

MODEL THEORY OF TANNAKIAN CATEGORIES

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In this lecture I explain how the model theoretic notion of internality leads to a quick and easy proof of the basic theorem in the Tannakian formalism. All the model theory appearing here is available in [Hru06], while a good account of the Tannakian formalism is in [DMOS82].

1. INTERNALITY

17 minutes

We are interested in the following situation: T_1 is a theory expanding a theory T by additional sorts. There is a sort X of T_1 such that for any sort Q of T_1 there is a sort C_Q of T , and a map $f_Q : Q \times X \rightarrow C_Q$, such that each fibre $f_{Q,x}$ is injective. In this situation, T_1 is said to be internal to T (the usual notion of internality is slightly more general than this, but for our purposes this is enough.) We note that though, over enough parameters, every sort is isomorphic to a sort of T , the interesting case is when this does not happen over 0.

Example 1. Let $T = ACF$, and let T_1 extend T by a single additional sort V , which stands for a vector space over the field, of a given finite dimension. A basis of V over the field (which is a tuple of elements of V) gives rise to a bijection of V with a power of the field. However, no such bijection exists over 0.

For any model M_1 of T_1 , we let $Aut(T_1/T)(M_1)$ be the group of automorphisms of M_1 fixing the sorts in M pointwise. A basic theorem, due to Poizat ([Poi83]) in the stable case, and to Hrushovski ([Hru02]) in general, says that $Aut(T_1/T)$ is an (infinite intersection of) group definable in T_1^{eq} : There is a definable group G , and a definable group action on every sort Q of T_1 , which identifies, for each model M_1 of T_1 , the elements of $Aut(T_1/T)(M_1)$ with $G(M_1)$. It follows that the action of G on X is free, and therefore, X contains a definable torsor (principle homogeneous space) for G . We note that T_1^{eq} is again internal to T .

Example 2. In the example above, $Aut(T_1/T)(M_1)$ is the group of linear automorphisms of V over the field. All such automorphisms are obtained by composing two of the bijections provided by the set of bases X . It is, thus, a quotient of $X \times X$. The set X itself is a torsor over the group.

The existence of a torsor gives rise to the following Galois correspondence: let $a \in X$, and let B be a definably closed set containing a (as usual, dcl is computed in T_1^{eq} .) If H is a B -definable subgroup of G such that G fixes all elements of B fixed by H , then $G = H$. The proof is by considering the class of a in X/H .

Given a T_1 automorphism of a model M_1 , we may consider its restriction to M . This restriction is a T -automorphism of M . If T_1 induces no new definable sets on T , and furthermore, the sorts of T are *stably embedded* in T_1 , then this restriction map is surjective. We note that if there is no new structure on T , and T is stable,

then so is T_1 , and in this case every set is stably embedded. In this case we thus get an exact sequence

$$(1) \quad 0 \rightarrow G(M_1) \rightarrow \text{Aut}(M_1) \rightarrow \text{Aut}(M) \rightarrow 0$$

If, furthermore, there is an element $a \in X$ such that $dcl(a) \cap M = 0$, we get a splitting of this sequence, by mapping an automorphism τ of M to the unique automorphism of M_1 extending it and fixing a (any 0-definable set of T_1 is mapped into 0 by a , hence is fixed by τ .) We shall call such an element a *splitting element*. Given such a splitting element a , we may associate with any definably closed set B of T , a definably closed set $B_a = dcl(B, a)$ of T_1 , whose restriction to T is B . In particular, if B is a model of T , B_a is a model of T_1 . In particular, we may consider, for any model M of T , the group $G_a(M) = G(M_a)$. This group is a definable group in T : it is precisely the group corresponding to G under the bijection a . As a matter of fact, $G_a(B) = G(B_a)$ is the automorphism group of B_a over B (this follows from the fact that B_a is definably closed, and contains the element a of X .)

2. AFFINE GROUP SCHEMES AND REPRESENTATIONS

17 minutes

An (affine) group scheme over a field \mathbb{k} is an affine scheme that has a group structure. Thus, an affine group scheme G corresponds to a \mathbb{k} -algebra A , and for every \mathbb{k} -algebra B , the set of B points of G , $\text{Hom}_{\mathbb{k}}(A, B)$ has a group structure, functorially in B . G is algebraic if A is finitely generated (as a \mathbb{k} -algebra.) It is reduced if A is reduced. It is a fact ([Wat79]) that in characteristic 0 any affine group scheme is reduced. A map of group schemes is a map of schemes that respects the group structure.

As an example, given a finite dimensional vector space v over \mathbb{k} , the group $GL(V)$ assigning to each \mathbb{k} -algebra B the group $\text{Aut}(B \otimes_{\mathbb{k}} V/B)$ is an affine group scheme (represented by $S^*(V \otimes V^*)_{det}$.) A *representation* of a group scheme G is a map from G to some $GL(V)$. By abuse of notation, we then say that V itself is a representation.

The Tannakian formalism gives a duality theory between affine group schemes and their categories of representations. Given G , its category of representations has the following structure: it is an abelian category, the tensor product of two representations is again a representation, the tensor product operation has a unit $\mathbf{1}$, satisfying $V \otimes \mathbf{1} \xrightarrow{\sim} V$ for any representation V (this is the trivial representation), and the dual of a representation is again a representation. It also has a faithful functor F into the category of (finite dimensional) \mathbb{k} -vector spaces (the forgetful functor), and this functor respects the tensor structure. A *tensor automorphism* of F is an automorphism of F that respects the tensor structure. More generally, given a \mathbb{k} -algebra B , we may consider the group of tensor automorphisms of the functor $F \otimes_{\mathbb{k}} B$. We thus get a functor $B \mapsto \text{Aut}^{\otimes}(F \otimes_{\mathbb{k}} B/B)$ from \mathbb{k} -algebras to groups. The first part of the Tannakian formalism states that this functor is isomorphic to G . In other words, we may recover the group scheme G from its category of representations, viewed with its tensor structure, together with the fibre functor.

The second part describes the categorical structure of the category of representations of a group scheme. It states that any category with the above structure (where the tensor product is replaced by an abstract bi-functor) is the category of representations of some group scheme. In other words, that the functor Aut^{\otimes}

described above is always representable. Such a category (together with the fibre functor) is called a *neutralised Tannakian category*.

We shall describe a model theoretic proof of this theorem in the case when \mathbb{k} is perfect and the group is reduced. We note that this includes the general case in characteristic 0. To do this, we need to mention some reductions and facts. First, any group scheme is a projective limit of algebraic ones (this can be seen model theoretically, or via Hopf algebras.) It follows that it is enough to prove the theorem for the algebraic case. An algebraic group always has a faithful representation X , and this faithful representation is a *tensor generator*, i.e., any other representation is isomorphic to a sub-quotient of a direct sum of tensor powers of X (all this is only done for simplicity, and to be consistent with the simplified definitions we used.)

A basic fact we require ([Poi83]) is that a definable group in ACF can be identified with a unique reduced algebraic group.

(proof of uniqueness, in the char. 0 case: we show that any functorial group isomorphism between algebraic groups $G(L) \rightarrow H(L)$, where L is a field, comes from an actual isomorphism. By Weil's group chunk theorem, it is enough to show that this gives an isomorphism $K(G) \rightarrow K(H)$. Setting $L = K(G)$, we get a dominant map from an open set U of G to a subvariety of H , with algebra A . The map from $\mathcal{O}(H)$ to $K(A)$ corresponds to a map $\mathcal{O}(G) \rightarrow K(A)$, which gives, when composed with the map $K(A) \rightarrow K(G)$, the embedding. Thus, $K(A)$ is isomorphic to $K(G)$, and in fact, $A = \mathcal{O}(H)$.)

3. THE THEORY ASSOCIATED WITH A TANNAKIAN CATEGORY

15 minutes

Let \mathcal{C} be a neutral Tannakian category over a perfect field \mathbb{k} , generated (as a tensor category) by one object (for example, the category of representations of an algebraic group.) We consider the following theory T_1 :

- (1) For any object X of \mathcal{C} , there is a sort V_X .
- (2) $V_{\mathbf{1}} = K$, where $\mathbf{1}$ is the trivial representation, is an algebraically closed field, with \mathbb{k} as constant elements.
- (3) For any object X , V_X is a vector space over K , of dimension $rk(X)$ (if the characteristic is positive, we pick a fibre functor, and use it to define a dimension.)
- (4) For any morphism $f : X \rightarrow Y$ in \mathcal{C} , there is a function symbol, also denoted by f , for a linear map from V_X to V_Y . The axioms ensure that the category structure is reflected by these maps.
- (5) For any two objects X and Y , there is a bilinear map $b_{X,Y} : V_X \times V_Y \rightarrow V_{X \otimes Y}$, inducing, for any model, an isomorphism $V_X \otimes V_Y \rightarrow V_{X \otimes Y}$. If $\psi : V_{X \otimes Y} \rightarrow V_{Y \otimes X}$ is the map induced by switching the coordinates, we require that $\psi \circ b_{X,Y} = b_{Y,X}$, and similarly for associativity.
- (6) The map

$$(2) \quad K \times K \xrightarrow{b_{\mathbf{1},\mathbf{1}}} V_{\mathbf{1} \otimes \mathbf{1}} \xrightarrow{\sim} K$$

coincides with the field multiplication.

Specifying a fibre functor for \mathcal{C} amounts to specifying a definably closed subset A of T_1 whose restriction to T is \mathbb{k} , and which contains a basis for each vector space: Given such an A , $X \mapsto V_X(A)$ together with the maps b is a fibre functor (it is faithful since $V_X(A)$ is of the right dimension.) Conversely, given a fibre functor F , let L be an algebraic closure of \mathbb{k} . Then $F(X) \otimes_{\mathbb{k}} L$ gives a model of T_1 , and any

automorphism of L extends to an automorphism of $F(X) \otimes L$ fixing $F(X)$, so the collection of $F(X)$ is definably closed. In particular, the existence of a fibre functor implies that T_1 is consistent.

The assumption that \mathcal{C} has a tensor generator implies that T_1 is internal to $T = ACF_{\mathbb{k}}$: any basis of V_X , where X is a tensor generator, can be used to internalise all V_Y : By replacing X with $X \oplus \check{X}$, we may assume that X is isomorphic to \check{X} . If v is a basis of V_X , if v internalises some U it also internalises any subspace, and since X is self-dual, any quotient. Similarly, if v internalises two spaces, it internalises their sum, and via b their tensor product and dual.

It follows from the basic result on internality that the automorphism group of T_1 over T is definable. It also follows from the description of fibre functors above, that any fibre functor gives rise to a splitting element. Thus, using the fibre functor we may identify the automorphism group with a T -definable group G , which, by the result cited above, is algebraic.

As explained above, the existence of a splitting element implies that T_1 induces no new structure on T . It now follows from [Hru06] that T_1 eliminates imaginaries once we expand the theory by all the projective spaces $P(V_X)$. Thus, given a fibre functor F , subgroups of G are determined uniquely by the set of points and lines they fix in each $F(X)$. Now assume that \mathcal{C} is the category of representations of a reduced algebraic group H . Since X is a faithful representation of H , H can be presented as a subgroup of $GL(X)$. Since every element of H (over any field L) acts as an automorphism (over L), and H is reduced, H is a subgroup of G . Thus, to show that H coincides with G , it is enough to show that any element of T_1^{eq} fixed by H belongs to $dcl(0)$. However, such elements are precisely fixed vectors and fixed subspaces (i.e., sub-representations) of H , and thus are encoded in the category structure.

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