathfrak{T}}$ the corresponding extension. See [8, section 5.3] or [10, chapter 12] for the notions of elimination of quantifiers and model completion. Assuming quantifier elimination, showing that $\widetilde{\mathfrak{T}}$ is the model completion amounts to showing that any model of $\mathfrak{T}$ embeds in a model of $\widetilde{\mathfrak{T}}$.

We begin with a few remarks concerning only the relation between $K$ and the $G^i$. We denote by $\mathcal{L}^-$ the language restricted to these sorts, and by $\mathfrak{T}^-$, $\widetilde{\mathfrak{T}}^-$ the theories $\mathfrak{T}$ and $\widetilde{\mathfrak{T}}$ restricted to $\mathcal{L}^-$. For $\varphi(\bar{x}, p_1, \dots, p_k)$ an $\mathcal{L}^-$ formula, let

$$\varphi^*(\bar{x}, Y_1, \dots, Y_k) \overset{\text{def}}{=} \varphi(\bar{x}, \pi(Y_1), \dots, \pi(Y_k))$$

$\varphi^*$ is an homogeneous formula, with the obvious structure.

We are going to make assertions regarding linear transformations on spaces like $K^n$ and $M^n$. Since we usually need the claims for matrices with arbitrary terms (and not just constants), we redefine 'linear map' to mean any definable map from $K^m$ to the set of $n_1 \times n_2$ matrices (some $m, n_i$), considered as linear transformations acting on the right for $K$ and on the left for $M$.

**Lemma 4.** *The following facts hold in $\mathfrak{T}^-$:*

a. *Any quantifier free formula is equivalent to a quantifier free formula without $\pi$.*

b. *Let $\varphi(\bar{x}, p_1, \dots, p_k)$ be a quantifier free formula. Then*

$$\varphi \equiv P_{\varphi^*} \tag{5}$$

*We note that $P_{\varphi^*}$ is defined, since $\varphi^*$, by part a, is equivalent to a pure field formula.*

c. *For any linear map $A : K^m \to K^n$, and for subspaces $p \subseteq K^m$ and $q \subseteq K^n$, the image $pA$ and the inverse image $qA^{-1}$ are again linear subspaces. Thus we have induced maps $A : G^m \to G^n$ and $A^{-1} : G^n \to G^m$ (these maps will be written on the left.)*

*Proof.*

a. It's enough to prove this for atomic formulas. There are two kinds of these:
   - $P_{\widetilde{\varphi}}(\pi(X), \dots)$, where $X$ is a matrix term of size $k_1 \times k_1$, and $\varphi(\bar{x})$ has shape $k_1, \dots, k_m$. By definition, this formula is equivalent to $P_{\hat{\varphi}}(X, \dots)$, where $\hat{\varphi}$ is $\varphi$ considered as a homogeneous formula of shape $k_2, \dots, k_m$.

- $\pi(X) = p$, where $X$ is a matrix term, and $p$ is a term with value in $G$. This one is equivalent to $P_{\langle Y\rangle=\langle Z\rangle}(X,p)$, where $\langle Y\rangle = \langle Z\rangle$ is the field formula expressing that $Y$ and $Z$ span the same space (as explained in 1.1, this is a quantifier free set), consider as homogeneous formula in $Z$. Hence this reduces to the first case.

b. For $\varphi = P_\psi$, this follows directly from the definitions. By part a, this is the only kind of atomic formulas we should check. On the other hand, $*$ is a homomorphism of boolean algebras, and so is $P$ restricted to formulas of the form $\psi^*$, so the result follows.

c. Is obvious.

□

**Corollary 5.** *The theory* $\widetilde{\widetilde{\mathfrak{T}}}^{-}$ *admits elimination of quantifiers. Thus, the equivalence* (5) *of lemma 4 holds for any formula* $\varphi$ *in this theory. Any formula in this theory is equivalent to the formula* $P_\varphi$ *for some field formula* $\varphi$*.*

*Proof.* By lemma 4.b, we need to show there is a quantifier free formula equivalent to

$$\exists A(P_\varphi(x,\overline{y},p_1,\dots,p_k))$$

where $A$ is either $x$ or $p_1$. In the first case, the above formula is equivalent to $P_{\exists x\varphi}(\overline{y},p_1,\dots,p_k)$, in the second to $P_{\exists Y_1\varphi}(y,\overline{x},p_2,\dots,p_k)$ (where $Y_1$ corresponds to $p_1$.) The formula $\phi$ is in the pure field language, and since the field is algebraically closed, $\exists A\phi$ is equivalent to a quantifier free formula.

The rest is just a summary of lemma 4. □

At this point we can prove that if $\widetilde{\mathfrak{T}}$ has quantifier elimination, then it is the model completion of $\mathfrak{T}$:

**Proposition 6.** *Any model of* $\mathfrak{T}$ *embeds into* $\widetilde{\mathfrak{T}}$*.*

*Proof.* Let $\mathbf{N}$ be a model of $\mathfrak{T} = \mathfrak{T}^n$ ($n$ can be $\infty$), with $\mathbf{K} = K(\mathbf{N})$ and $\mathbf{M} = M(\mathbf{N})$ (the other sorts are uniquely determined by $\mathbf{K}$.) Let $\widetilde{\mathbf{K}}$ be an algebraic closure of $\mathbf{K}$, let $\widetilde{\mathbf{M}}$ be a vector space of dimension $n$ over $\widetilde{\mathbf{K}}$ containing $\widetilde{\mathbf{K}}\otimes_{\mathbf{K}}\mathbf{M}$) (which exists since $\dim(\mathbf{M}) \le n$), and let $\widetilde{\mathbf{N}}$ be the model of $\widetilde{\mathfrak{T}}$ determined by $(\widetilde{\mathbf{K}},\widetilde{\mathbf{M}})$. We claim that the unique map of $\mathbf{N}$ into $\widetilde{\mathbf{N}}$ restricting to the inclusion of $\mathbf{K}$ in $\widetilde{\mathbf{K}}$ (and of $\mathbf{M}$ in $\widetilde{\mathbf{K}}\otimes_{\mathbf{K}}\mathbf{M}$), and commuting with the projections $\pi$, is an embedding.

Assume that $P = P_{\widetilde{\varphi}}(\overline{x},\overline{p},D(\overline{v}_1),\dots,D(\overline{v}_n))$ holds in $\mathbf{N}$. Since $\mathbf{N}$ is a model of $\mathfrak{T}$, there are matrices $Y_i$ over $\mathbf{K}$ whose rows span $p_i$. Similarly, there are matrices $Z_i$ whose rows span $D(\overline{v}_i)$. The statement $P$ is then equivalent to $\varphi(\overline{x},\overline{Y},\overline{Z})$, together with the fact that the rows of $Z_i$ span the space of tuples $\overline{z}$ such that $\sum z_iv_i = 0$. The formula $\varphi$ is quantifier free, and linear dependence does not change when extending scalars, so these statements hold in $\widetilde{N}$ as well, hence so does $P$. □

It thus remains to prove that $\widetilde{\mathfrak{T}}$ eliminates quantifiers. We shall give two proofs of this fact. The first, explained directly below, exhibits an explicit quantifier free formula equivalent to an arbitrary one. The other one, starting with criterion 8, uses a general criterion for quantifier elimination and an analysis of quantifier free types. This proof is extended in section 3 to the general case.

For $A_1,\dots,A_k$ linear maps, and $\varphi(\overline{x},p_1,\dots,p_k)$ a formula, we set

$$(A_1,\dots,A_k)^*\varphi = \overline{A}^*\varphi \overset{\text{def}}{=} \varphi(\overline{x},A_1(q_1),\dots,A_k(q_k))$$

and

$$(A_1, \dots, A_k)_* \varphi = \overline{A}_* \varphi \overset{\text{def}}{=} \varphi(\overline{x}, A_1^{-1}(q_1), \dots, A_k^{-1}(q_k))$$

Note that these formulas will depend on the additional variables of the $A_i$ (Strictly speaking, the operators $A_i$ do not actually exist in the language, but the formulas exist, and are quantifier free.)

We now go back to the full $\widetilde{\mathfrak{T}}$, and the next step is to analyse the quantifier free formulas in the theory. The main lemma we need is:

**Lemma 7.** *Any quantifier free formula is equivalent to a quantifier free formula in which for every term of the form $D(t_1, \dots, t_n)$, each $t_i$ is either a module variable or a module constant (We assume that the base set is a substructure)*

*Proof.* The claim follows from the fact that for any linear map $A$, we have $D(A\overline{v}) = A^{-1}(D(\overline{v}))$: Indeed, both sides are equal to the space of all $\overline{x}$ such that $\overline{x}A\overline{v} = 0$. Now, any quantifier free formula $\varphi$ has the form

$$\varphi'(\overline{x}, D(A_1\overline{v}), \dots, D(A_k\overline{v}), \overline{p})$$

so by the above equality, $\varphi$ is equivalent to

$$(\overline{A}_*\varphi')(\overline{x}, D(\overline{v}), \dots, D(\overline{v}), \overline{p})$$

□

We can now prove quantifier elimination:

*Proof of proposition 3.* Let $\varphi(\overline{x}, \overline{p}, \overline{v})$ be some quantifier free formula. We need to find a quantifier free formula equivalent to $\exists A\varphi$, where $A$ is one of $x_0$, $p_0$ or $v_0$. Now, by lemma 7, there is some formula $\varphi'(\overline{x}, \overline{p}, q)$ such that $\varphi$ is equivalent to $\varphi'(\overline{x}, \overline{p}, D(\overline{v}))$. Hence, for the cases that $A$ is either $x_0$ or $p_0$, $\exists A\varphi$ is equivalent to $\exists A\varphi'(\overline{x}, \overline{p}, D(\overline{v}))$, and $\exists A\varphi'$ is equivalent to a quantifier free formula by lemma 4. Thus the only case left is $\exists v_0\varphi$.

Let $\varphi_0'$ be the formula

$$\varphi'(\overline{x}, \overline{p}, q) \wedge \exists \overline{y}((1, \overline{y}) \in q)$$

(i.e., the set of $q$ satisfying $\varphi$ whose projection to the first coordinate is not 0), and let $\varphi_1' = \varphi' \wedge \neg\varphi_0'$. Since existential quantifiers commute with disjunction, it is enough to prove for each of these cases separately.

Assume first that $\varphi' = \varphi_0'$. Then $\exists v_0\varphi$ is equivalent to

$$\exists \overline{y}\varphi'(\overline{x}, \overline{p}, D(-\sum_{i>0} y_i v_i, v_1, \dots, v_n))$$

In the case $\varphi' = \varphi_1'$, $\varphi$ says that $v_0$ is independent of the other vectors, and therefore $D(\overline{v})$ coincides with $i(D(v_1, \dots, v_n))$, where $i : K^n \mapsto K^{n+1}$ is the inclusion as the last $n$ coordinates. Hence $\exists v_0\varphi$ is equivalent to

$$\varphi'(\overline{x}, \overline{p}, i(D(v_1, \dots, v_n))) \wedge (< v_1, \dots, v_n > \neq M)$$

Since the theory determines the dimension of $M$, the statement that the $v_i$ span $M$ depends only on the dimensions of $D$ applied to subsets of the $v_i$, hence is quantifier free. □

Our next goal is to analyse the quantifier free types. Since we don't use quantifier elimination here, we will be able to use this to give a second proof of quantifier elimination. Then, because of quantifier elimination, this will give information about the spaces of types, and eventually $\omega$-stability will be shown.

To prove quantifier elimination, we will use the following criterion, which is a variant of [10, 17.5]:

**Criterion 8.** *A theory $T$ eliminates quantifiers if for any model $M$ and any $A \subseteq M$, any quantifier free $1$-type over $A$ is also a type (i.e. consistent) with respect to any extension of $T_A$ (where $T_A$ is the theory obtained by adding to $T$ all quantifier free sentences over $A$ that hold in $M$.)*

*Proof.* We use [10, 17.5]. Let $\mathbf{B}$ and $\mathbf{C}$ be models of $T$ containing $\mathbf{A}$, and let $\mathbf{B}^*$ be a $(\operatorname{card}\mathbf{C})^+$ saturated extension of $\mathbf{B}$. Let $p$ be the quantifier free type over $\mathbf{A}$ of some element $c \in \mathbf{C}$. By assumption, $p$ is consistent with the theory of $\mathbf{B}^*$ over $\mathbf{A}$. By saturation, $p$ is realised by some element $b \in \mathbf{B}^*$. Thus we may extend the inclusion of $\mathbf{A}$ in $\mathbf{B}^*$ to $c$. ☐

We begin by analysing the substructures of a model of $\widetilde{\mathfrak{T}}$, and first, as before, we consider only the restriction to the sorts $K$ and $G^i$. For this restricted theory we will assume elimination of quantifiers (as proved in lemma 4.)

Let $\mathbf{A}$ be a substructure of a model of $\widetilde{\mathfrak{T}}^-$. Then $K(\mathbf{A})$ is an integral domain, whose fraction field we denote by $\mathbf{L}$, and $G^i(\mathbf{A})$ contains all the subspaces of $\mathbf{L}^i$ (but maybe more.)

**Claim 9.** *There is a minimal extension $\mathbf{B}$ of $\mathbf{A}$ such that $K(\mathbf{B})$ is a field, and for each $i$, $\pi_i : K(\mathbf{B})^{i^2} \to G^i(\mathbf{B})$ is onto.*

The extension is minimal in the sense that any other such extension factors uniquely through it. In other words, if $\mathbf{A}$ is definably closed, then $K(\mathbf{A})$ is a field, and the projections $\pi_i$ are onto.

*Proof.* First, we may assume that $K(\mathbf{A})$ is a field by passing to the fraction field. Consider the subset $\mathbb{P}(\mathbf{A})$ of $G(\mathbf{A})$ consisting of the one-dimensional subspaces (these are the $\mathbf{A}$ points of the projective space inside $G$, which we also denote by $G_1(\mathbf{A})$.) A point of this subset corresponds to a line in some affine space $K^i$ (not necessarily defined over $K(\mathbf{A})$.) For $\pi_i$ to be onto, this line should have a point in $\mathbf{B}$. This will happen if and only if the unique point on this line whose $k$-th coordinate is 1 has its other coordinates in $\mathbf{B}$. Such points correspond to intersection of this line with the standard cover of $\mathbb{P}$ (cf. section 1.1.) This cover corresponds to some elements (over 0) of $G$, and the intersections are encoded in the structure of $G$, as in example 1.2.

We thus get a finite set of points in affine space, one for each such intersection. Using the standard projections, we get a finite set of points in $\mathbb{A}^1$. The type of these points as field elements is well defined, since both the field $K(\mathbf{A})$ and the field operations can be viewed as part of the structure of $G$, as in example 1.3.

In other words, every line appearing in $\mathbb{P}(\mathbf{A})$ has a definable point in the model. Extending the field by all such points, we get a field extension $K(\mathbf{B})$, which, by construction, contains a point in each element of $\mathbb{P}(\mathbf{A})$, and is obviously minimal with this property. We let $\mathbf{B}$ be the structure whose field part is the field just constructed, and whose $G$ parts consist of the original points $G(\mathbf{A})$, together with the image of the projections $\pi$ from powers of $K(\mathbf{B})$.

It remains to show that $K(\mathbf{B})$ contains a basis for any other element of $G(\mathbf{B})$. Consider the elements $G^n_k(\mathbf{A})$ in $G(\mathbf{A})$ corresponding to $k$-dimensional subspaces of $K^n$. This set has a natural embedding (over $\mathbb{Z}$) into $\mathbb{P}(\bigwedge^k K^n)$, corresponding to the natural map $(K^n)^k \to \bigwedge^k K^n$ (as in section 1.1.) We may assume that $\mathbf{A}$ is closed under these maps (cf. example 1.4.) The image of a point $U$ of $G^n_k(\mathbf{A})$ in the above projective space contains, by the definition of $\mathbf{B}$, a point $\overline{x}$ of $K(\mathbf{B})$. Thus, as subspace of $K^n$, $U$ is defined by a quantifier free formula in the language of fields (namely, $\bigwedge^k X = \overline{x}$.)

Thus the problem reduces to showing that, for any field $\mathbf{L}$, any non-zero point of $\bigwedge^k \mathbf{L}^n$ has a pre-image in $(\mathbf{L}^n)^k$ under the natural map $W : (\mathbf{L}^n)^k \to \bigwedge^k \mathbf{L}^n$, provided it has such a pre-image in a field extension (i.e., provided that it corresponds to a point in the image of the embedding of $G^n_k$ in projective space.) Two such pre-images $X$ and $Y$ (in an extension) span the same $k$-dimensional vector space, and so are obtained from each other by multiplication by a unique element of $GL_k$. Furthermore, this unique element lies in $SL_k$, since $W(X) = W(Y)$ (rather than just being colinear.) Conversely, if $X$ is obtained from $Y$ by multiplying by an element of $SL_k$, then $W(X) = W(Y)$. Hence, the pre-image set forms an $SL_k$-torsor, but any such torsor has an $\mathbf{L}$ point (see, e.g., [5, Lemma III.4.10] for $GL_k$, and then use the exact sequence $1 \to SL_k \to GL_k \to G_m \to 1$.) ☐

Next, we extend the statement to the sort $M$ in the full theory $\widetilde{\mathfrak{T}}$. If $\mathbf{A}$ is a substructure of a model of this theory, then $M(\mathbf{A})$ is a flat $K(\mathbf{A})$ module, which therefore embeds into the $\mathbf{L}$-vector space $M(\mathbf{A}) \otimes_{K(\mathbf{A})} \mathbf{L}$.

**Claim 10.** *Let* $\mathbf{A}$ *be a substructure of a model of* $\widetilde{\mathfrak{T}}$*, and let* $\mathbf{B}$ *be as promised by claim 9 (for the restriction of* $\mathbf{A}$ *to* $\mathfrak{T}^-$*.) Then there is a minimal vector space* $\mathbf{V}$ *over* $K(\mathbf{B})$ *such that* $(\mathbf{B}, \mathbf{V})$ *is an extension of* $\mathbf{A}$ *as a substructure.*

*Proof.* Let

$$V = K(\mathbf{B}) \otimes_{K(\mathbf{A})} M(\mathbf{A}) / \langle \sum x_i \otimes v_i | \overline{x} \in D(\overline{v}) \rangle$$

Since $D(\sum x^1_i \otimes v^1_i, \dots, \sum x^m_i \otimes v^m_i)$ is determined by $D(\overline{v^1}, \dots, \overline{v^m})$, this already defines a structure. It is obvious that this is what we want. ☐

Let us say that $\mathbf{A}$ is a *good substructure* if $K(\mathbf{A})$ is a field and $G^i(\mathbf{A})$ is the set of subspaces of $K(\mathbf{A})^i$ (in other words, it is a model of $\mathfrak{T}$). Then the above claims say that any substructure has a unique minimal extension to a good substructure (in other words, definably closed structures are good.)

**Claim 11.** *Let* $\mathbf{A}$ *be a good substructure,* $\mathbf{B}$ *an extension of* $\mathbf{A}$*. Then* $M(\mathbf{B})$ *contains* $K(\mathbf{B}) \otimes_{K(\mathbf{A})} M(\mathbf{A})$*.*

*Proof.* Since $M(\mathbf{B})$ is a vector space over $K(\mathbf{B})$ containing $M(\mathbf{A})$, there is a canonical map $i : K(\mathbf{B}) \otimes_{K(\mathbf{A})} M(\mathbf{A}) \to M(\mathbf{B})$. Assume that $\sum x_j \otimes v_j$ goes to 0 in $M(\mathbf{B})$ ($v_j \in M(\mathbf{A})$). Then $\overline{x} \in D(\overline{v})$, but according to the assumption, $D(\overline{v})$ has a basis with coordinates in $K(A)$, so $\sum x_j \otimes v_j$ is 0 already in $K(\mathbf{B}) \otimes_{K(\mathbf{A})} M(\mathbf{A})$. ☐

This implies that for good substructures, statements regarding the vector space are unambiguous: In general, for example, the statement "$v_1, \dots, v_n$ are linearly independent" might mean either that it is independent over the field part of the structure, or that $D(\overline{v}) = 0$. For good substructures, this is the same.

We can now give a

*Second proof of quantifier elimination.* We use criterion 8. Let $\mathbf{A}$ be a substructure. Any model of $\widetilde{\mathfrak{T}}$ containing $A$ will also contain the substructure given by claim 10, and $\mathbf{K}$ will contain an algebraic closure of $K(\mathbf{A})$, so by claim 11 we may assume that

- $K(\mathbf{A})$ is an algebraically closed field.
- each $\pi$ is onto.
- $M(\mathbf{A})$ is a vector space over $K(\mathbf{A})$.

Let $\mathfrak{T}_1$ be a theory extending $\widetilde{\mathfrak{T}}_{\mathbf{A}}$, and let $\mathfrak{p}(v)$ be a quantifier free 1-type over $\mathbf{A}$ with respect to $\widetilde{\mathfrak{T}}$, in the module sort (quantifiers on other sorts are eliminated as before). Consider the set of formulas $D(v, v_1, \dots, v_n) \neq 0$ with $v_i \in M(\mathbf{A})$, satisfying $D(v_1, \dots, v_n) = 0$. Assume first that there is no such formula in $\mathfrak{p}$. Then (since $\mathfrak{p}$ is consistent), the vector space is either $\infty$-dimensional, or of dimension greater than the dimension of $M(\mathbf{A})$. In any case, there is a model of $\mathfrak{T}_1$ which has a member outside the space generated by $M(\mathbf{A})$. Any such member will satisfy the type.

Now assume, conversely, that there are formulas as above, and assume that $n$ is minimal. Then for any $u_i$ with $D(v, \overline{u}) \neq 0$, the space $V$ spanned by $v_1, \dots, v_n$ is contained in the space spanned by $\overline{u}$ (Otherwise, the intersection of these spaces is properly contained in $V$, and any basis of it is a contradiction to the minimality of $n$). We claim that the set

$$\{P_\varphi(D(v, \overline{v})) \in \mathfrak{p}\}$$

determines the type. Let $\psi(D(v, \overline{u}_1), \dots, D(v, \overline{u}_k))$ be a formula in $\mathfrak{p}$. We first note, that if $\overline{w}_1$ spans the same subspace as $\overline{u}_1$, then $\psi$ is equivalent to some formula $\psi'(D(v, \overline{w}_1), \dots, D(v, \overline{u}_k))$: by assumption, there is some matrix $U$ (over $K(A)$!) such that $(v, \overline{u}_1) = U(v, \overline{w}_1)$. Hence the equivalence follows from lemma 7. In particular, we may assume that the first $n$ vectors in each $\overline{u}_i$ coincide with $\overline{v}$, and that each $\overline{u}_i$ is linearly independent. But then, letting $i_m : K^n \hookrightarrow K^{l_m}$ be the inclusion of the first $n$ coordinates (where $l_m$ is the length of $\overline{u}_m$), it is clear that $(i_1, \dots, i_k)^*\psi$ is the formula we seek.

Let $\mathfrak{q} = \{P_\varphi(p) : P_\varphi(D(v, \overline{v})) \in \mathfrak{p}\}$. Since $ACF$ eliminates quantifiers, there is a model of $\mathfrak{T}_1$ in which $\mathfrak{q}$ has a realisation, $q$. Since $\overline{v}$ is independent, $q$ will be of dimension either 1 or 0. If it's 1, let $x_1, \dots, x_n$ be the unique tuple with $(1, \overline{x}) \in q$. Then $v = -\sum x_i v_i$ satisfies $\mathfrak{p}$. If $q = 0$, then the dimension of $M$ must be more than $n$ (otherwise $\mathfrak{p}$ would be inconsistent, since the dimension is given already in $\mathfrak{T}$). Then any $v$ independent of $\overline{v}$ satisfies $\mathfrak{p}$. $\square$

Let us record the result in the proof as a separate claim:

**Claim 12** (description of the vector space types)**.** *A* 1*-type* $\mathfrak{p}(v)$ *over a good structure* $\mathbf{A}$ *is determined by either a sequence* $v_1, \dots, v_n \in M(\mathbf{A})$ *of minimal length such that*

$$D(v, v_1, \dots, v_n) = 0$$

*is in* $\mathfrak{p}$*, together with the type* $\mathfrak{q}$ *(in the field and Grassmanian sorts) such that*

$$\mathfrak{p} = \mathfrak{q}(D(v, v_1, \dots, v_n))$$

*or by the fact that there is no such sequence (In other words, it is determined by the minimal subspace to which v belongs, together with the minimal field over which it happens).*

*Remark* 13. In the proof we dealt only with quantifier free types, but now we know that this is all there is.

Recall that a theory is *$\omega$-stable* if the set of types over any countable set is countable. As a corollary of the description of types we get

**Corollary 14.** *$\widetilde{\mathfrak{T}}$ is $\omega$-stable.*

*Proof.* This follows by counting the types, using the above claim and $\omega$-stability of $ACF$. ☐

We note that in the case that $M$ is finite-dimensional, this corollary already follows from the $\omega$-stability of $ACF$, since, after adding a basis, $M$ is interpretable in the field. However, the quantifier elimination result holds without adding any parameters.

## 3. The general case

Unlike the case $n = 0$, for $n > 0$, $\mathfrak{T}_I$ is far from being complete (unless $I$ is maximal), even if the field is algebraically closed. Nevertheless, quantifier elimination in the field (and $G$) variables follows automatically from the case $n = 0$. For the full quantifier elimination, we consider an extended theory $\widetilde{\mathfrak{T}}$, whose models are those models $\mathbf{N}$ of $\mathfrak{T}$ satisfying the following property: let $\mathbb{A} = \mathbb{A}_N$ be the algebra $K(\mathbf{N})[T_1, \dots, T_n]/\langle I\rangle$. Then any (finite) set of conditions:

$$f_i v = v_i \tag{6}$$

$$g_j v \notin U_j \tag{7}$$

where $f_i$,$g_i$ are in $\mathbb{A}$, $v_i \in M(\mathbf{N})$ are module elements, and $U_i$ are finite dimensional subspaces of $M(\mathbf{N})$, has a solution $v$, provided that they satisfy the following conditions:

- If

$$\sum t_i f_i = 0 \tag{8}$$

  then

$$\sum t_i v_i = 0 \tag{9}$$

  for any $t_i \in \mathbb{A}$.
- No $g_i$ is in the ideal generated by the $f_i$.

Note, that these conditions are necessary for a solution to exist.

*Remark* 15. We no longer consider extensions of $\mathfrak{T}$ that bound the dimension of $M$ (or other definable subspaces), as we did in the $n = 0$ case. However, as before, a finite dimensional module is still a model of $\mathfrak{T}$, and therefore will embed in a model of $\widetilde{\mathfrak{T}}$.

Since, as they are written, these conditions involve quantifying over all elements of $\mathbb{A}$, it is not clear that this is a first order condition. Thus we need to show that such a theory $\widetilde{\mathfrak{T}}$ indeed exists, that it eliminates quantifiers, and that any model of $\mathfrak{T}_I$ can be embedded in a model of this kind.

The fact that the above condition is actually first order, follows from the following theorem of [11] (by the degree of a polynomial we mean the *total* degree):

**Fact 16.** *Let $A$ be the polynomial algebra in $n$ variables over an arbitrary field, $d$ a fixed degree. There is a degree $e$ depending only on $n$ and $d$ (and not on the base field), such that for any $p_1, \dots, p_m \in A$ of degree at most $d$:*

1. *For any $f \in A$ of degree at most $d$, if $f$ is in the ideal generated by the $p_i$, then $f = \sum h_i p_i$ for $h_i$ of degree at most $e$.*
2. *The module of tuples $(s_1, \dots, s_m)$ such that $\sum s_i p_i = 0$ is generated by tuples of elements of degree at most $e$.*

*More generally, the same results hold when $A$ is replaced by $A^k$. Here, the degree of $(t_1, \dots, t_k) \in A^k$ is the maximum of the degrees, and $e$ depends also on $k$.*

*Remark* 17.

1. For the polynomial algebra, the set of polynomials of a given degree forms, in a natural way, a definable set. For the more general algebra $\mathbb{A}$ we may define the degree of an element $r$ to be the minimal degree of a pre-image of $r$ in $K[X_1, \dots, X_n]$. A priori, it is not clear that the set of elements of a given degree in $\mathbb{A}$ is again definable, since an element is represented in more than one way as a polynomial. However, since (according to fact 16) membership in $I$ is a first order property (of the coefficients), we have formulas whose free variables represent an element of $\mathbb{A}$ of a given degree. Alternatively, for the purpose of describing members of $\mathbb{A}$ we may assume that $\mathbb{A}$ is actually the polynomial algebra, since $I$ only appears as a condition on the modules.
2. Elements of $K^m$ will usually be considered as coefficients of polynomials in the $T_i$. This means that we fix an order on the monomials in the $T_i$, and for $\overline{x} \in K^m$, $x_i$ is the coefficient of the $i$-th monomial. Multiplication of polynomials induces an operation $* : K^m \times K^l \to K^N$.

   The same is true for elements of the $G^i$: if two such elements $p, q$ corresponds to vector spaces $V_p$ and $V_q$ of polynomials in the $T_i$, $p*q$ corresponds to the image of $V_p \otimes V_q$ in the polynomial algebra.

   Sometimes, instead of thinking of a tuple as a polynomial, we think of it as a tuple of polynomials (it will be clear from the context.) In that case, multiplication (by a polynomial or one vector space) is done term-wise.
3. Here is an instance of the above notation: Let $J$ be an ideal in $\mathbb{A}$. Then $J$ is finitely generated; let $p$ be the vector space generated by a finite set of generators. It follows from fact 16, that given a degree $d$ there is a degree $e$ such that, setting $q = K^e$, the set of elements of degree $d$ in $J$ is precisely the set of elements of degree $d$ in $q * p$.
4. Some more notation: $D_m(v, \overline{v})$ will denote $D(T^{\overline{i}}v, \dots, v, \overline{v})$, where the $\dots$ stands for all monomials of total degree at most $m$ (with the prescribed order).
5. Recall from the case $n = 0$ that over a good substructure, the type of $D(v, \overline{v})$ determines the type of $D(v, \overline{u})$ whenever both are not 0 and $D(\overline{v}) = D(\overline{u}) = 0$. The passage to a good substructure is done precisely as in the previous case.

The fact that the condition on the $v_i$ is first order now follows from the second item of fact 16, since it is enough to state the conditions on the $f_i$ for generators of the tuples $(t_i)$. The fact that the condition on the $g_i$ is first order follows from the first item of fact 16.

Using the last point of remark 17, we may obtain a description of the types:

**Claim 18** (Description of types, general case)**.** *For any quantifier free* 1*-type* $\mathfrak{p}(v)$ *over a good substructure, either there are* $m$ *and* $\overline{v}$ *such that* $\mathfrak{p}$ *is determined by the formulas in it of the form* $\varphi(D_m(v,\overline{v}))$ *(where* $\varphi$ *does not involve any module stuff), or* $\mathfrak{p}$ *is the unique quantifier free type determined by the set of formulas* $D_m(v,\overline{v})=0$ *for all* $m$ *and* $\overline{v}$*.*

*Proof.* Let $\mathbf{N}$ be a model realising $\mathfrak{p}$, $v\in M(\mathbf{N})$ a realisation, and let $\mathbb{A}=\mathbb{A}_N$ be as before. Since we are working over a good substructure $\mathbf{N_0}$, we may view $K(\mathbf{N})\otimes_{K(\mathbf{N_0})}M(\mathbf{N_0})$ as a sub $\mathbb{A}$-module of $M(\mathbf{N})$. Let $J$ be the ideal in $\mathbb{A}$ of elements $f$ such that $fv\in K(\mathbf{N})\otimes_{K(\mathbf{N_0})}M(\mathbf{N_0})$. If this ideal is 0, we are in the second case. Otherwise, let $f_1,\dots,f_n$ generate $J$, and let $\overline{v}_i\in M(\mathbf{N_0})$, for $i$ between 1 and $n$, be bases for the minimal $K(\mathbf{N_0})$ subspace containing $f_iv$. We set $m$ to be the maximum of the degrees of the $f_i$ and $\overline{v}=(\overline{v}_1,\dots,\overline{v}_n)$. A different choice of $\mathbf{N}$ and $v$ will result in choosing $f_i$ of the same form, with coefficients satisfying the same type over $K(\mathbf{N_0})$. Thus $m$ and $\overline{v}$ do not depend on the choice of $\mathbf{N}$ and $v$.

Let $\mathbf{V_i}$ be the vector space spanned by $\overline{v_i}$. Using the basis $\overline{v_i}$ we identify $V_i$ with a power of the field. Let $g\in\mathbb{A}$ be such that $\mathfrak{p}$ says that $D(gv,u_1,\dots,u_k)\neq 0$ for some module elements $u_i$. Then $g=\sum h_if_i$ for some $h_i\in\mathbb{A}$. Since the base is a structure, applying the operators $T_i$ to elements of $\mathbf{V_i}$ is well defined. If $\psi(v)=\varphi(D(gv,\overline{u}))$ is a formulas in $\mathfrak{p}$, consider the definable set

$$\{(w_1,\dots,w_n)\in\mathbf{V_1}\oplus\cdots\oplus\mathbf{V_n}|\varphi(D(\sum h_iw_i,u_1,\dots,u_k))\}$$

(Using the identification of $\mathbf{V_i}$ mentioned above, this is a subset of the $G$ sorts.) Since we are over a good substructure, the type of this space is determined by the base. Also, $v$ satisfies $\psi$ if and only if $(f_1v,\dots,f_nv)$ belongs to this set. But this is determined by the type of $D_m(v,\overline{v})$ (over the base $\mathbf{N_0}$) □

As in the case $n=0$, the result we seek easily follows from this:

**Theorem 19.** *Let* $\widetilde{\mathfrak{T}}$ *be the theory extending* $\mathfrak{T}$*, and stating, in addition, that the field is algebraically closed, and that any system*

$$\bigwedge_i f_ix=v_i\wedge\bigwedge_i D(g_ix,\overline{u})=0$$

*has a solution* $x$*, where* $f_i,g_i\in\mathbb{A}$*,* $v_i,\overline{u}\in M$*, provided that each* $g_i$ *is not in the ideal generated by the* $f_i$*, and that if* $\sum t_if_i=0$ *for some* $t_i\in\mathbb{A}$*, then* $\sum t_iv_i=0$ *as well.*

*Then* $\widetilde{\mathfrak{T}}$ *eliminates quantifiers.*

*Proof.* Using criterion 8, and the above claim, we need to show that given a good substructure $\mathbf{M_0}$, and a quantifier free type $\mathfrak{p}$ over $K(\mathbf{M_0})$ and $G(\mathbf{M_0})$, we may satisfy $\mathfrak{p}(D_m(v,\overline{v}))$ in any theory extending $\mathfrak{T}_{\mathbf{M_0}}$.

Since $\mathfrak{p}$ is a type in the $G$ sorts over $K(\mathbf{M_0})$, it follows from section 2 that $\mathfrak{p}$ is consistent. Let $p$ satisfy $\mathfrak{p}$. Again, by the case $n=0$, we may assume that $p$ is in $\mathbf{M_0}$, and we may extend the field so that $p$ corresponds to some subspace of $K^l$. This means that satisfying $\mathfrak{p}(D_m(v,\overline{v}))$ amounts to satisfying conditions of the form

$$fv = \sum x_i v_i$$
$$gv \notin \langle v_j \rangle$$

Where $f$ is an element of $\mathbb{A}$ and $x_i \in K$. Since $\mathfrak{p}$ was consistent to start with, the conditions appearing in the axioms are satisfied for any set of conditions like this that appears in $\mathfrak{p}$. Hence, the axioms imply that these equations have a solution. □

**Corollary 20.** *$\widetilde{\mathfrak{T}}$ is $\omega$-stable*

*Proof.* from the theorem, by counting the types. □

*Remark* 21. Following through the proofs, one sees that they work just as well with the commutativity assumption on the generators replaced by some other axioms, provided that the resulting algebra is (left) Noetherian, and the class of modules satisfying the solvability conditions is first order. In particular, using the more general version of fact 16, we see that the same result holds for algebras finite over their centre (where the field is contained in the centre.)

The last thing is to prove that any module over $\mathbb{A}$ (considered as a model of $\mathfrak{T}$) can be embedded in a model of $\widetilde{\mathfrak{T}}$. First note that the axioms can be split into two parts:

(1) There is a solution for any finite set of equations $f_i v = u_i$, provided that if $\sum t_i f_i = 0$ then $\sum t_i u_i = 0$.
(2) There is a solution for any finite set of formulas $f_j v = 0$, $g_i v \notin U_i$ (where $U_i$ is a finite dimensional vectors space), provided that no $g_i$ is in the ideal generated by the $f_j$.

This is true since a solution of a general set of equations of the type considered is the sum of a solution of the corresponding equations of the first kind, and of the second kind.

We claim:

**Claim 22.** *Let $\mathbb{A}$ be a Noetherian ring. An $\mathbb{A}$ module $M$ satisfies condition (1) above (for $f_i, t_i \in \mathbb{A}$ and $v, u_i \in M$) if and only if $M$ is injective.*

*Proof.* Let $M$ be an injective $\mathbb{A}$-module, let $U \subseteq M$ be the sub-module generated by the $u_i$, and let $V = (U \oplus \mathbb{A})/ < f_i - u_i >$. The condition

$$\sum t_i f_i = 0 \implies \sum t_i u_i = 0$$

is equivalent to the map $U \to V$ being injective. Therefore, the inclusion map of $U$ in $M$ extends to $V$, and the image of $1 \in V$ in $M$ is a solution.

Conversely, by a result of Baer (see [2, Lemma A3.4]), it is enough to check the condition of injectivity for the inclusion of an ideal $I$ in $\mathbb{A}$. Since $\mathbb{A}$ is Noetherian, $I$ is finitely generated, say by $f_i$. Let $u_i$ be the images of $f_i$ in $M$. Then $f_i$, $u_i$ satisfy the assumption of (1), so there is some $v \in M$ such that $f_i v = u_i$ for all $i$. Now, the map from $\mathbb{A}$ to $M$ that takes 1 to $v$ extends the given map. □

Regarding the other condition, consider the module $M = \prod M_I$, where $M_I = (\mathbb{A}/I)^\omega$, for $I$ an arbitrary ideal of $\mathbb{A}$ (so the product is over all ideals.) We claim that any module containing $M$ satisfies the second condition. To see this, we first note that it is enough to show that $M$ itself satisfies the condition. Indeed, given arbitrary finite dimensional vector spaces $U_i$ in a module containing $M$, any solution in $M$ to the problem, with $U_i$ replaced by $U_i \cap M$ will solve the original problem.

For $M$ itself, let $I$ be the ideal generated by the $f_j$. For the same reason as before, it is enough to find a solution in $M_I$ (note that the condition is non-trivial only if $I$ is a proper ideal.)

Now, any element of $M_I$ is a solution to the equations. Thus we only need to satisfy the inequalities. Since the $g_i$ are not in $I$, they are non-zero in each $\mathbb{A}/I$. Hence, almost all of the unit vectors in $M_I$ satisfy the inequalities. This solves the problem.

We now can prove:

**Claim 23.** *Any module over $\mathbb{A}$ embeds into a model of $\widetilde{\mathfrak{T}}$.*

*Proof.* Let $N$ be any module. Then $N \oplus M$ can be embedded into some injective module $I$ (where $M$ is the module constructed above.) Then $I$ contains $N$, satisfies the first condition since it is injective, and satisfies the second condition since it contains $M$. ☐

Finally, combining theorem 19 and claim 23, we get:

**Corollary 24.** *The theory $\widetilde{\mathfrak{T}}$ is the model completion of the theory $\mathfrak{T} = \mathfrak{T}_I$.*

In algebraic geometry, one is mostly interested in *finitely generated* modules (see [3, part 2].) The following example shows the theory of a finitely generated module is far from having quantifier elimination. Therefore, such definability results can not be derived directly by considering the theory of the module, but should probably be obtained by interpreting the module in our theory $\widetilde{\mathfrak{T}}$.

**Example 25.** Let $\mathbb{A} = K[T]$, the polynomial algebra in one variable over a field $K$ of characteristic 0, and consider $M = \mathbb{A}$ as a module over itself. We will show that the semi-ring of natural numbers can be interpreted in this theory (a very similar argument is used in [9] to show that the polynomial algebras themselves, considered as rings, are undecidable.)

For elements $v$ of $M$ and $x$ of $K$, denote by $v(x) = 0$ the formula $\exists u((T-x)u = v)$. Now consider the formula:

$$\begin{aligned}
&\phi(v,y) = \\
&\qquad v \neq 0 \wedge v(0) = 0 \wedge \\
&\qquad \forall x(v(x) = 0 \implies (x = y \vee v(x+1) = 0))
\end{aligned}$$

We claim that for any $y$, the fibre $\phi(v,y)$ is non-empty (in $M$) if and only if $y$ is a natural number. Indeed, assume that $y$ is not in $\mathbb{Z}$, and that $v$ satisfies $\phi(v,y)$. Then $v$ is a non-zero element that is divisible by $T+n$ for any natural $n$. There is no such element in $K[T]$ (here we use that the characteristic is 0.) Conversely, when $y$ is natural, the element $T(T-1)\dots(T-y)$ satisfies the formula.

The conclusion is that $\exists v \in M(\phi(v,y))$ defines the set of natural numbers. The ring operations are automatically defined, since this formula actually defines the copy of the natural numbers contained in $K$.

This example holds more generally: If $\mathbb{A}$ is any finitely generated algebra over a field $K$ of characteristic 0, and $M$ is any finitely generated module over $\mathbb{A}$ of infinite dimension over $K$, there is dominant map of *spec*$(\mathbb{A})$ to the affine line, that makes

$M$ into a $K[T]$ module, which, after a localisation becomes free (and in particular, torsion free.) The fact that $M$ is infinite dimensional means that the support of $M$ has dimension at least 1, so that this map can be chosen so that the resulting free module is non-zero. Now we may repeat the above example to interpret (all but finitely many of) the natural numbers.

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