

A CATEGORICAL APPROACH TO INTERNALITY

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ABSTRACT. Model theoretic internality provides conditions under which the group of automorphisms of a model over a reduct is itself a definable group. In this paper we formulate a categorical analogue of the condition of internality, and prove an analogous result on the categorical level. The model theoretic statement is recovered by considering the category of definable sets.

1. INTRODUCTION

In model theory, one works with an abstract notion of a structure, or a model. Such a structure can be a set \mathcal{M} (or a collection of sets), together with a distinguished collection of subsets of \mathcal{M} and its powers, the definable subsets. An example could be an algebraically closed field, with the definable sets being the field operations, solutions to equations derived from them (polynomial equations).

There is a natural notion of maps of such structures, and so one may form the group $\text{Aut}(\mathcal{M})$ of automorphisms (invertible such self-maps) of \mathcal{M} . This group and certain subgroups of it are an important tool in the study of definable sets and other model-theoretic objects. In general, the elements of this group have nothing to do with the elements of the structure \mathcal{M} itself. However, under special circumstances, certain such automorphism groups are *themselves definable*: there is a canonical correspondence between the elements of the group and a definable subset in \mathcal{M} , and the (graph of the) group operation is also definable. This condition is known as internality. Internality also provides (under suitable assumptions) a definable Galois theory, that provides a correspondence between certain intermediate structures and definable subgroups.

Internality was first discovered in Zil'ber [15], where it was used to study strongly minimal sets. It can be used in an abstract setting to study the relation between definable sets that are “close” to each other. However, it was also discovered in Poizat [13] that, when applied to a particular theory (differentially closed fields), the definable Galois group and the associated Galois theory specialise to a classical construction, the Galois group (and Galois theory) of linear differential equations. It has since been applied to provide and study Galois theory in various situations (e.g. Pillay [12]). A more detailed review of internality, along with some references appears in Section 3.

Whereas the original definition of internality appeared under rather specific model theoretic assumptions (strongly minimal structures), it was subsequently generalised and simplified. In Hrushovski [3] the construction appeared without any restrictions on the theories in question.

It is shown in Kamensky [7], that a certain categorical construction, the algebraic group associated with a Tannakian category, can also be viewed as an instance

of this formalism. Since the theory of internality no longer depends on “stability theoretic” assumption, it was natural to ask whether it is possible to go in the other direction, and recover the theory of internality (or some variant) from a categorical statement.

The present paper is an attempt to answer this question. In the spirit of Makkai and Reyes [10], we replace a theory \mathcal{T} by its category of definable sets, and interpretations and models by suitable functors. We do not use the precise characterisation provided by Makkai and Reyes [10], since our result holds in greater generality, but the motivation comes from there. On the other hand, though the category of definable sets is not a topos, when the theory eliminates imaginaries it is rather close being that, and we use some ideas from topos theory, and more generally from category theory, in the style of Mac Lane and Moerdijk [9], that remain relevant in this case.

The main results of this paper appear in sections 4 and 5. In Section 4, we make some observations on the category of definable sets of a theory (some of these appear in Makkai and Reyes [10]), and formulate categorical versions of stably embedded definable sets, and internality. We then prove a weak version of the existence of definable Galois groups in the categorical context (Corollary 4.13). In Section 5 we introduce additional assumptions on the data, that allow us to prove the full analogy to the model theoretic case. In Section 6 we provide a sketch of two possible applications of these theorems.

The main point of this paper is to explicate some analogies between model theoretic notions and constructions with categorical counterparts. It is therefore my hope that this paper is readable by anyone with minimal knowledge in these areas. To this end, the paper includes two sections which are almost completely expository. Section 2 reviews some basic notions from category theory, and discusses the categorical assumptions we would like to make. The only new result there is Proposition 2.7 (which is a direct application of ends, and is possibly known) and its corollary. Section 3 contains a brief, and rather informal review of basic model theory, as well as internality and the associated Galois group. Hopefully, there are no new results there.

2. CATEGORICAL BACKGROUND

We review some notions from category theory that will be used in the description of internality. The classical reference for these is Mac Lane [8], though Wikipedia [14] is also sufficient. All categories in this paper are (essentially) small.

A contra-variant functor from a category \mathcal{C} to the category of sets will be called a *presheaf* (on \mathcal{C}). The functor that assigns to an object \mathbf{X} the presheaf $\mathbf{Y} \mapsto \text{Hom}(\mathbf{Z}, \mathbf{X})$ it represents is called the *Yoneda embedding*, and the Yoneda lemma states that this embedding is fully faithful (induces a bijection on sets of morphisms). Hence we also denote by \mathbf{X} the presheaf it represents, and view \mathcal{C} as a subcategory of its presheaf category whenever this is convenient. More generally, the Yoneda lemma states that $\mathcal{F}(\mathbf{X}) = \text{Hom}(\mathbf{X}, \mathcal{F})$ for any presheaf \mathcal{F} .

All categories we consider, with the exception of index categories, will have all finite (inverse) limits. If \mathcal{C} is such a category, we denote by $\mathbf{1}$ the terminal object (empty product) of \mathcal{C} , and by Γ (global sections) the functor $\mathbf{X} \mapsto \text{Hom}(\mathbf{1}, \mathbf{X})$.

2.1. Pro- and Ind- objects. Our categories will not, usually, be closed under infinite limits or colimits. However, certain such limits appear naturally in what follows.

We recall that a *filtering category* is a category \mathcal{I} in which any two objects map into a common object, and for any two maps $f, g : \mathbf{A} \rightarrow \mathbf{B}$ there is some map $h : \mathbf{B} \rightarrow \mathbf{C}$ with $h \circ f = h \circ g$. A *filtering system* in a category \mathcal{C} is a functor from a filtering category to \mathcal{C} . The category $\text{Ind}(\mathcal{C})$ of *ind-objects* of \mathcal{C} is a category that contains \mathcal{C} as a full sub-category, admits the colimits of all filtering systems, and satisfies

$$\text{Hom}(\mathbf{X}, \text{Ind}_{\mathcal{I}}(\mathbf{Y}_i)) = \varprojlim_{\mathcal{I}} (\text{Hom}(\mathbf{X}, \mathbf{Y}_i)) \quad (1)$$

where \mathbf{X} and \mathbf{Y}_i are objects of \mathcal{C} , and $\text{Ind}_{\mathcal{I}}(\mathbf{Y}_i)$ is the colimit of the system in $\text{Ind}(\mathcal{C})$ (note that if \mathcal{C} itself happens to admit the colimit $\varinjlim_{\mathcal{I}} \mathbf{Y}_i$ of this system, this object will not be isomorphic to $\text{Ind}_{\mathcal{I}}(\mathbf{Y}_i)$ in $\text{Ind}(\mathcal{C})$). A construction of this category appears in Grothendieck et al. [2].

The category $\text{Pro}(\mathcal{C})$ of pro-objects is the dual notion: it is the opposite of the category of ind-objects of the opposite of \mathcal{C} . Thus it consists of inverse limits $\text{Pro}(\mathbf{X}_i)$ of co-filtering systems in \mathcal{C} , with \mathcal{C} embedded as a full sub-category, and satisfying

$$\text{Hom}(\text{Pro}_{\mathcal{I}}(\mathbf{X}_i), \mathbf{Y}) = \varinjlim_{\mathcal{I}} (\text{Hom}(\mathbf{X}_i, \mathbf{Y})) \quad (2)$$

with similar conventions to the above.

Because of the above properties, the objects of \mathcal{C} will be called *compact* when considered as objects of $\text{Ind}(\mathcal{C})$ or $\text{Pro}(\mathcal{C})$. A functor (or a presheaf) will be called pro-representable (ind-representable) if it is represented by a pro- (ind-) object. In other words, it is a filtered colimit of representable functors (presheaves).

By a *pro-group* (resp., a pro-monoid, etc.) in \mathcal{C} we mean a pro-object of \mathcal{C} that can be represented by an inverse filtering system of group (resp. monoid) objects of \mathcal{C} . Such a pro-group is a group object in $\text{Pro}(\mathcal{C})$, but the converse is not true, in general: there may be group objects in $\text{Pro}(\mathcal{C})$ which are not of this form.

2.2. Closed and Ind-closed categories. Let \mathcal{C} be a category with finite inverse limits. Given two objects \mathbf{X} and \mathbf{Y} of \mathcal{C} , we denote by $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ the presheaf $\mathbf{Z} \mapsto \text{Hom}(\mathbf{Z} \times \mathbf{X}, \mathbf{Y})$. There is, therefore, a natural evaluation map $\text{ev} : \underline{\text{Hom}}(\mathbf{X}, \mathbf{Y}) \times \mathbf{X} \rightarrow \mathbf{Y}$, where \mathbf{X} and \mathbf{Y} are identified with the presheaf they represent. Applying the definition with $\mathbf{Z} = \mathbf{1}$ (and using Yoneda's lemma), we see that $\Gamma(\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})) = \text{Hom}(\mathbf{X}, \mathbf{Y})$.

The category \mathcal{C} is called (Cartesian) *closed* (Mac Lane [8, § IV.6]) if all such $\underline{\text{Hom}}$ presheaves are representable (in which case the representing object is denoted by the same symbol, and is called the *internal Hom*). In other words, the functor $\mathbf{Z} \mapsto \mathbf{Z} \times \mathbf{X}$ has a right adjoint $\mathbf{Y} \mapsto \underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$. By the Yoneda lemma, the evaluation map is also in \mathcal{C} in this case.

Definition 2.3. We will say that a category \mathcal{C} with finite inverse limits is *ind-closed* if $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ is ind-representable for all objects \mathbf{X} and \mathbf{Y} .

Note that $\text{Ind}(\mathcal{C})$ again has all finite inverse limits, but the condition does not imply that $\text{Ind}(\mathcal{C})$ is closed. Instead, we know that $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ exists (and is constructed in the obvious way) whenever \mathbf{X} is compact. Again, the evaluation map is in $\text{Ind}(\mathcal{C})$.

It may sometimes be difficult to determine whether a category is closed. However, for ind-closed, we have the following formal criterion.

Proposition 2.4. *Assume that \mathcal{C} has all finite limits and colimits. Then it is ind-closed if and only if, for any object \mathbf{X} of \mathcal{C} , the functor $\mathbf{Z} \mapsto \mathbf{Z} \times \mathbf{X}$ commutes with finite colimits.*

Proof. The ‘‘Grothendieck construction’’ presents every presheaf as a colimit of representable presheaves (Mac Lane and Moerdijk [9, § VII]). The system produced by the construction is filtering if the presheaf is left-exact (takes finite colimits to the corresponding inverse limits). Hence in a category that has finite colimits, the ind-representable presheaves are precisely the left-exact ones (it is clear that any ind-representable presheaf is left-exact).

Since any representable presheaf is left-exact, $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ is left-exact if $- \times \mathbf{X}$ preserves finite colimits.

Conversely, if \mathcal{C} is ind-closed, then $- \times \mathbf{X}$ commutes with colimits in $\text{Ind}(\mathcal{C})$, since it has a right adjoint there. The inclusion of \mathcal{C} in $\text{Ind}(\mathcal{C})$ preserves finite colimits, so the result follows. \square

In our case, the requirement that finite colimits exist is too strong. It will be replaced by a more direct condition, elimination of imaginaries, ensuring that the system produced by the Grothendieck construction is filtering (see Proposition 4.2).

2.5. Closed functors. If $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is any functor, it extends by composition to a functor $\mathcal{F} : \text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{D})$. If \mathcal{C} is ind-closed, and \mathcal{F} preserves finite inverse limits, we may apply \mathcal{F} to the evaluation map, to obtain a map $\mathcal{F}(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})) \times \mathcal{F}(\mathbf{X}) \rightarrow \mathcal{F}(\mathbf{Y})$. This induces a map $\mathcal{F}(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})) \rightarrow \underline{\mathbf{Hom}}(\mathcal{F}(\mathbf{X}), \mathcal{F}(\mathbf{Y}))$, and we say that such a functor \mathcal{F} is *closed* if this map is an isomorphism.

2.6. Endomorphisms of left exact functors. We recall that a map (natural transformation) θ from a functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ to another functor $\mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ consists of a collection morphisms $\theta_{\mathbf{X}} : \mathcal{F}(\mathbf{X}) \rightarrow \mathcal{G}(\mathbf{X})$, one for each object \mathbf{X} of \mathcal{C} , commuting with all maps that come from \mathcal{C} .

Let \mathcal{C} and \mathcal{D} be two categories with finite inverse limits, and let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be a left exact functor. For an object \mathbf{T} of \mathcal{D} , we denote by $\text{End}_{\mathbf{T}}(\mathcal{F})$ the set of natural transformations from $\mathbf{T} \times \mathcal{F}$ to \mathcal{F} (where $(\mathbf{T} \times \mathcal{F})(\mathbf{X}) = \mathbf{T} \times \mathcal{F}(\mathbf{X})$). Each such natural transformation induces a transformation of $\mathbf{T} \times \mathcal{F}$ to itself, and therefore $\mathbf{T} \mapsto \text{End}_{\mathbf{T}}(\mathcal{F})$ is a presheaf of monoids on \mathcal{D} . We denote by $\text{Aut}_{\mathbf{T}}(\mathcal{F})$ the subset of $\text{End}_{\mathbf{T}}(\mathcal{F})$ consisting of transformations where the induced endomorphism above is invertible. Thus it is a presheaf of groups on \mathcal{D} .

More generally, if $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$ is another functor, we denote by $\text{End}_{\mathbf{T}}(\mathcal{F}/\mathcal{J})$ and $\text{Aut}_{\mathbf{T}}(\mathcal{F}/\mathcal{J})$ the subsets of $\text{End}_{\mathbf{T}}(\mathcal{F})$ and $\text{Aut}_{\mathbf{T}}(\mathcal{F})$ consisting of those endomorphisms whose ‘‘restriction’’ to the composition $\mathcal{F} \circ \mathcal{J}$ is the identity. These are again presheaves on \mathcal{D} . The main observation in this section is that if \mathcal{D} is closed, these presheaves are represented by a pro-monoid and a pro-group, respectively, of \mathcal{D} .

Proposition 2.7. *Let $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$ and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ be functors. Assume that \mathcal{D} admits finite inverse limits, and that $\underline{\mathbf{Hom}}(\mathcal{F}(\mathbf{X}), \mathcal{F}(\mathbf{Y}))$ exists for all objects \mathbf{X} and \mathbf{Y} of \mathcal{C} . Then the functors $\mathbf{T} \mapsto \text{End}_{\mathbf{T}}(\mathcal{F}/\mathcal{J})$ and $\mathbf{T} \mapsto \text{Aut}_{\mathbf{T}}(\mathcal{F}/\mathcal{J})$ are pro-representable, by a pro-monoid \mathbf{M} and pro-group \mathbf{G} , respectively.*

If $\mathcal{H} : \mathcal{D} \rightarrow \mathcal{E}$ is any closed functor (at least when restricted to the essential image of \mathcal{F}), then $\mathcal{H}(\mathbf{M})$ and $\mathcal{H}(\mathbf{G})$ represent $\mathbf{T} \mapsto \text{End}_{\mathbf{T}}(\mathcal{H} \circ \mathcal{F} / \mathcal{J})$ and $\mathbf{T} \mapsto \text{Aut}_{\mathbf{T}}(\mathcal{H} \circ \mathcal{F} / \mathcal{J})$, respectively.

The proof is a direct consequence of the end construction, as explained below.

Corollary 2.8. *Let $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$ be a functor into an ind-closed category \mathcal{C} . Then there are pro-monoid \mathbf{M} and pro-group \mathbf{G} in $\text{Ind}(\mathcal{C})$, such that for any closed functor $\mathcal{H} : \mathcal{C} \rightarrow \mathcal{E}$, $\text{End}(\mathcal{H} / \mathcal{J}) = \Gamma(\mathcal{H}(\mathbf{M}))$ and $\text{Aut}(\mathcal{H} / \mathcal{J}) = \Gamma(\mathcal{H}(\mathbf{G}))$.*

Proof. The embedding \mathcal{F} of \mathcal{C} into $\mathcal{D} = \text{Ind}(\mathcal{C})$ satisfies the condition of the proposition. \square

In the model theoretic application, \mathcal{H} will be extension of constants (base-changes), so $\Gamma \circ \mathcal{H}$ is “taking \mathcal{N} -points” for some model \mathcal{N} (or simply a model, in the language of Makkai and Reyes [10]; see §4.6).

2.9. Ends. The following construction is presented in Mac Lane [8, § IX.5]. Let $\mathcal{S} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ be a bi-functor (so for any object \mathbf{X} of \mathcal{C} , $\mathcal{S}(\mathbf{X}, -)$ and $\mathcal{S}(-, \mathbf{X})$ are a functor and a pre-sheaf, resp., with values in \mathcal{D}). The *end* of \mathcal{S} is an object e of \mathcal{D} , together with a morphism $p_{\mathbf{X}} : e \rightarrow \mathcal{S}(\mathbf{X}, \mathbf{X})$ for each object \mathbf{X} of \mathcal{C} , such that for any morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$ of \mathcal{C} , the diagram

$$\begin{array}{ccc}
 e & \xrightarrow{p_{\mathbf{X}}} & \mathcal{S}(\mathbf{X}, \mathbf{X}) \\
 p_{\mathbf{Y}} \downarrow & & \downarrow \mathcal{S}(\mathbf{X}, f) \\
 \mathcal{S}(\mathbf{Y}, \mathbf{Y}) & \xrightarrow{\mathcal{S}(f, \mathbf{Y})} & \mathcal{S}(\mathbf{X}, \mathbf{Y})
 \end{array} \tag{3}$$

commutes, and which is universal with these properties.

Example 2.10. Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ be two functors, and let $\mathcal{S} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$ be the functor $(\mathbf{X}, \mathbf{Y}) \mapsto \text{Hom}(\mathcal{F}(\mathbf{X}), \mathcal{G}(\mathbf{Y}))$. The set $\text{Hom}(\mathcal{F}, \mathcal{G})$ of natural transformations between \mathcal{F} and \mathcal{G} maps to each $\mathcal{S}(\mathbf{X}, \mathbf{X})$ by restriction, and the commuting diagrams as above amount to the statement that a collection of such restrictions amalgamates to a natural transformation. Hence $\text{Hom}(\mathcal{F}, \mathcal{G})$ is the end of \mathcal{S} .

The end of a functor is in a natural way an inverse limit of diagrams as above. Hence it exists if the category \mathcal{D} admits all inverse limits. The proof of Proposition 2.7 is a repetition of the above example internally:

Proof of Proposition 2.7. Assume first that \mathcal{C}_0 is the empty category. By assumption, we have a functor $\mathcal{S} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$, $\mathcal{S}(\mathbf{X}, \mathbf{Y}) = \underline{\text{Hom}}(\mathcal{F}(\mathbf{X}), \mathcal{F}(\mathbf{Y}))$. Since the category $\text{Pro}(\mathcal{D})$ admits all filtered inverse limits, and \mathcal{D} (and therefore $\text{Pro}(\mathcal{D})$) admits all finite inverse limits, $\text{Pro}(\mathcal{D})$ admits all inverse limits (Mac Lane [8, § IX.1]). Therefore, the end \mathbf{M} of \mathcal{S} exists. The fact that \mathbf{M} represents $\mathbf{T} \mapsto \text{End}_{\mathbf{T}}(\mathcal{F})$ follows directly from the definitions.

The fact that \mathbf{M} can be given a system of monoid objects follows from the fact that $\text{End}_{\mathbf{T}}(\mathcal{F})$ is the filtered inverse limit of $\text{End}_{\mathbf{T}}(\mathcal{F}_{\alpha})$, where \mathcal{F}_{α} are restrictions of \mathcal{F} to finite sub-categories \mathcal{C}_{α} of \mathcal{C} .

When \mathcal{C}_0 is not empty, the pro-monoid in the theorem is the pullback of \mathbf{M} constructed above along the product of the maps $\mathbf{1} \rightarrow \underline{\text{Hom}}(\mathcal{F}(\mathcal{J}(\mathbf{X})), \mathcal{F}(\mathcal{J}(\mathbf{X})))$ corresponding to the identity $\mathcal{F}(\mathcal{J}(\mathbf{X})) \rightarrow \mathcal{F}(\mathcal{J}(\mathbf{X}))$. Hence it is again of the same type. Likewise, the object \mathbf{G} is a suitable pullback.

The last statement follows since in this case, by assumption, the functors $\mathcal{H} \circ \mathcal{S}$ and $(\mathbf{X}, \mathbf{Y}) \mapsto \underline{\mathbf{Hom}}(\mathcal{H}(\mathcal{F}(\mathbf{X})), \mathcal{H}(\mathcal{F}(\mathbf{Y})))$ are isomorphic, hence so are their ends. \square

3. MODEL THEORY AND INTERNALITY

In this section we summarise some basic notions from model theory. Most of them are classical, and can be found, for example, in Marker [11]. The exception is the notion of internality, which is the central notion for us, and which is recalled here following Hrushovski [4]. The first three subsections are illustrated in Example 3.4.

3.1. Theories, models and definable sets. Logic is concerned with studying mathematical theories as mathematical objects. Thus, the data of a *theory* consists of a formal language, together with a collection of statements (axioms) written in this language. A *model* of the theory \mathcal{T} is a structure consisting of an interpretation of the symbols in the language of \mathcal{T} , in which all the axioms of \mathcal{T} hold. For example, the theory of fields, and the theory ACF of algebraically closed fields can both be written in a language constructed of the symbols $(0, 1, +, -, \cdot)$, and a model of ACF is a particular algebraically closed field (see Example 3.4).

A *formula* is written in the same formal language, but has *free variables*, into which elements of the model can be plugged. Any such formula $\phi(x_1, \dots, x_n)$ (where x_1, \dots, x_n contain all free variables of ϕ) thus determines a subset $\phi(\mathcal{M})$ of \mathcal{M}^n , for any model \mathcal{M} , namely, the set of all tuples \bar{a} for which $\phi(\bar{a})$ holds. Two formulas ϕ and ψ are *equivalent* if $\phi(\mathcal{M}) = \psi(\mathcal{M})$ for all models \mathcal{M} . An equivalence class under this relation is called a *definable set*. Thus, if \mathbf{X} is a definable set, there is a well defined set $\mathbf{X}(\mathcal{M})$ for any model \mathcal{M} (but the syntactic information is lost). If we have a fixed model \mathcal{M} in mind, the set $\mathbf{X}(\mathcal{M})$ is also called definable.

If \mathcal{M} is the model of some theory, the set of all statements (in the underlying language) that are true in \mathcal{M} is a theory $\mathcal{T}(\mathcal{M})$, and \mathcal{M} is a model of $\mathcal{T}(\mathcal{M})$.

We will assume all our theories to be multi-sorted, i.e., the variables of a formula can take values in any number of disjoint sets. In fact, if \mathbf{X} and \mathbf{Y} are sorts, we view $\mathbf{X} \times \mathbf{Y}$ as a new sort. In particular, any definable set is a subset of some sort (for example, in ACF the sorts correspond to the affine spaces \mathbb{A}^n). By a statement such as $\mathbf{a} \in \mathcal{M}$ we will mean that \mathbf{a} is an element of one of the sorts. Likewise, a subset of \mathcal{M} is collection of subsets of $\mathbf{X}(\mathcal{M})$, one for each sort \mathbf{X} .

3.2. Parameters and definable closure. Let \mathcal{M} be a model of a theory \mathcal{T} , \mathbf{X} and \mathbf{Y} definable sets. A function $f : \mathbf{X}(\mathcal{M}) \rightarrow \mathbf{Y}(\mathcal{M})$ is *definable* if its graph is definable. An element $\mathbf{b} \in \mathcal{M}$ is definable over another element \mathbf{a} if $f(\mathbf{a}) = \mathbf{b}$ for some definable function f . The *definable closure* $\text{dcl}(\mathcal{A})$ of a subset \mathcal{A} of \mathcal{M} is the set of elements definable over some tuple \mathbf{a} of elements of \mathcal{A} . The set \mathcal{A} is *definably closed* if $\mathcal{A} = \text{dcl}(\mathcal{A})$. We denote by $\mathbf{X}(\mathcal{A})$ the set $\mathbf{X}(\mathcal{M}) \cap \text{dcl}(\mathcal{A})$.

More generally, we may consider formulas $\phi(x, \mathbf{a})$ with parameters in \mathcal{A} , obtained by plugging a tuple \mathbf{a} from \mathcal{A} in a regular formula $\phi(x, \mathbf{y})$. These determine subsets in models containing \mathcal{A} , give rise to definable sets over \mathcal{A} . If $\mathcal{T} = \mathcal{T}(\mathcal{M})$ is the theory of \mathcal{M} , it is possible to expand \mathcal{T} by constant symbols and suitable axioms to obtain a theory $\mathcal{T}_{\mathcal{A}}$, whose models are the models of \mathcal{T} containing \mathcal{A} , and whose definable sets the \mathcal{A} -definable sets of \mathcal{T} .

An *elementary map* from a model \mathcal{M} to another model \mathcal{N} is a map α from \mathcal{M} to \mathcal{N} such that for each definable set \mathbf{X} , $\mathbf{X}(\mathcal{M}) = \alpha^{-1}(\mathbf{X}(\mathcal{N}))$. Such a map is necessarily an embedding, and when it is fixed, we identify elements of \mathcal{M} with their image in

\mathcal{N} . An *automorphism* of \mathcal{M} is an invertible elementary map. The automorphism α of \mathcal{M} is over $\mathcal{A} \subseteq \mathcal{M}$ if α fixes each element of \mathcal{A} .

An automorphism over \mathcal{A} will also fix pointwise any element of $\text{dcl}(\mathcal{A})$. Hence $\text{dcl}(\mathcal{A})$ is contained in the set \mathcal{M}^G of elements of \mathcal{M} fixed pointwise by all elements of the group $G = \text{Aut}(\mathcal{M}/\mathcal{A})$ of automorphisms of \mathcal{M} fixing \mathcal{A} pointwise. This inclusion might be strict, in general, but any model admits an elementary embedding in a *homogeneous* model, where this inclusion becomes an equality whenever the cardinality of \mathcal{A} is smaller than that of the model.

3.3. Imaginaries and interpretations. A *definable equivalence relation* on a definable set \mathbf{X} is a definable subset of $\mathbf{X} \times \mathbf{X}$ that determines an equivalence relation in any model. The theory *eliminates imaginaries* if any equivalence relation has a quotient, i.e., any equivalence relation $\mathbf{E}(x, y)$ can be represented as $f(x) = f(y)$ for some definable function f on \mathbf{X} .

A *definition* of a theory \mathcal{T}_1 in another theory \mathcal{T}_2 is specified by assigning, to each definable set \mathbf{X} of \mathcal{T}_1 a definable set \mathbf{X}_2 of \mathcal{T}_2 , such that for any model \mathcal{M}_2 of \mathcal{T}_2 , the sets $\mathbf{X}_2(\mathcal{M}_2)$ comprise a model \mathcal{M}_1 of \mathcal{T}_1 (when interpreted as a \mathcal{T}_1 structure in the obvious way). The expansion $\mathcal{T}_{\mathcal{A}}$ of \mathcal{T} mentioned above is an example of a definition of \mathcal{T} in $\mathcal{T}_{\mathcal{A}}$.

For any theory \mathcal{T} there is a theory \mathcal{T}^{eq} and a definition of \mathcal{T} in \mathcal{T}^{eq} , where \mathcal{T}^{eq} eliminates imaginaries, and the definition is universal with this property. Any model of \mathcal{T} can be expanded uniquely to a model of \mathcal{T}^{eq} . One can often assume that a theory admits elimination of imaginaries via this process. Through the rest of this section, we assume that our theories eliminate imaginaries.

Example 3.4. We illustrate some of the notions discussed above for the theory ACF of algebraically closed fields. All the facts in this example appear in Marker [11].

As mentioned above, this theory is given in the language determined by functions symbols $+$, $-$, \cdot , and constant symbols 0 and 1 . The properties of an algebraically closed field can be expressed by (infinitely many) statements in this language. For instance, commutativity of addition is written as $\forall x \forall y (x + y = y + x)$.

An example of a model is the field \mathbb{C} of complex numbers. The full theory $\mathcal{T}(\mathbb{C})$ of the complex numbers as a field turns out to be ACF together with the (infinitely many) axioms stating that the characteristic is 0 .

The sorts in this theory are all Cartesian powers of the field sort, hence can be identified with \mathbb{A}^n , the affine n -space. A formula consists of quantifiers (\forall and \exists) applied to boolean combinations of polynomial equations. For example $\exists y (y^2 = x)$. Such a formula determines a subset of some \mathbb{A}^n (where n is the number of free variables). Whereas the formulas may have an arbitrary number of quantifiers, the subset it defines can also be defined by a formula without quantifiers. In other words, the definable sets can all be represented by quantifier-free formulas, i.e., they are the constructible subsets.

An automorphism is a bijection of the field with itself that preserves the operations, 0 and 1 . Hence it is the same as a field automorphism. A set of parameters is a subset of some algebraically closed field. Since the field operations are definable, a definably closed set is a field, which is also perfect, since extracting p -th roots (when $p > 0$ is the characteristic) is again a definable function. By considering the automorphism group of the model, we find that perfect fields are precisely the definably closed sets: The definable closure of a set \mathcal{A} is contained in the fixed field

of the group of automorphisms of the algebraically closed field fixing \mathcal{A} . This fixed field is, by field theory, the perfect field generated by \mathcal{A} .

Let \mathbf{X} be the definable set of pairs (x, y) . The definable subset $\mathbf{E}((x, y), (z, w))$ of $\mathbf{X} \times \mathbf{X}$ given by $(x = z \wedge y = w) \vee (x = w \wedge y = z)$ is an equivalence relation, which is definable not only in ACF, but in any theory (since the formula mentions only equality). In the theory of equality (with no other symbols) this relation does not have a quotient. However, if \mathbf{X} is a field, the definable function $f(x, y) = (x + y, xy)$ is a quotient. In fact, ACF eliminates imaginaries.

3.5. Internality. Let \mathcal{T} be an expansion of \mathcal{T}_0 (so we have an interpretation of \mathcal{T}_0 in \mathcal{T}), and let \mathcal{M} be a model of \mathcal{T} , with restriction \mathcal{M}_0 to \mathcal{T}_0 . An automorphism of \mathcal{M} over \mathcal{M}_0 consists certain bijections from $\mathbf{X}(\mathcal{M})$ to itself, for each definable set \mathbf{X} of \mathcal{T} . These bijections will rarely come from *definable* bijections in \mathcal{T} (even with parameters). Internality, which is the main notion of interest for us, provides a situation where all such automorphisms are, in fact, definable (with parameters in \mathcal{M}).

A definable set \mathbf{X} in a theory \mathcal{T} is *stably embedded* if for any model \mathcal{M} of \mathcal{T} , any \mathcal{M} -definable subset of \mathbf{X}^n (for any n) is definable over $\mathbf{X}(\mathcal{M})$. An interpretation of a theory \mathcal{T}_0 in \mathcal{T} is a *stable embedding* if any definable set of \mathcal{T}_0 is stably embedded when viewed as a definable set in \mathcal{T} .

Given a stable embedding of \mathcal{T}_0 in \mathcal{T} , let \mathcal{M} be a model of \mathcal{T} of cardinality bigger than the language, and let \mathcal{M}_0 be the induced model of \mathcal{T}_0 . Then \mathcal{T} is an *internal cover* of \mathcal{T}_0 if for any such model there is a small subset \mathcal{A} of \mathcal{M} (smaller than the cardinality of \mathcal{M}), such that $\text{del}(\mathcal{M}_0 \cup \mathcal{A}) = \mathcal{M}$. A set \mathcal{A} of this form can (and will) always be taken to be definably closed, and will be called a set of *internality parameters*. We will assume for simplicity that such an \mathcal{A} can be found with $\mathcal{A}_0 = \mathcal{M}_0 \cap \mathcal{A}$ contained in the definable closure of 0 in \mathcal{T}_0 (the theory works also without this assumption, but then one has to deal with definable groupoids, rather than groups. Cf. Hrushovski [4]).

Example 3.6. Let \mathcal{T} be the theory generated by two sorts, \mathbf{L} and \mathbf{X} , and stating that \mathbf{L} is an algebraically closed field, and \mathbf{X} is a vector space over \mathbf{L} , of a fixed dimension n (in the natural language for such a structure). Then the theory \mathcal{T}_0 of algebraically closed fields is stably embedded in \mathcal{T} via \mathbf{L} , and \mathcal{T} is an internal cover of \mathcal{T}_0 , since in any model \mathcal{M} of \mathcal{T} , the definable closure of any basis of $\mathbf{X}(\mathcal{M})$ over $\mathbf{L}(\mathcal{M})$ is a set of internality parameters.

The internal covers are interesting because of the following theorem, which was first proved by Zil'ber (Zil'ber [15]) in the context of strongly minimal structures. The version here was proved in Hrushovski [3, appendix B], but was reformulated in (approximately) our language in Hrushovski [4].

Theorem 3.7. *Let \mathcal{T} be an internal cover of \mathcal{T}_0 . There is a pro-group \mathbf{G} in \mathcal{T} , together with a definable action $m_Q : \mathbf{G} \times \mathbf{Q} \rightarrow \mathbf{Q}$ of \mathbf{G} on every definable set \mathbf{Q} of \mathcal{T} , such that for any model \mathcal{M} of \mathcal{T} , $\mathbf{G}(\mathcal{M})$ is identified with $\text{Aut}(\mathcal{M}/\mathcal{M}_0)$ through this action (with \mathcal{M}_0 the restriction of \mathcal{M} to \mathcal{T}_0).*

Another part of this theorem, which will not be discussed in this paper, provides a Galois correspondence. As mentioned in the introduction, Poizat [13] developed this Galois correspondence to recover the Galois theory of a linear differential equation. Other applications appear in Pillay [12], Hrushovski [3] and Kamensky [5].

In Kamensky [7], this theorem is the main tool in a model theoretic proof of the Tannakian formalism.

4. CATEGORICAL INTERNALITY

Let \mathcal{T} be a first order theory. The collection of definable sets, with definable maps between them, forms a category, $\mathcal{D}_{\mathcal{T}}$. Our aim in this section is to reformulate some of the notions and results of Section 3, and especially the notion of internality, in terms of the category $\mathcal{D}_{\mathcal{T}}$. We in fact work with more general categories, and obtain a weaker result than Theorem 3.7. In Section 5, we introduce additional assumptions on the category, and deduce the full strength of the theorem.

4.1. The category of definable sets. The structure of the category $\mathcal{D}_{\mathcal{T}}$ was described in Makkai and Reyes [10]. We will not need the full characterisation, since our result holds under more general and simpler assumptions. We mention, however, that a category of the form $\mathcal{D}_{\mathcal{T}}$ admits all finite inverse limits, and finite co-products¹. A *logical functor* between such categories preserves these limits (and has additional properties). When \mathcal{T} eliminates imaginaries, $\mathcal{D}_{\mathcal{T}}$ is a special kind of a *pre-topos*, in the sense of Grothendieck et al. [2]. In fact, the construction of \mathcal{T}^{eq} appeared first in Makkai and Reyes [10], as a universal embedding of a logical category in a pre-topos.

The categories $\text{Pro}(\mathcal{D}_{\mathcal{T}})$ and $\text{Ind}(\mathcal{D}_{\mathcal{T}})$ were discussed in Kamensky [6]. Essentially the results are that one may evaluate \mathcal{M} points of a pro- or ind- definable set by evaluating them for each member in a system, and taking the corresponding limit. Definable maps between such sets are then precisely those that come from maps in the corresponding category, and if the model is sufficiently saturated, a definable map is an isomorphism if it is bijective on \mathcal{M} -points.

We denote by **1** and **2** the definable sets with 1 and 2 (named) elements. These are the final object in $\mathcal{D}_{\mathcal{T}}$ and its coproduct with itself, respectively.

Proposition 4.2. *The theory \mathcal{T} eliminates imaginaries if and only if $\mathcal{D}_{\mathcal{T}}$ is ind-closed (in the sense of Definition 2.3).*

Proof. Assume $\mathcal{D}_{\mathcal{T}}$ is ind-closed, and let $\mathbf{E} \subseteq \mathbf{X} \times \mathbf{X}$ be an equivalence relation. The corresponding characteristic function $\chi_{\mathbf{E}} : \mathbf{X} \times \mathbf{X} \rightarrow \mathbf{2}$ corresponds, by assumption, to some $\mathbf{t} : \mathbf{X} \rightarrow \mathcal{P}(\mathbf{X}) := \underline{\text{Hom}}(\mathbf{X}, \mathbf{2})$. If $\mathcal{P}(\mathbf{X})$ is represented by some system $(\mathbf{Y}_a, f_{a,b})$, the map \mathbf{t} is represented by some $\mathbf{t}_a : \mathbf{X} \rightarrow \mathbf{Y}_a$, and so the relation $\mathbf{E}(x, y)$ is equivalent to the intersection over all \mathbf{b} of the conditions $f_{a,b}(\mathbf{t}_a(x)) = f_{a,b}(\mathbf{t}_a(y))$, hence, by compactness, by one such condition. The map $f_{a,b} \circ \mathbf{t}_a$ onto its image is then the quotient.

Conversely, assume that \mathcal{T} eliminates imaginaries, and let \mathbf{X} and \mathbf{Y} be two definable sets. Let \mathcal{I} be the category whose objects are definable maps $\mathbf{Z} \times \mathbf{X} \rightarrow \mathbf{Y}$, and whose morphisms are definable maps of the \mathbf{Z} coordinate that make the obvious diagram commute. For each such object, we view the elements of \mathbf{Z} as maps from \mathbf{X} to \mathbf{Y} . Let $\bar{\mathbf{Z}}$ be the quotient of \mathbf{Z} by the definable equivalence relation saying that $z_1, z_2 \in \mathbf{Z}$ define the same map. The quotient map determines a map in \mathcal{I} from $\mathbf{Z} \times \mathbf{X} \rightarrow \mathbf{Y}$ to $\bar{\mathbf{Z}} \times \mathbf{X} \rightarrow \mathbf{Y}$, which co-equalises any two maps of \mathcal{I} into its domain. Since \mathcal{I} also inherits the co-products from $\mathcal{D}_{\mathcal{T}}$, it has all finite colimits,

¹We allow disjoint unions of definable sets as a basic logical operation. This is automatic if there are at least two constant symbols, or if \mathcal{T} eliminates imaginaries, and is convenient otherwise. It implies that $\mathcal{D}_{\mathcal{T}}$ has finite co-products.

and is therefore filtering category. The “forgetful” functor that assigns to an object $\mathbf{Z} \times \mathbf{X} \rightarrow \mathbf{Y}$ of \mathcal{I} the definable set \mathbf{Z} in \mathcal{T} is therefore a filtering system in $\mathcal{D}_{\mathcal{T}}$. It is clear that this system represents $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$. \square

Remark 4.3. The system used in the second part of the above proof is the one used in the Grothendieck construction, to show that any presheaf is a colimit of sheaves. The existence of quotients produces the co-equalisers in this system, despite the fact that the original category $\mathcal{D}_{\mathcal{T}}$ might not have admitted them.

A model \mathcal{M} of \mathcal{T} determines a functor $\mathbf{X} \mapsto \mathbf{X}(\mathcal{M})$ from $\mathcal{D}_{\mathcal{T}}$ to the category of sets, which preserves all inverse limits, co-products, and images. Conversely, any such functor is (isomorphic to) a model. An elementary map from \mathcal{M} to \mathcal{N} as models is the same as a map of the corresponding functors. A subset \mathcal{A} of \mathcal{M} is definably closed if and only if it defines a left-exact sub-functor of \mathcal{M} that preserves co-products.

Similarly, a definition of the theory \mathcal{T}_1 in another theory \mathcal{T}_2 is the same as a logical functor \mathcal{J} (in the same sense as above) from $\mathcal{D}_{\mathcal{T}_1}$ to $\mathcal{D}_{\mathcal{T}_2}$. Given a model \mathcal{M} of \mathcal{T}_2 (viewed as a functor), the corresponding model of \mathcal{T}_1 is $\mathcal{M} \circ \mathcal{J}$ (See also Makkai and Reyes [10, § 7]). We will from now on identify models with the corresponding functors (so that $\mathbf{X}(\mathcal{M}) = \mathcal{M}(\mathbf{X})$ for a definable set \mathbf{X} and a model \mathcal{M}), and likewise for definably closed sets and definitions.

The following result is essentially the same as Makkai and Reyes [10, Theorem 7.1.8].

Proposition 4.4. *Let $\mathcal{J} : \mathcal{D}_{\mathcal{T}_1} \rightarrow \mathcal{D}_{\mathcal{T}_2}$ be a definition, where \mathcal{T}_1 eliminates imaginaries, and let \mathcal{J}^* be the induced functor from models of \mathcal{T}_2 to models of \mathcal{T}_1 (obtained by composition). Then \mathcal{J} is an equivalence of categories if and only if \mathcal{J}^* is.*

Proof. Assume that \mathcal{J}^* is an equivalence. We will show that \mathcal{H} is essentially surjective. Let \mathbf{X} be a definable set of \mathcal{T}_2 , and assume first that $\mathbf{X} \subseteq \mathcal{J}(\mathbf{Y})$ for some definable set \mathbf{Y} of \mathcal{T}_1 . Then for any two models \mathcal{M} and \mathcal{N} of \mathcal{T}_2 such that $\mathcal{M} \circ \mathcal{J} = \mathcal{N} \circ \mathcal{J}$, we have $\mathcal{M}(\mathbf{X}) = \mathcal{N}(\mathbf{X})$ (by equivalence). Hence by Beth’s definability theorem, $\mathbf{X} = \mathcal{J}(\mathbf{X}_0)$ for some \mathbf{X}_0 .

We next claim that for any model \mathcal{M}_2 of \mathcal{T}_2 , any element of \mathcal{M}_2 is definable over the restriction $\mathcal{M}_1 = \mathcal{M}_2 \circ \mathcal{J}$. Indeed, \mathcal{M}_2 embeds into a sufficiently saturated model \mathcal{N}_2 , and if there is an automorphism of \mathcal{N}_2 that fixes pointwise \mathcal{M}_1 but not \mathcal{M}_2 , we get two different embeddings of \mathcal{M}_2 in \mathcal{N}_2 , which agree on \mathcal{M}_1 , contradicting the equivalence.

It follows, by compactness, that any definable set \mathbf{Y} of \mathcal{T}_2 is the image of a definable map from a definable set $\mathcal{J}(\mathbf{X})$ that comes from \mathcal{T}_1 . By the previous point, the kernel of this map also comes from \mathcal{T}_1 , hence, since \mathcal{T}_1 eliminates imaginaries, \mathbf{Y} itself comes from \mathcal{T}_1 as well.

This shows that \mathcal{J} is essentially surjective. It is full since any map is a definable set, and it is faithful since for equality of definable sets is first order. Consequently, \mathcal{J} is an equivalence. The other direction is trivial. \square

As indicated in §3.2, one often considers sets definable with parameters in a set \mathcal{A} . These can be viewed as definable sets in an expansion $\mathcal{T}_{\mathcal{A}}$ of \mathcal{T} . The expansion corresponds to a functor $\mathcal{J}_{\mathcal{A}} : \mathcal{D}_{\mathcal{T}} \rightarrow \mathcal{D}_{\mathcal{T}_{\mathcal{A}}}$.

Proposition 4.5. *Assume \mathcal{T} eliminates imaginaries, and let \mathcal{M} be a model of \mathcal{T} . Then for any definably closed $\mathcal{A} \subseteq \mathcal{M}$, the functor $\mathcal{J}_{\mathcal{A}}$ is closed (in the sense of §2.5).*

Proof. A definable map f in \mathcal{T}_A from \mathbf{Z} to $J_A(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}))$ is given by formulas $\phi(w, z) \subseteq \mathbf{W} \times \mathbf{Z}$ and $\mathbf{F} \subseteq \mathbf{W} \times \mathbf{Z} \times J_A(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}))$, such that $\mathbf{Z} = \phi(\mathbf{a}, -)$, $f = \mathbf{F}(\mathbf{a}, -, -)$, $\mathbf{a} \in \mathbf{W}(A)$ and $\mathbf{F}, \mathbf{W}, \phi$ are in \mathcal{T} . There is a \mathcal{T} definable subset $\mathbf{W}_0 \subseteq \mathbf{W}$ such that $\mathbf{F}(w, -, -)$ is a function from $\phi(w, -)$ to $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ for all $w \in \mathbf{W}_0$. We may assume that $\mathbf{W} = \mathbf{W}_0$, so that \mathbf{F} gives a \mathcal{T} definable function from ϕ to $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$. Thus it corresponds to a map $\tilde{\mathbf{F}} : \phi \times \mathbf{X} \rightarrow \mathbf{Y}$. The restriction $\tilde{f} : \mathbf{Z} \times \mathbf{X} \rightarrow \mathbf{Y}$ of this map to $\phi(\mathbf{a}, -) \times \mathbf{X}$ corresponds to the original map f .

Reversing this argument gives, for each map $f : \mathbf{Z} \times \mathbf{X} \rightarrow \mathbf{Y}$ in \mathcal{T}_A a corresponding map $\tilde{f} : \mathbf{Z} \rightarrow J_A(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}))$. This shows that $J_A(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}))$ represents the correct functor in \mathcal{T}_A , and therefore coincides with $\underline{\mathbf{Hom}}(J_A(\mathbf{X}), J_A(\mathbf{Y}))$.

(More abstractly, \mathcal{T}_A can be viewed as a full sub-category of $\text{Pro}(\mathcal{T})$, and J_A as the inclusion of \mathcal{T} in $\text{Pro}(\mathcal{T})$. This inclusion is always closed.) \square

4.6. Since the definable closure of the empty set can be identified with the set of one element definable subsets, we get that for each definable set, $\mathbf{X}(0) = \Gamma(\mathbf{X}) = \underline{\mathbf{Hom}}(\mathbf{1}, \mathbf{X})$. Likewise, for any subset A of a model \mathcal{M} we have

$$A(\mathbf{X}) = \mathbf{X}(A) = \Gamma_A(J_A(\mathbf{X})) \quad (4)$$

(where Γ_A is global sections in $\mathcal{D}_{\mathcal{T}_A}$). In other words, $A = \Gamma_A \circ J_A$. We deduce that $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ gives the expected result when evaluated on parameters.

Corollary 4.7. *Assume \mathcal{T} eliminates imaginaries. For any definably closed set A and any definable sets \mathbf{X} and \mathbf{Y} , the evaluation map identifies $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})(A)$ with the set of A -definable maps from \mathbf{X} to \mathbf{Y} . In particular, $\mathcal{P}(\mathbf{X})(A)$ is the set of A -definable subsets of \mathbf{X} .*

Proof. According to Proposition 4.5 and the above remark,

$$\begin{aligned} \underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})(A) &= \Gamma_A(J_A(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}))) = \\ &= \text{Hom}_A(\mathbf{1}, \underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})) = \text{Hom}_A(\mathbf{X}, \mathbf{Y}) \end{aligned} \quad (5)$$

\square

Finally, we can characterize the stable embeddings.

Proposition 4.8. *Let $J : \mathcal{D}_{\mathcal{T}_1} \rightarrow \mathcal{D}_{\mathcal{T}_2}$ be an interpretation, where both theories eliminate imaginaries. Then the interpretation is stably embedded if and only if J is closed.*

Proof. Assume that the interpretation is stably embedded, let \mathbf{X}, \mathbf{Y} be definable sets in \mathcal{T}_1 , and let \mathcal{M} be a model of \mathcal{T}_2 . The map $t : J(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})) \rightarrow \underline{\mathbf{Hom}}(J(\mathbf{X}), J(\mathbf{Y}))$ determines a map

$$t : J(\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y}))(\mathcal{M}) \rightarrow \underline{\mathbf{Hom}}(J(\mathbf{X}), J(\mathbf{Y}))(\mathcal{M})$$

By Corollary 4.7, the co-domain is the set of \mathcal{M} -definable maps from $J(\mathbf{X})$ to $J(\mathbf{Y})$ in \mathcal{T}_2 , while the domain is the set of definable maps from \mathbf{X} to \mathbf{Y} in \mathcal{T}_1 with parameters from the model $\mathcal{M} \circ J$ (and t assigns to each map the same map viewed as a map between $J(\mathbf{X})$ and $J(\mathbf{Y})$). Since \mathcal{T}_1 asserts that all maps in $\underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{Y})$ are distinct, the same has to hold in \mathcal{T}_2 . Therefore, t is injective. Since \mathcal{T}_1 is stably embedded, any \mathcal{M} -definable map between $J(\mathbf{X})$ and $J(\mathbf{Y})$ is definable with parameters in $\mathcal{M} \circ J$, hence t is surjective. Since t is a bijection in every model, t is an isomorphism.

Assume conversely that J is closed, let \mathbf{X} be a definable set in \mathcal{T}_1 , and let \mathcal{M} be a model of \mathcal{T}_2 . Any \mathcal{M} -definable subset $\mathbf{Z} \subseteq J(\mathbf{X})$ (in \mathcal{T}_2), corresponds to an \mathcal{M} -point

of $\mathcal{P}(\mathcal{J}(\mathbf{X}))$ (by Corollary 4.7), hence to an $\mathcal{M} \circ \mathcal{J}$ point of $\mathcal{P}(\mathbf{X})$, which can be used to define \mathbf{Z} . \square

4.9. Internality. We now return to discuss categories \mathcal{C} which are ind-closed and admit finite co-products. Thus, $\mathcal{C} = \mathcal{D}_{\mathcal{T}}$ for a theory \mathcal{T} with elimination of imaginaries is an example. Our aim is to formulate the notion of internality in §3.5 in this setting. From Proposition 4.8, we already have the notion of a stably embedded interpretation: This is simply a closed functor between two such ind-closed categories.

Let \mathcal{T} be an internal cover of \mathcal{T}_0 (both eliminating imaginaries), and let \mathbf{X} be a definable set in \mathcal{T} . By assumption (and compactness), there is a parameter \mathbf{a} , and an \mathbf{a} -definable surjective map $f_{\mathbf{a}}$ from some sort in \mathcal{T}_0 onto \mathbf{X} . Since \mathcal{T}_0 eliminates imaginaries, we may assume $f_{\mathbf{a}}$ to be bijective. The domain \mathbf{X}_0 of $f_{\mathbf{a}}$ is a definable set in \mathcal{T}_0 , hence (since \mathcal{T}_0 is stably embedded) it can be defined with a (canonical) parameter from a model of \mathcal{T}_0 , and therefore (because of the assumption in the end of §3.5) with no parameters at all.

Thus, after fixing suitable parameters of internality, we may associate to each definable set \mathbf{X} in \mathcal{T} , a definable set \mathbf{X}_0 in \mathcal{T}_0 . Since this process commutes with products and inclusions and complements, this is an interpretation of \mathcal{T} in \mathcal{T}_0 . Furthermore, this is a stably embedded interpretation: any subset of \mathbf{X}_0 definable with parameters can be translated via the same function $f_{\mathbf{a}}$ to a subset definable with parameters of \mathbf{X} . This motivates the following definition.

Definition 4.10. Let \mathcal{C}_0 be an ind-closed category. An *internal cover* of \mathcal{C}_0 consists of an ind-closed category \mathcal{C} , together with closed functors $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$ and $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}_0$, and an isomorphism of $\mathcal{F} \circ \mathcal{J}$ with the identity functor (we will ignore this isomorphism in the notation and assume that $\mathcal{F} \circ \mathcal{J}$ is the identity functor).

Remark 4.11. The assumption that $\mathcal{F} \circ \mathcal{J}$ is the identity is equivalent (under the other conditions) to \mathcal{F} being “internally left adjoint” to \mathcal{J} , in the sense that the canonical map $\underline{\mathrm{Hom}}(\mathcal{F}(\mathbf{X}), \mathbf{Y}_0) \rightarrow \mathcal{F}(\underline{\mathrm{Hom}}(\mathbf{X}, \mathcal{J}(\mathbf{Y}_0)))$ is an isomorphism (this can be seen by applying the isomorphism with $\mathbf{X} = \mathbf{1}$). The “unit of adjunction” obtained by setting $\mathbf{Y}_0 = \mathcal{F}(\mathbf{X})$ (which does not formally exist, in general) is, in the model theoretic setting, the collection of maps $f_{\mathbf{a}}$ described above (which “do not exist” because they require parameters).

We now wish to prove an analogue of Theorem 3.7. According to §4.6, a “model” of \mathcal{C}_0 is given by a composition of functors $\Gamma_{\mathcal{D}} \circ \mathcal{H}$, where \mathcal{H} is a closed functor into an ind-closed category \mathcal{D} , and $\Gamma_{\mathcal{D}}$ is the functor $\mathrm{Hom}(\mathbf{1}, -)$ in \mathcal{D} . A model of \mathcal{C} is thus given by $\Gamma_{\mathcal{D}} \circ \mathcal{H} \circ \mathcal{F}$. We are interested in automorphisms of this functor that fix \mathcal{J} . We also know from Corollary 2.8 that the group $\mathrm{Aut}(\mathcal{H} \circ \mathcal{F}/\mathcal{J})$ of “definable” automorphisms comes from a pro-group, but $\Gamma_{\mathcal{D}}$ is far from being closed. Regardless, we have the following statement.

Theorem 4.12. *Let $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$, $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}_0$ be an internal cover (so \mathcal{C}_0 and \mathcal{C} are ind-closed). Then for any left-exact $\mathcal{H} : \mathcal{C}_0 \rightarrow \mathcal{D}$, the map of groups*

$$\mathrm{Aut}(\mathcal{H} \circ \mathcal{F}/\mathcal{J}) \rightarrow \mathrm{Aut}(\Gamma_{\mathcal{D}} \circ \mathcal{H} \circ \mathcal{F}/\mathcal{J})$$

obtained by applying $\Gamma_{\mathcal{D}}$ is an isomorphism.

We note that \mathcal{H} need not be closed in this theorem.

Proof. Let θ be an automorphism of $\mathcal{R} = \Gamma_{\mathcal{D}} \circ \mathcal{H} \circ \mathcal{F}$ fixing $\mathcal{R} \circ \mathcal{J}$, let \mathbf{Q} be an object of \mathcal{C} , and set $\mathbf{X} = \underline{\mathbf{Hom}}(\mathbf{Q}, \mathcal{J}(\mathcal{F}(\mathbf{Q})))$. Thus there is an evaluation map $\underline{\mathbf{ev}} : \mathbf{X} \times \mathbf{Q} \rightarrow \mathcal{J}(\mathcal{F}(\mathbf{Q}))$. By the remark above, we have $\mathcal{F}(\mathbf{X}) = \underline{\mathbf{Hom}}(\mathcal{F}(\mathbf{Q}), \mathcal{F}(\mathbf{Q}))$, and hence an evaluation map $\mathcal{F}(\mathbf{X}) \times \mathcal{F}(\mathbf{Q}) \rightarrow \mathcal{F}(\mathbf{Q})$. Since \mathcal{R} is left exact, we thus get a map

$$\mathcal{R}(\underline{\mathbf{ev}}) : \mathcal{R}(\mathbf{X}) \times \mathcal{R}(\mathbf{Q}) \rightarrow \mathcal{R}(\mathbf{Q})$$

In particular, each element of $\mathcal{R}(\mathbf{X})$ determines a map of $\mathcal{R}(\mathbf{Q})$ to itself. We claim that $\theta_{\mathbf{Q}}$ is represented by an element of $\mathcal{R}(\mathbf{X})$, $\mathfrak{g} : \mathbf{1} \rightarrow \mathcal{H}(\mathcal{F}(\mathbf{X}))$. This will show that θ comes from an automorphism of $\mathcal{H} \circ \mathcal{F}$, namely the composition

$$\mathcal{H}(\mathcal{F}(\mathbf{Q})) = \mathbf{1} \times \mathcal{H}(\mathcal{F}(\mathbf{Q})) \xrightarrow{\mathfrak{g} \times \text{Id}} \mathcal{H}(\mathcal{F}(\mathbf{X})) \times \mathcal{H}(\mathcal{F}(\mathbf{Q})) \rightarrow \mathcal{H}(\mathcal{F}(\mathbf{Q}))$$

We first note that there is a map $\mathbf{1} \rightarrow \mathcal{F}(\mathbf{X})$ in \mathcal{C}_0 corresponding to the identity on $\mathcal{F}(\mathbf{Q})$. Applying $\Gamma \circ \mathcal{H}$, we deduce that the identity map is represented by an element $\mathbf{1}$ in $\mathcal{R}(\mathbf{X})$. Since \mathbf{X} , \mathbf{Q} , $\mathcal{J}(\mathcal{F}(\mathbf{Q}))$ and the evaluation map are from \mathcal{C} , the evaluation map $\mathcal{R}(\underline{\mathbf{ev}})$ is preserved by θ . Also, $\theta_{\mathcal{J}(\mathcal{F}(\mathbf{Q}))}$ is the identity. Hence, for all $\mathfrak{q} \in \mathcal{R}(\mathbf{Q})$, we get $\mathcal{R}(\underline{\mathbf{ev}})(\theta_{\mathbf{X}}(\mathbf{1}), \theta_{\mathbf{Q}}(\mathfrak{q})) = \mathcal{R}(\underline{\mathbf{ev}})(\mathbf{1}, \mathfrak{q}) = \mathfrak{q}$. It follows that $\theta_{\mathbf{X}}(\mathbf{1})$, as an endomorphism of $\mathcal{R}(\mathbf{Q})$ is invertible, with inverse $\theta_{\mathbf{Q}}$. This complete the proof of the surjectivity. The injectivity is similar, using the composition map instead of the evaluation. \square

Corollary 4.13. *Let $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$, $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}_0$ be an internal cover. There is a pro-group \mathbf{G} in $\text{Ind}(\mathcal{C})$, and an action of \mathbf{G} on each object of \mathcal{C} , such that for any closed functor $\mathcal{H} : \mathcal{C}_0 \rightarrow \mathcal{D}$ into an ind-closed category, $\Gamma_{\mathcal{D}}(\mathcal{H}(\mathcal{F}(\mathbf{G}))) = \text{Aut}(\Gamma_{\mathcal{D}} \circ \mathcal{H} \circ \mathcal{F} / \mathcal{J})$.*

Proof. By Corollary 2.8 we have such a \mathbf{G} with the property that $\Gamma_{\mathcal{D}}(\mathcal{H}(\mathcal{F}(\mathbf{G}))) = \text{Aut}(\mathcal{H} \circ \mathcal{F} / \mathcal{J})$. By Theorem 4.12, the last group is equal to $\text{Aut}(\Gamma_{\mathcal{D}} \circ \mathcal{H} \circ \mathcal{F} / \mathcal{J})$. \square

As explained above, an internal cover of theories gives rise to an internal cover in our sense. Applying the corollary with \mathcal{H} an expansion by constants in a model recovers Theorem 3.7, but with a pro-group in ind-definable, rather than definable sets.

5. RECOVERING A PRO-GROUP

The model theoretic statement in Theorem 3.7 produces a pro-definable group, i.e., a pro-group in the category \mathcal{C} of definable sets, whereas the more general statement of Corollary 4.13 provides only a pro-group in $\text{Ind}(\mathcal{C})$. This difference seems unavoidable in general, but in this section we would like to formulate additional conditions that ensure the stronger statement (these conditions are easily seen to hold in the first order setting).

The idea, which appears in the model theoretic proof, is that the objects $\mathbf{X} = \underline{\mathbf{Hom}}(\mathbf{Q}, \mathbf{Q})$ which participate in the construction of \mathbf{G} are unnecessarily big. If \mathbf{Y} is an object in a system that represents \mathbf{X} , it will not, in general, be closed under composition. However, if $\mathfrak{f}_{\mathbf{Q}}$ is an “element” of \mathbf{Y} that is part of a natural automorphism, then composition with it remains within \mathbf{Y} , since the map from \mathbf{Y} to \mathbf{X} is preserved by the natural automorphism. Hence, to construct the natural automorphisms, it is possible to replace \mathbf{X} by a compact sub-object.

For this to work, we need to know that the internal Homs can be constructed from compact sub-objects. This is captured by the following definitions.

Definition 5.1. A map $f : \mathbf{X} \rightarrow \mathbf{Y}$ in $\text{Ind}(\mathcal{C})$ is called *proper* if for any $g : \mathbf{Z} \rightarrow \mathbf{Y}$ with \mathbf{Z} compact, the pullback $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ is also compact.

An ind-object \mathbf{Y} is called *semi-strict* if any map from a compact object to \mathbf{Y} is proper.

For example, if \mathbf{Y} is compact, then $f : \mathbf{X} \rightarrow \mathbf{Y}$ is proper if and only if it is in \mathcal{C} .

Recall that an ind-object \mathbf{Y} is *strict* if any map $\mathbf{X} \rightarrow \mathbf{Y}$, with \mathbf{X} compact, factors as $\mathbf{X} \rightarrow \mathbf{Z} \xrightarrow{i} \mathbf{Y}$, with \mathbf{Z} compact and i a monomorphism (equivalently, \mathbf{Y} can be represented by a system of monomorphisms).

Lemma 5.2. *Let \mathbf{Y} be an ind-object.*

- (1) *If \mathbf{Y} is strict, then it is semi-strict*
- (2) *\mathbf{Y} is semi-strict if and only if for any two ind-objects \mathbf{X} and \mathbf{Z} over \mathbf{Y} , the map $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z} \rightarrow \mathbf{X} \times \mathbf{Z}$ is proper.*

The converse of (1) seems to require additional assumptions (for example, quotients by equivalence relations).

Proof. (1) Since \mathbf{Y} is strict, we may assume that both arrows from compact objects \mathbf{X} and \mathbf{Z} factor through the same monomorphism $\mathbf{W} \rightarrow \mathbf{Y}$, with \mathbf{W} compact. Hence $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z} = \mathbf{X} \times_{\mathbf{W}} \mathbf{Z}$ is compact as well.
 (2) This immediately reduces to the case when \mathbf{X} and \mathbf{Z} are compact, in which case we need to show that $\mathbf{X} \times_{\mathbf{Y}} \mathbf{Z}$ is compact, which is precisely the definition of \mathbf{Y} being semi-strict. \square

For two objects \mathbf{X} and \mathbf{Y} , let $\underline{\text{Bi}}(\mathbf{X}, \mathbf{Y})$ be the ind-object consisting of invertible maps from \mathbf{X} to \mathbf{Y} , and let $\underline{\text{Bi}}(\mathbf{X}) = \underline{\text{Bi}}(\mathbf{X}, \mathbf{X})$ (this is a group ind-object).

Theorem 5.3. *In the situation of Corollary 4.13, assume in addition:*

- (1) *For any two objects \mathbf{X} and \mathbf{Y} of \mathcal{C} , the ind-object $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ is semi-strict.*
- (2) *In \mathcal{C}_0 , each $\underline{\text{Bi}}(\mathbf{X}, \mathbf{Y})$ is strict.*
- (3) *If \mathbf{X} is an object of \mathcal{C} such that $\mathcal{F}(\mathbf{X})$ is compact, then \mathbf{X} is compact.*
- (4) *For any monomorphism $k : \mathbf{X} \rightarrow \mathbf{Y}$ and object \mathbf{Z} , all in \mathcal{C}_0 , the induced map $\text{ok} : \underline{\text{Hom}}(\mathbf{Z}, \mathbf{X}) \rightarrow \underline{\text{Hom}}(\mathbf{Z}, \mathbf{Y})$ is proper.*

Then the group \mathbf{G} of the conclusion of Corollary 4.13 is a pro-group in \mathcal{C} .

Note that assumption (3) holds if \mathcal{F} reflects isomorphisms, in the sense that i is an isomorphism if $\mathcal{F}(i)$ is.

Proof. The proof is analogous to the model theoretic proof. The idea is that by internality, for any object \mathbf{Q} of \mathcal{C} there is an object \mathbf{C} of \mathcal{C}_0 with $\underline{\text{Bi}}(\mathbf{Q}, \mathbf{C})$ “non-empty”. The group \mathbf{G} acts freely on this ind-object, and therefore, by assumption, on a compact sub-object \mathbf{X} . \mathbf{G} is then obtained by a sequence of pullbacks as in the proof of Proposition 2.7, except that now the functor whose end we are taking has values in \mathcal{C} itself for objects on the diagonal. The pullbacks are again compact by the properness assumption.

More precisely, in the construction of the end, the pro-group \mathbf{G} is constructed as an inverse system of pullbacks of the form $\mathbf{G}_3 = \mathbf{G}_1 \times_{\underline{\text{Hom}}(\mathbf{X}_1, \mathbf{X}_2)} \mathbf{G}_2$, where \mathbf{G}_i is a subgroup of $\underline{\text{Bi}}(\mathbf{X}_i)$, and the product is with respect composition maps with a given morphism $f : \mathbf{X}_1 \rightarrow \mathbf{X}_2$ in \mathcal{C} (\mathbf{G}_3 is then viewed as a subgroup of $\underline{\text{Bi}}(\mathbf{X}_1 \times \mathbf{X}_2)$). Since we are assuming that $\underline{\text{Hom}}(\mathbf{X}_1, \mathbf{X}_2)$ is semi-strict, \mathbf{G}_3 will be compact if \mathbf{G}_1

and \mathbf{G}_2 are. Hence, it is enough to prove that each $\underline{\mathbf{Bi}}(\mathbf{Q})$ contains a compact subgroup \mathbf{G}_Q , which is in the system.

Let $\mathbf{C} = \mathcal{J}(\mathcal{F}(\mathbf{Q}))$ (so that $\mathcal{F}(\mathbf{C}) = \mathcal{F}(\mathbf{Q})$), and let $\tilde{t} : \mathbf{X} \rightarrow \underline{\mathbf{Bi}}(\mathbf{Q}, \mathbf{C})$ be a map from a compact object in \mathcal{C} , such that the map $\mathbf{1} \rightarrow \underline{\mathbf{Hom}}(\mathcal{F}(\mathbf{C}), \mathcal{F}(\mathbf{C}))$ corresponding to the identity map on $\mathcal{F}(\mathbf{C})$ factors through $\mathcal{F}(\tilde{t})$. Thus, we have a corresponding evaluation map $t : \mathbf{X} \times \mathbf{Q} \rightarrow \mathbf{C}$, and a map $i : \mathbf{1} \rightarrow \mathcal{F}(\mathbf{X})$, such that $\mathcal{F}(t) \circ (i \times \text{id}_{\mathcal{F}(\mathbf{Q})})$ is the identity. We also have a ‘‘composition with the inverse’’ map $m : \mathbf{X} \times \mathbf{X} \rightarrow \underline{\mathbf{Bi}}(\mathbf{C})$, given on points $(x, y) \in \mathbf{X} \times \mathbf{X}$ by $m(x, y) = x \circ y^{-1}$. We let \mathbf{G}_Q be the ‘‘subgroup’’ of $\underline{\mathbf{Bi}}(\mathbf{X})$ preserving m ; More precisely, $\underline{\mathbf{Bi}}(\mathbf{X})$ embeds in $\underline{\mathbf{Bi}}(\mathbf{X} \times \mathbf{X})$ by acting diagonally, and composition with m gives in total a map $\underline{\mathbf{Bi}}(\mathbf{X}) \rightarrow \underline{\mathbf{Hom}}(\mathbf{X} \times \mathbf{X}, \mathbf{C})$. \mathbf{G}_Q is the pullback of this map along the map $\tilde{m} : \mathbf{1} \rightarrow \underline{\mathbf{Hom}}(\mathbf{X} \times \mathbf{X}, \mathbf{C})$ corresponding to m .

$$\begin{array}{ccc} \mathbf{G}_Q & \longrightarrow & \underline{\mathbf{Bi}}(\mathbf{X} \times \mathbf{X}) \\ \downarrow & & \downarrow \circ m \\ \mathbf{1} & \xrightarrow{\tilde{m}} & \underline{\mathbf{Hom}}(\mathbf{X} \times \mathbf{X}, \mathbf{C}) \end{array} \quad (6)$$

The ind-group \mathbf{G}_Q is one of the groups that appear in the system defining the total group of automorphisms. Thus, it is enough to prove that it is compact. For that, it is sufficient, by the assumption, to prove that $\mathcal{F}(\mathbf{G}_Q)$ is compact. Hence we may assume that \mathbf{X} is a subobject $k : \mathbf{X} \rightarrow \underline{\mathbf{Bi}}(\mathcal{F}(\mathbf{C}))$ containing the identity $i : \mathbf{1} \rightarrow \mathbf{X}$, and \mathbf{G}_Q is a group acting on \mathbf{X} . The identity map i and the action μ of \mathbf{G}_Q on \mathbf{X} induce a map $j = \mu \circ (\text{id} \times i) : \mathbf{G}_Q \rightarrow \mathbf{X}$. The pullback diagram defining \mathbf{G}_Q implies that we have a pullback diagram

$$\begin{array}{ccc} \mathbf{G}_Q & \xrightarrow{\tilde{\mu}} & \underline{\mathbf{Hom}}(\mathbf{X}, \mathbf{X}) \\ j \downarrow & & \downarrow \circ k \\ \mathbf{X} & \xrightarrow{\tilde{\sigma}} & \underline{\mathbf{Hom}}(\mathbf{X}, \underline{\mathbf{Bi}}(\mathcal{F}(\mathbf{C}))) \end{array} \quad (7)$$

Since k is a monomorphism, the vertical map $\circ k$, induced by composition with k , is by assumption proper. Therefore \mathbf{G}_Q , being the pullback over a compact object of a proper map, is itself compact. \square

6. EXAMPLES

As explained above, Corollary 4.13 and 5.3 are categorical analogues of the classical model theoretic Theorem 3.7. I now briefly mention two additional examples where the more general results may be applied.

6.1. Partial automorphisms. Instead of considering automorphisms of the full model \mathcal{M} , as in Theorem 3.7, one may be interested in automorphisms of a definable set \mathbf{Q} in \mathcal{T} that preserve only part of the structure on \mathbf{Q} . This question was studied in Kamensky [5], where one is ultimately interested in automorphisms of a linear difference equation that preserve only the quantifier free formulas.

The categorical version may be applied in this case by taking \mathcal{C} to be the subcategory consisting of the definable sets one wishes to preserve. The condition that this category eliminates imaginaries corresponds to the requirement that the internality datum that witnesses the fact that $\text{dcl}(\mathcal{A} \cup \mathcal{M}_0) = \mathcal{M}$, should be preserved.

6.2. Tannakian categories. As mentioned above, the model theoretic theorem can be applied to prove a version the Tannakian reconstruction theorem. Below I outline the “composition” of the model theoretic argument with the categorical formulation of internality described above. This is not a complete example, but a “work plan”, since some of the assumptions may be difficult to verify. The idea is only to show how the language used here is translated in this case.

Let \mathbb{k} be a field of characteristic 0. Recall (Deligne [1, § 7]) that if $\tilde{\mathcal{C}}$ is a (symmetric, rigid, \mathbb{k} -linear) tensor category, one may construct the category $\mathcal{S}ch_{\tilde{\mathcal{C}}}$ of $\tilde{\mathcal{C}}$ -schemes as the opposite category of the category of (commutative, unital) algebras in $\text{Ind}(\tilde{\mathcal{C}})$ (such an algebra consists of an ind-object A of $\tilde{\mathcal{C}}$, together with morphisms $m : A \otimes A \rightarrow A$ and $u : \mathbf{1} \rightarrow A$, satisfying obvious axioms). This category admits finite inverse limits, since a pullback corresponds to a tensor product of algebras (over a base), as well as finite co-products.

Furthermore, there is a faithful functor $\mathbf{X} \mapsto \hat{\mathbf{X}}$ from $\tilde{\mathcal{C}}$ to $\mathcal{S}ch_{\tilde{\mathcal{C}}}$, sending \mathbf{X} to the symmetric algebra on the dual $\check{\mathbf{X}}$. Let \mathcal{C} be the full sub-category of $\mathcal{S}ch_{\tilde{\mathcal{C}}}$ containing all object of the form $\hat{\mathbf{X}} \times_{\check{\mathbf{Y}}} \hat{\mathbf{Z}}$, with \mathbf{X} , \mathbf{Y} and \mathbf{Z} objects of $\tilde{\mathcal{C}}$, as well as their finite co-products (these are the “finitely presented” $\tilde{\mathcal{C}}$ -schemes). It again admits all finite inverse limits and co-products. It is possible to show, like in usual algebraic geometry, that $\underline{\text{Hom}}(\mathbf{X}, \mathbf{Y})$ is a strict ind-object for all objects \mathbf{X} and \mathbf{Y} of \mathcal{C} .

In particular, we take $\tilde{\mathcal{C}}_0$ to be the category of finite-dimensional vector spaces over \mathbb{k} , so that \mathcal{C}_0 is the category of finitely presented affine schemes over \mathbb{k} . An exact tensor functor between tensor categories induces a left-exact functor between the corresponding categories of schemes, which restricts to a functor between the finitely presented sub-categories. Furthermore, the construction of the $\underline{\text{Hom}}$ ind-schemes shows that the resulting functor is a stable embedding. In particular, the (essentially unique) tensor functor from $\tilde{\mathcal{C}}_0$ to $\tilde{\mathcal{C}}$ induces a functor $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{C}$.

If $\tilde{\mathcal{C}}$ is neutral Tannakian, i.e., we have an exact tensor functor ω to $\tilde{\mathcal{C}}_0$, this functor again induces a stable embedding $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}_0$, with $\mathcal{F} \circ \mathcal{J}$ isomorphic to the identity. We thus get an internal cover, in the sense of Definition 4.10.

If \mathbf{T} is any (affine) scheme over \mathbb{k} , setting $\mathcal{H}_{\mathbf{T}}$ to be base change by \mathbf{T} (i.e., $\mathcal{H}_{\mathbf{T}}(\mathbf{U}) = \mathbf{T} \times \mathbf{U}$ as a functor from \mathbb{k} -schemes to \mathbf{T} -schemes), one gets from Corollary 4.13 that any automorphism of the functor $\mathbf{X} \mapsto \mathcal{F}(\mathbf{X})(\mathbf{T})$ (as a functor from \mathcal{C} to sets) comes from a unique compatible collection of automorphisms of the \mathbf{T} -schemes $\mathbf{T} \times \mathcal{F}(\mathbf{X})$; and furthermore, all such automorphisms correspond to the \mathbf{T} -points of (with the help of Theorem 5.3, whose assumptions should be verified) a pro-group-scheme over \mathbb{k} , acting on all $\mathcal{F}(\mathbf{X})$. In fact, this group comes from a pro-group in \mathcal{C} (which is called the fundamental group of \mathcal{C} in Deligne [1]).

Note that if \mathbf{X} is the $\tilde{\mathcal{C}}$ -scheme associated to an object $\check{\mathbf{X}}$ of $\tilde{\mathcal{C}}$, then $\mathcal{F}(\mathbf{X})$ is $\omega(\check{\mathbf{X}})$ with its \mathbb{k} -scheme structure. Hence, if $\mathbf{T} = \text{spec}(A)$ is an affine scheme over \mathbb{k} , then $\mathcal{F}(\mathbf{X})(\mathbf{T}) = \mathcal{F}(\mathbf{X})(A)$ is identified (at least as a set) with $A \otimes_{\mathbb{k}} \omega(\check{\mathbf{X}})$. Furthermore, the linear space and tensor structures are described by suitable maps between schemes, all of which come from \mathcal{C} . Therefore, an automorphism of the functor $\mathbf{X} \mapsto \mathcal{F}(\mathbf{X})(\mathbf{T})$ as a functor to sets is the same as a tensor automorphism of the functor $\mathbf{X} \mapsto A \otimes_{\mathbb{k}} \omega(\mathbf{X})$.

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