# RESEARCH DESCRIPTION

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## 1. Past research

In this section I describe my research as a PhD student, under the supervision of Prof. Ehud Hrushovski, at the department of mathematics of the Hebrew University. In my research I attempted to apply model theory (a branch of mathematical logic) to problems in other fields of mathematics, mainly algebra and algebraic geometry. I ended up doing two separate works, which I describe below.

### 1.1. Elimination of quantifiers in modules over finitely generated algebras.

Let $A$ be a commutative reduced algebra, finitely generated over an algebraically closed field $k$. Such an algebra corresponds to an affine variety $X$ over $k$, and a module over $A$ corresponds to a (quasi-coherent) sheaf over $X$. Whereas varieties are reasonably accessible within the framework of model theory, for example as definable sets in the theory of algebraically closed fields, modules (or sheaves) do not appear so naturally. The purpose of this work was to find a framework for studying such modules.

The basic theory we used includes a sort $K$ for the field, a sort $V$ for the module with a vector space structure over the field, and a finite number of commuting linear operators on $V$, possibly satisfying some relations. Given a presentation of the algebra $A$, a model of this theory is obtained by interpreting $V$ as the vector space structure on the module $M$, and the linear operators as the action of the generators. Thus, this theory represents a finitely generated algebra with a given presentation, and a module over it. Since we are interested in the model completion, the field is algebraically closed.

In the case when the number of operators is 0, this yields, up to the dimension of $V$, a complete theory. This theory eliminates quantifier after expanding the language with sorts for the Grassmanians of each $K^n$, and functions from $V^n$ to $Gr(K^n)$ describing the dependence relations.

With the full language (with more than zero generators), the theory is not complete. However, I proved that in the extended language (with the Grassmanians), the theory admits a model completion (and thus, quantifier elimination). The model completion axiomatises a certain class of ("large") injective modules. To show that this is first order axiomatisable, I used a finiteness result from Dries and Schmidt [5]. As a side result, the class of all injective modules is also first order axiomatisable. An easy corollary of this elimination of quantifiers is that the theory is $\omega$-stable.

The results from this part of my thesis appear as a separate paper, accepted for publication in the Journal of Symbolic Logic [12].

### 1.2. Definable automorphism groups and difference fields.

The main motivation for the second part of my thesis was to extend the results of Hrushovski [7] to difference fields. In that paper, the automorphism group of a linear differential equation (or a system of such equations) over $\mathbb{Q}(t)$ is shown to be computable. In model theoretic terms, one considers the theory of *Differentially Closed Fields* (DCF), with constant symbols for $\mathbb{Q}(t)$ (with $Dt = 1$). The linear equation defines a set $X$, which is a finite dimensional vector space over the field of constants $C$ (itself defined by the equation $Dx = 0$). Given a model $M$, the automorphism group of $X(M)$ over $C(M)$ (and $\mathbb{Q}(t)$) turns out to be (canonically) the group of $M$ points of a definable group in this theory. This is an instance of a general result, developed in works of Zilber and Poizat (Poizat [17]), and explained in Appendix B of Hrushovski [7]. It is further explained there that this group has a variant $G$ that is contained in (a power of) the field $C$, and is therefore a linear algebraic group. This group can be identified with the Galois group obtained by algebraic methods (Put and Singer [19]).

In the context of difference equations, the algebraic treatment appears in Put and Singer [18]. In model theoretic terms, the problem is defined analogously: A difference field is a field endowed with an automorphism $\sigma$. The theory of such fields admits a model companion called ACFA (Chatzidakis and Hrushovski [2]). A difference equation again gives rise to a definable set in this theory, and the solution set of a linear difference equation is again a finite dimensional vector space over the fixed field, the field defined by the equation $\sigma(x) = x$.

However, there are several essential differences with the differential case. First, the theory ACFA does not eliminate quantifiers. Therefore, the model theoretic automorphism group is, in general, properly contained in the algebraic one (which takes into account only the quantifier free structure). Further, the field $C$ is not algebraically closed in this case, and therefore the general theory of definable groups in algebraically closed fields can not be applied to groups definable in $C$. Finally, the algebraically defined Galois group is harder to identify with a model theoretic one, since its construction involves algebras that may contain zero-divisors.

To apply model theory to this situation, the definition of the model theoretic Galois group should be slightly modified. To this end, we give an elementary description of the general model theoretic construction of the Galois group of a definable set $X$ *internal* to another definable set $C$, and show that any such group occurs as a definable subgroup of a group constructed purely from the internality datum — an injective map from $X$ to $C$, defined with parameters. We describe this last group $Aut_0(X/C)$, which is a group of bijections of $X$ onto itself fixing pointwise the set $C$, and its action on $X$ explicitly in terms of the internality data. We also describe explicitly the equations that define subgroups of $Aut_0$ corresponding to preserving arbitrary formulas. We show that when this procedure is applied to the set of all formulas, we obtain the previously known construction. On the other hand, by considering the set of quantifier free formulas in ACFA, one obtains the group of automorphisms preserving the quantifier free structure.

We then show that in the case of difference equations, this group is given by a set of equations of the form $f(X) = f(Y)$, where $f$ is a rational function invariant under the action of $\sigma$ (and more generally, a similar result holds for theories obtained from a stable theory by adding a generic automorphism). Finally, we describe explicitly the connection with the algebraically defined automorphism group. In particular, this work extends the definition of the Galois group of linear difference equations to a broader class of equations (where the fixed field of the base is not algebraically closed).

The main result of this part was submitted for publication in "Selecta Mathematica" [9]. Some general results about pro-definable and ind-definable sets that were required for the presentation were described in a separate paper [10].

## 2. Future and Current Research

2.1. **Tannakian Categories.** Recently my research focused mainly on applying and connecting the model theoretic notions of internality and definable automorphism groups to the Tannakian formalism and its variants. The Tannakian formalism provides an abstract description of the category of representations of an affine group scheme, which can be viewed as an analogue, for non-commutative groups, of Pontryagin duality.

The description involves a *rigid tensor category*, an abelian category endowed with a formal tensor operation on the objects, and associativity and commutativity morphisms, satisfying certain axioms, all satisfied by the category of (finite dimensional) representations of an algebraic group. There is a natural notion of morphisms between such categories (called tensor functors). A *neutral Tannakian category* is then a rigid tensor category, together with a faithful tensor functor ("fibre functor") to the category of vector spaces (over a fixed base field $k$). The category of representations of an affine group scheme over $k$, together with the forgetful functor, is an example. The fundamental result of this theory (Saavedra-Rivano [20] and Deligne et al. [4]) is that, conversely, to any such category corresponds an affine group scheme $G$, such that the fibre functor is an equivalence with the category of $G$-representations, and the group can be reconstructed from this structure.

I show that results of this kind can be viewed as an instance of the construction of the group of automorphisms from internality data. From this point of view, the vector space associated via the fibre functor to each object is internal to the base field. Thus, the general theory gives rise to a definable group of automorphisms. Since this group is definable in an algebraically closed field, the general theory of such groups (Poizat [17]) shows that this is an algebraic group.

In the other direction, I showed that the notion of internality has a natural interpretation in a categorical setting, and that (a weak form of) the model theoretic formalism can be viewed as a categorical construction. This result appears in Kamensky [8], and I expect it to be useful, in particular, in model theoretic situations where the assumptions of the classical model theoretic formalism are too strong (as in the case of general fields with operators, indicated below).

The model theoretic formulation gives rise to natural generalisations of this theory to other kinds of groups. Replacing the field by a differential field, I defined the notion of a *differential* tensor category, and showed that the model theoretic technique applies there as well. These results are available in Kamensky [11].

In my current research, I am working on extending these results to a wider context where differential fields are replaced by an arbitrary action of "operator". This uses a formalism for such actions developed in Moosa and Scanlon [15] and Moosa and Scanlon [14], and which includes the differential and difference cases, as well as examples in positive characteristic, and their mixtures. A corresponding theory of tensor categories with operations is developed, which describes categories of representations for linear groups definable in such fields.

Two other theorems of Deligne (Deligne [3]) state that if a tensor category has a fibre functor over an extension of the base field, it has one over the algebraic closure over the field, and that, in characteristic 0, a fibre functor exists, provided the rank of each object (an element of the base field defined intrinsically for an object) is a natural number. As explained in Kamensky [11], both of these results can be formulated naturally in the model theoretic language. For example, the second result states that the theory associate with this tensor category is consistent. The first result would follow from quantifier elimination. In my future research, I hope to turn these observations into proofs, which I then expect will be easy to generalise to the other contexts discussed above.

2.2. **Compact complex manifolds.** This is a description of work I started with Rahim Moosa, from the University of Waterloo. A (reduced) complex analytic space $M$ can be viewed as a first order structure, with basic relations given by analytic subsets of $M$ and its powers. While in general the resulting theory $T_M$ is complicated, Zil′ber [21] noticed that if $M$ is compact, $T_M$ eliminates quantifiers and has finite Morley rank, and the structure itself is $\omega_1$-compact (every partial type given by countably many formulae is realised).

Let $M$ be a compact complex manifold. By the definition of $T_M$, every analytic subset of $M$ is definable without parameters. In particular, every $M$-point is 0-definable. It is natural to ask if there is a countable reduct of $T_M$ in which every such subset is definable with parameters from $M$. Moosa [13] showed that this condition is equivalent to the statement that each component of the *Douady space* of $M$ is compact. The Douady space $\mathcal{D}(M)$ is the analogue for complex spaces of the Hilbert scheme in algebraic geometry: it is a space whose points correspond to the compact subspaces of $M$.

A space $M$ is called *essentially saturated* if the above condition holds for every Cartesian power of $M$. In other words, $M$ is essentially saturated if there is a countable sub-theory $T$ of $T_M$ such that every 0-definable set in $T_M$ is definable in $T$ with parameters from $M$.

It is known that every component of the Douady space of a compact Kähler manifold is compact, and since the class of such manifolds is closed under products, a Kähler manifold is essentially saturated. More generally, holomorphic images of such manifolds (Kähler-type varieties) are again essentially saturated. On the other hand, there is an example of a space $H$ (a Hopf surface), such that the components of $\mathcal{D}(H)$ are compact, but the same is false for $H \times H$. For some time it was not known whether there are essentially saturated manifolds that are not of Kähler type. An example of such a manifold was given in Moosa et al. [16].

The aim of the present work is thus to study the class of essentially saturated spaces. Grauert et al. [6] define a class $\mathfrak{C}$ of compact complex spaces to be *geometrically stable* if it is closed under products, images of holomorphic maps and pre-images of Moishezon maps, and if the components of the cycle spaces of spaces in $\mathfrak{C}$ are compact (a map $f : X \to Y$ is Moishezon if it is bimeromorphic to a projective map. The assumption then says that if such an $f$ is surjective, and $Y$ is in $\mathfrak{C}$, then so is $X$). For example, the class of Kähler-type varieties is the smallest geometrically stable class containing the compact Kähler manifolds. It follows from these axioms, together with a theorem of Campana, that the components of the cycle spaces of spaces in $\mathfrak{C}$ are also in $\mathfrak{C}$.

A natural question is then: is the class of essentially saturated spaces geometrically stable? It follows from the model theoretic description that this class is closed under images of holomorphic maps, and by definition, the components of the cycle space of an essentially saturated space are compact. Both of the other conditions seem to follow from the following statement: if $f : P \to X$ is a projective bundle over an essentially saturated space $X$, then $P$ is essentially saturated. This is the statement we are trying to prove. It will follow, in particular, that the class of essentially saturated spaces is closed under bimeromorphisms.

2.3. **The Riemann–Hilbert correspondence.** In this section I describe some longer term plans, whose goal is to understand the Riemann–Hilbert correspondence through model theoretic techniques, and to related the algebraic theory of differential equations via $D$-modules to the model theoretic approach.

For a complex space $X$, the Riemann–Hilbert problem is concerned with the following kinds of objects:

- Local systems on $X$ (locally constant sheaves of finite dimensional complex vector spaces on $X$).
- Finite dimensional complex representations of $\pi_1(X)$
- (Algebraic) vector bundles on $X$, equipped with a flat connection

The equivalence between local systems and representations of the fundamental group holds already for (nice) topological spaces $X$. For the other equivalence, if $X$ is a complex manifold, and $M$ is a holomorphic vector bundle with a flat connection $\nabla$ on $X$, a local system is obtained by taking "horizontal sections" (the kernel of $\nabla$). This gives an equivalence between local systems and holomorphic flat connections. The equivalence follows from the local existence theorem for linear differential equations. If $X = Y^{an}$ is the complex manifold associated with a smooth projective algebraic variety $Y$, then the GAGA principle gives a further equivalence between the holomorphic data and algebraic vector bundles on $Y$, equipped with a flat connection.

If $Y$ is a smooth algebraic variety, a vector bundle on $Y$ with a flat connection is the same as an $\mathcal{O}_Y$-coherent $D$-module over $Y$, i.e., a module over the algebra $D_Y$ of differential operators on $Y$, which is coherent over the sub-algebra $\mathcal{O}_Y$ of regular functions on $Y$ (or the analogous "sheafified" objects when $Y$ is not affine).

If $M$ is a $D_Y$-coherent $D_Y$-module, any set of generators for $M$ determines a system of linear differential equations with polynomial coefficients (given by the relations among the generators). If $M$ is actually $\mathcal{O}_Y$-coherent, these relations are generated by relations that involve each vector field separately: given a differential operator $v$ and a generator $e$, $v(e)$ can be expressed as an $O_Y$-linear combination of the generators.

Thus, to any algebraic vector field on $Y$, the module $M$ (with the specified generators) associates an ordinary linear differential equation. In other words, an $O_Y$-coherent $D$-module with a given set of generators can be viewed as determining either a definable set in a theory of differential fields with several commuting derivations, or a family of definable sets in DCF. Changing the set of generators amounts changing the definable sets by a 0-definable bijection. Thus $M$ itself corresponds to a definable isomorphism class.

My goal is to determine more precisely the relation between $\mathcal{O}$-coherent $D$-modules and definable sets in DCF, and describe model theoretically (some aspects of) the Riemann–Hilbert correspondence. For example, the analytic part of the correspondence appears to imply that local systems correspond to definable finite dimensional vector spaces over the constants. Thus some version of the Riemann–Hilbert correspondence can be viewed as part of quantifier elimination for DCF.

From another point of view, one may try to recover the connection between linear differential equations and the fundamental group. In the context of algebraic geometry, the topological fundamental group cannot usually be recovered; at most, one recovers an algebraic analogue, such as its pro-finite completion. However, in the context of model theory, the topological fundamental group can be recovered within a theory of real-closed fields. Thus, it is possible that the correspondence can be recovered inside a theory of real-closed fields with derivations.

The Riemann–Hilbert correspondence can be extended to the case of singular varieties (cf. Borel et al. [1]). This requires a suitable definition of $D$-modules over a variety with singularities $Y$. Such a definition exists, but the $D$-modules are no longer modules over the algebra of differential operators on $Y$. To work with them, one must relate such "modules" to $D$-modules on locally closed subsets of $Y$, and on smooth varieties containing $Y$. In other words, one has to define operations of restriction and extension of $D$-modules along algebraic morphisms. One of the problems with these operations is that they do not, in general, preserve coherence. The class of $\mathcal{O}$-coherent $D$-modules is thus replaced by the larger class of *holonomic* $D$-modules, which is preserved under all operations. An interesting problem is to expand the correspondence between coherent modules and definable sets to the class of holonomic $D$-modules. I believe this will require understanding the restriction and extension operations (at least for a certain class of morphisms).

Another problem with the extension of the Riemann–Hilbert correspondence is related to the fact that if $Y$ is not projective, the GAGA principle can no longer be used (directly). The solution is to consider $D$-modules on a compactification of $Y$, that have "regular singularities" on the complement of $Y$. The definition of regular-singular modules is via reduction to the case of curves, so it is given in the framework of one derivation. On the other hand, it is concerned with the order of the singularity on the complement, so the notion can possibly be interpreted within a theory of valued differential fields.

I would like to mention one possible motivation for this study. The full version of the Riemann–Hilbert correspondence asserts that there is an equivalence between regular-singular holonomic $D$-modules and perverse sheaves (cf. Borel et al. [1]), which play the role of local systems in the original correspondence. Although the theory of perverse sheaves exists in any characteristic, the theory of $D$-modules is only in characteristic 0. Hopefully, a model theoretic interpretation can lead to an analogous interpretation in positive characteristic.

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