OVERVIEW OF SOME MOTIVIC INTEGRATION

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1. Motivation

30 minutes

Motivic integration originated from uniformity phenomena when computing p-adic integrals, so we start by reviewing these first. The reference to the whole first lecture (and part of the second) is [Loe03] or [Loe06].

15 minutes

1.1. **p-adic integration.** We work over a (non-Archemedian) local field L of characteristic 0, with residue field $\mathbb{F}_q$, valuation ring $\mathcal{O}$, and maximal ideal $\wp$. L^n is, thus, a locally compact topological group, with a Haar measure μ_n, determined uniquely by the normalisation $\mu_n(\mathcal{O}^n) = 1$. We thus may integrate functions $f : L^n \to \mathbb{R}$, and various zeta functions tend to be expressible as such integrals.

Example 1. Let $f : L^n \to L$ be a polynomial function over $\mathcal{O}$. Then

$$\int_{\mathcal{O}^n} |f(x)|^s \mathrm{d}x = \sum_{i\geq 0} \mu_n(\{x|val(f(x)) = i\})q^{-is} \tag{1}$$

Now, for each $i \geq 0$, $\mu_n(\{x|val(f(x)) \geq i\})$ is the number N_i of solutions in $\mathcal{O}/\wp^i$ (a finite ring) of $f(x) = 0$, multiplied by the measure of the set giving such a solution, which is q^{-in}. Since

$$\mu_n(\{x|val(f(x)) = i\}) = \mu_n(\{x|val(f(x)) \geq i\}) - \mu_n(\{x|val(f(x)) \geq i+1\})$$

We get

$$\begin{aligned}\int_{\mathcal{O}^n} |f(x)|^s \mathrm{d}x &= \sum_{i\geq 0}(N_i q^{-in} - N_{i+1}q^{-(i+1)n})q^{-is} = \\ &= 1 + \sum_{i\geq 1} N_i(q^{-in-is} - q^{-in-(i-1)s}) = \\ &= 1 + (1-q^s)\sum_{i\geq 1}\frac{N_i}{q^{i(n+s)}}\end{aligned} \tag{2}$$

Substituting $T = q^{-n-s}$ in this expression we get (approximately) a zeta function for the N_i (the *Poincaré series.*)

Example 2. Taking $f(x) = x$ in the last example, we get $n = 1$, $N_i = 1$, hence

$$\int_{\mathcal{O}} |x|^s \mathrm{d}x = 1 + (1-q^s)\sum_{i\geq 1}\frac{1}{q^{i(1+s)}} = \frac{1-q^{-1}}{1-q^{-1-s}} \tag{3}$$

In particular, it is a rational function of q^s.

More generally, Igusa ([Igu00]) managed to prove that the integral in example 1 is alway a rational function of q^s. The idea is that if the variety defined by f is smooth, then it can be viewed as a manifold, and in local coordinates, f looks like example 2. If the variety is not smooth, one uses resolution of singularities first (this is the place where $char(L) = 0$ is used) to reduce to the smooth case. Both of these steps use the fact that there is a change of variable formula for p-adic integration.

Example 3. Instead of N_i, we may consider the number $\tilde{N}_i$ of solutions in $\mathcal{O}/\wp^i$ that lift to a solution in $\mathcal{O}$. These numbers can be described in terms of measures as follows: The pullback to $\mathcal{O}^n$ of the set of points contributing to $\tilde{N}_i$ is the set of points of distance at most q^{-i} from the variety V defined by f. The zeta function for these numbers can therefore be expressed in terms of the integral

$$\int_{\mathcal{O}^n} |d(x, V)|^s \mathrm{d}x \tag{4}$$

where $d(x, V)$ is the distance of a point x from V.

This example shows that there are interesting functions produced by p-adic integrals that involve non-analytic functions. Which such integrals give rise to rational functions? In [Den84], Denef proved:

Theorem 4. *The series* $\sum n_i T^i$ *is rational whenever* n_i *is obtained as an integral*

$$n_i = \int_{X_i} p^{-\alpha(x,i)} \mathrm{d}x \tag{5}$$

with X_i *and* $\alpha(x, i)$ *definable in the theory of* $\mathbb{Q}_p$, $\alpha(x, i) \geq 0$ *(Here* i *is a multi-index.)*

The theory of $\mathbb{Q}_p$ is expressed in a two sorted language, for the field and for the value group, with additional symbols for divisibility. The central axiom of $\mathbb{Q}_p$ in this language is that it is Henselian. The main ingredient in the proof of Denef's theorem is the fact, due to Macintyre, that Q_p eliminates quantifiers. Note that this theorem implies both examples above.

Remark 5.

1. This theorem has applications for various other zeta functions, for example counting subgroups of nilpotent groups.
2. There is no analogue of this result for characteristic $p > 0$, since there is no known analogue of Macintyre's theorem.

10 minutes

1.2. **The topological zeta function.** Let $f(x_1, \dots, x_n)$ be a polynomial over $\mathbb{Z}$. Pulling back to $\mathcal{O}$, where $\mathcal{O}$ is the valuation ring of an unramified extension of $\mathbb{Q}_p$, we consider example 1 again. Let $D = f^{-1}(0)$ be the variety defined by f. As mentioned above, there is a change of variables formula that gives rise to a connection between $\int_{\mathcal{O}^n} |f(x)|^s \mathrm{d}x$ and the same integral applied to a resolution of singularities of D. Let $\pi : E \to D$ be such a resolution. Then this formula has the following explicit form:

$$\int_{\mathcal{O}^n} |f(x)|^s \mathrm{d}x = q^{-n} \sum_{I \subseteq J} \sharp E_I(\mathbb{F}_q) \prod_{i \in I} \frac{(q-1)q^{-(N_i s + \nu_i)}}{1 - q^{-(N_i s + \nu_i)}} \tag{6}$$

Where J is the set of irreducible components of E, for each subset $I \subseteq J$, E_I is some boolean combination of the elements of I, and N_i, ν_i satisfy $E = \sum_{i \in J} N_i i$ and $\omega_Y = \omega_{\mathbb{A}^n} + \sum_{i \in J} (\nu_i - 1)i$, with Y the space containing E.

We note the following two features of the above formula: The left hand side does not depend on the resolution of singularities, and the right hand side is (almost) uniform in q. This observation led Denef and Loeser (in [DL92]) to consider "taking the limit" $q = 1$. This should thus yield a function where q and integration no longer appear, but which is independent of the resolution of singularities, and is thus an invariant of D (and the space $\mathbb{A}^n$ where it is embedded.)

The main obstacle is interpreting $\sharp E_I(\mathbb{F}_q)$ as uniformly dependent on q. This is done as follows: by the Weil conjectures, this number is the alternating sum of the traces of Fr^k, where $q = p^k$, acting on the l-adic cohomology (with compact support) of E_I. Therefore, when $k = 0$, $\sharp E_I(\mathbb{F}_q)$ should be replaced by the Euler characteristic of E_I, which coincides with the topological Euler characteristic when E_I are interpreted over $\mathbb{C}$. This approach works, in fact, yielding the topological zeta function,

$$Z_{top}(X, D)(s) = \sum_{I \subseteq J} \frac{Eu(E_I)}{\prod_{i \in I}(N_i s + \nu_i)} \tag{7}$$

where X is a smooth algebraic variety, and D a divisor on X (in the example above, $X = \mathbb{A}^n$), which is independent of the resolution.

Thus, a topological result, the independence of Euler characteristic on the choice of resolution is proved by means of p-adic integration. Results of this sort brought Kontsevich ([Kon95]) to suggest, that such equations should hold already in the Grothendieck ring of algebraic varieties, and there should be an integration theory for functions into this ring.

2. GROTHENDIECK GROUPS AND RINGS

10 minutes

A common property of key concepts we considered so far is additivity: One assigns numbers n_X to sets X in such a way that if X is the disjoint union of X_1 and X_2, then $n_X = n_{X_1} + n_{X_2}$. This is true for counting points and for the Euler characteristic. Such an assignment is called an additive invariant. The result of the last section can be summarised by saying, that there is an expression, associated with D, which allows us to compute these additive invariants. Formally, this expression lies in a universal ring of additive invariants, called the Grothendieck ring.

The Grothendieck group can be defined for any category with a concept of decomposition of objects. For example, given an abelian category, we form the free abelian group on isomorphism classes of objects, and then divide by the relation $[B] = [A] + [C]$, whenever there is an exact sequence $0 \to A \to B \to C \to 0$. If the category has a product structure (for example, tensor product), the Grothendieck group has a corresponding ring structure.

In our case, we are interested in the category of algebraic varieties (over a fixed ring k). The Grothendieck group $K_0(Var_k)$ is thus defined as isomorphism classes of varieties, divided by the relation $[X] = [Z] + [U]$, where $Z \subseteq X$ is a closed sub-variety, and U is its complement. The Cartesian product of varieties gives rise to the ring structure.

More generally, for any first order theory, the Grothendieck group of definable sets is defined as the free abelian group generated by definable sets up to definable bijections, divided by the relation $[X] = [Z] + [U]$, where X is the disjoint union of Z and U, and ring structure as before. If K is a perfect field, we recover the previous example as the Grothendieck group of ACF_K the theory of algebraically closed fields containing K. This follows from elimination of quantifiers in ACF_K.

The Grothendieck group (or ring) of a category is thus equipped with a canonical additive invariant, which is universal by construction.

Remark 6. There is a theorem of Bittner ([Bit04]) that if k is a field of characteristic 0, then $K_0(Var_k)$ is generated by projective varieties and blow-up relations. This means that an additive invariant is determined by specifying it on projective varieties, provided it respects blow-up relations. In particular, it follows that there is a well defined homomorphism from $K_0(Var_k)$ into the Grothendieck group of the category of Chow motives.

3. MOTIVIC INTEGRATION 25 minutes

We now describe the realisation of the idea of integration with values in a Grothendieck group in the works of Denef-Loeser and Hrushovski-Kazhdan.

10 minutes

3.1. The motivic topological zeta function.

In [Loe03], the theory of integration is defined for functions on arc spaces. This allows one to work only with the Grothendieck group of algebraic varieties. With a view to the future (of the lecture), we shall formulate the result in terms of the Grothendieck group of a first order theory, namely, the theory of $k[[t]]$, $char(k) = 0$ in the Denef-Pas language. This language has sorts for the valued field, residue field and the value group, as well as an angular component map $ac : K^* \to k^*$. This theory eliminates quantifiers in the value field sort ([Pas89]).

Let $\mathbf{L}$ be the class of $\mathbb{A}^1$ in $K_0(Var_k)$ (k a field of characterisitc 0, which, for simplicity, we assume to be algebraically closed.) Measure and integrals of definable sets will take values in a ring $\mathcal{M}$ which is a localisation of $K_0(Var_k)$. Let V be a valued field of residue characteristic 0. We would like to assign a measure to definable subsets in such a way that for any closed subset of X of k^n,

$$\mu_n(\pi^{-1}(X)) = \mu_n(\wp^n)[X] \tag{8}$$

We also want to have, as in the p-adic integration, that $\mu_n(\mathcal{O}^n) = 1$. This immediately forces that $\mu(\wp) = \mathbf{L}^{-1}$. Thus $\mathbf{L}$ should be inverted. Similarly to allow for convergent series, one should invert $1 - L^{-i}$, for all i. This localised ring is $\mathcal{M}$, and the measure extends in a natural way to a measure on all definable sets into $\mathcal{M}$.

Similarly, one defines, for definable set A and function $\alpha : A \to \mathbb{Z}$

$$\int_A \mathbf{L}^{-\alpha} \mathrm{d}\mu = \sum_n \mu(A \cap \alpha^{-1}(n))\mathbf{L}^{-n} \tag{9}$$

and $\mathbf{L}^{-\alpha}$ is called integrable if the above sum converges (i.e., is an element of $\mathcal{M}$.) More generally, L can be replaced with an elements of an $\mathcal{M}$-module. In particular, given a definable function $f : \mathcal{O}^n \to \mathcal{O}$, setting $A = \mathcal{O}^n$, $\alpha(x) = val(f(x))$ and substituting $T = \mathbf{L}^{-s}$ instead of $\mathbf{L}$, we get

$$\begin{aligned}\int_{\mathcal{O}^n} |f(x)|^s \mathrm{d}x &= \int_{\mathcal{O}^n} \mathbf{L}^{-sval(f(x))} = \sum_m \mu_n(val(f(x) = m))T^m \\ &= \sum_m [X_m]\mathbf{L}^{-mn}T^m\end{aligned} \tag{10}$$

where X_m is the definable subset of $\mathcal{O}/\wp^m$ consisting of simple zeroes of f.

One of the main ingredients in the definition of the topological zeta function was the change of variable formula of p-adic integration. A similar formula exists in the motivic case. This allowed Denef and Loeser to deduce the following motivic analogue:

$$\sum_{m\geq 0}[X_m]\mathbf{L}^{-mn}T^m = \mathbf{L}^{-n}\sum_{I\subseteq J}[E_I]\prod_{i\in I}\frac{(\mathbf{L}-1)\mathbf{L}^{-\nu_i}T^{N_i}}{1-\mathbf{L}^{\nu_i}T^{N_i}} \tag{11}$$

with the notation defined above.

The topological zeta function, as well as the p-adic analogues now follow by setting $T = \mathbf{L}^{-s}$.

15 minutes

3.2. **The results of** [HK05]**.** Going back to equation 1, we see that its uniformity translates to the motivic equation 10. The same formula is preserved when taking *unramified* field extensions, since these correspond to residue field extensions. However, these formulas become incorrect for ramified extensions. The reason is that the theory as it is is only uniform in the residue field part.

To solve this problem, as well as some lesser problems in the original theory, Hrushovski and Kazhdan developed in [HK05] a theory of integration in valued fields which is geometric in all parts. This is done by replacing the theory of Henselian fields with (a variant of) $ACVF$.

The language used in [HK05] consists of two "official" sorts: a sort VF for the valued field K, and a sort RV for $K^*/(1+\wp)$. This sort contains the multiplicative group of the residue field k, and the quotient is the value group. The language contains a symbol for k^* in RV, a partial ordering on RV, the pullback of the order on the value group Γ, and all the abelian group structures.

The paper studies the relationship between (several variants of) the Grothendieck groups of definable subsets of VF, RV, k (and some finite dimensional vector spaces over it; this replaces the angular component map in the Denef-Pas language), and Γ. As explained above, motivic integration roughly amounts, in the discrete case, to giving a map from the Grothendieck group of the valued field, to an algebra over the Grothendieck ring of the residue field. This algebra was specified by inverting some elements of the ring in a rather ad-hoc way. It turns out that this is described canonically via the Grothendieck group of the value group.

Let $rv : VF^x \to RV$ be the projection. Any element X of $K(RV)$ gives rise to an element $rv^{-1}(X)$ in $K(VF)$ (when the groups are suitably defined.) The inverse image of $1 = rv(1) \in RV$ is $1+\wp$, while the inverse image of the positive part of Γ is $\wp \setminus 0$. Since translations are definable bijections, this shows that $rv^{-1}(1) - rv^{-1}(\Gamma_{>0}) = 1$ in $K(VF)$, where 1 is the singleton set, identified with $rv^{-1}(*)$, where $* = RV^0$. Let I_{sp} be the ideal generated by this relation. The first result is:

Theorem 7 (Theorem 8.8)**.** *The homomorphism* rv^{-1} *is surjective, with kernel* I_{sp}*.*

The next step is to analyse $K(RV)$. As before, there is a map $val^{-1} : K(\Gamma) \to K(RV)$. The inclusion of Res (the collection of the residue field and Γ-rational vector spaces over it) in RV gives another map. Thus we have a map $K(Res) \otimes K(\Gamma) \to K(RV)$. The class of 0 in Γ and of k^x are the same in $K(RV)$. More generally, any finite subset Γ is identified in RV with a subset of Res. Thus we get a map $K(Res) \otimes_{K(\Gamma^{fin})} K(\Gamma) \to K(RV)$, where $K(\Gamma^{fin})$ is the Grothendieck group of finite subsets of Γ.

Theorem 8 (Corollary 10.3)**.** *This last map is an isomorphism*

Combined together, these two theorems give an isomorphism μ from $K(VF)$ to $K(Res) \otimes_{K(\Gamma^{fin})} K(\Gamma)/I_{sp}$. This isomorphism can be viewed as motivic integration, in the same way as before. In particular, for a polynomial f, we get, setting $|f| = \mathbf{L}^{-val(f)}$, the following informal expression

$$\int_{\mathcal{O}^n} |f|^s \mathrm{d}x = \sum_{\gamma \geq 0} \mu(val(f) = \gamma)\mathbf{L}^{-s\gamma} \tag{12}$$

Where the expression $\mu(val(f) = \gamma)$ is a finite sum of elements of the form $[E] \otimes \Delta$, with E the class of a constructible set (over the residue field), and Δ the class of a definable subset of Γ. Rearranging terms, this recovers a formula as in the motivic zeta function, with the explicit rational function in T replaced with a uniform expression in terms of a definable subset of Γ (Theorem 1.3 of [HK05])

4. APPLICATIONS

20 minutes

In this section I list some applications, most of which appear in the paper.

8 minutes

4.1. **Passage to subfields.** Before discussing most application it is important to understand how the results described above can be applied to non-algebraically closed valued fields. This is done in section 12 of [HK05].

Restriction to a subfield F may mean two different things: The first is to assign volume to definable sets where quantifiers are interpreted in the theory of F. The second is to consider F rational points.

For the first, assume we have a language obtained by adding relations to the RV sort only. It turns out that elimination of quantifiers reduces to elimination of RV quantifiers. In other words, any formula is equivalent to one with no value field quantifiers. As a corollary, there is still a surjective integration map from $K(VF)$ to $K(RV)/I_{sp}$ (which need not be injective, since the new structure on RV may introduce bijections not liftable to VF.) The example of discrete valuation fields is obtained in this way by adding a relation for a subgroup of Γ, satisfying the theory of $\mathbb{Z}$, and then adding relations for the subgroups of finite index to obtain elimination of quantifiers.

The second point concerns with the fact that two definable sets can have the same points in a given valued field F without being equal as definable sets. For example, the open ball of radius 1 and the closed ball of radius γ are very different as definable sets, but their points are the same in any field where γ is the immediate successor of 1. If F is a definably closed substructure such that rv restricted to F

is surjective, then the integration map induces integration on the level of F points. This condition on F translates to the statement that F is Henselian. In particular, the quantifier elimination in the Denef-Pas language now follows from these two statements.

4 minutes

4.2. **The Ax-Kochen-Eršov principle.** The Ax-Kochen-Eršov theorem is a special case of the above theory. It states:

Theorem 9 (Ax-Kochen-Eršov principle)**.**
(1) *The theory of an Henselian valued field of residue characteristic* 0 *is determined by the theories of the residue field and the value group.*
(2) *For almost all p, any sentence (in the language of valued fields) holds in $\mathbb{Q}_p$ if and only if it holds in $\mathbb{F}_p((t))$.*

The second part follows from the first by compactness, since $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ have the same residue fields and value groups. A special case of the first part (usually proved using Galois theory) is that a valued field (of residue characteristic 0) with algebraically closed residue field and divisible value group is algebraically closed.

We give an application of the second statement to a conjecture of Artin. A field F is called C_i if any homogeneous polynomial of degree d has a non-trivial zero, provided the number of variables n is bigger than d^i (thus, a field is C_0 iff it is algebraically closed.) Chevalley proved that any finite field is C_1, and Lang ([Lan52]) proved that if F is C_i, then $F((t))$ is C_{i+1}. The conjecture of Artin was that any local field is C_2, and Lang thus proved it for function fields. It now follows asymptotically for $\mathbb{Q}_p$ from the above theorem (and it is known to have exceptions, depending on d.)

3 minutes

4.3. **The grothendieck group of $\mathbb{Q}_p$.** Recall that the theory pCF of $\mathbb{Q}_p$ ("p-adically closed fields") eliminates quantifiers in the language with relations for finite index subgroups of the value group. Since this does not give any definable function on RV, we get by restriction to subfields an isomorphism $K(pCF) \rightarrow K(RV_{pCF})/I_{sp}$. However, RV_{pCF} is a finite extension of $\mathbb{Z}$, and $K(\mathbb{Z})$ is 0. Hence $K(pCF)$ is 0.

4 minutes

4.4. **The grothendieck group of ACF.** The following theorem is proved in section 13 of [HK05]:

Theorem 10. *Let X, Y be two smooth d-dimensional subvarieties of a smooth projective n-dimensional variety V, and assume $V \setminus X$, $V \setminus Y$ are isomorphic. Then $X \times \mathbb{A}^{n-d}$, $Y \times \mathbb{A}^{n-d}$ are birationally equivalent. If X, Y contain no rational curves, then X, Y are birationally equivalent*

The last statement is a corollary of the first, that does not use motivic integration. In the case where X, Y are complete curves, it follows that X and Y are isomorphic.

The proof works by considering the base field as a valued field with trivial valuation. Hence varieties can be viewed either over the value field or the residue field. Since V is projective, $V(VF) = V(\mathcal{O})$, so we have a map $\rho_V : V(VF) \rightarrow V(k)$. The isomorphism $F : V \setminus X \rightarrow V \setminus Y$ gives rise to a bijection $V(VF) \setminus X(VF) \rightarrow V(VF) \setminus Y(VF)$, which restricts to a bijection $\rho_V^{-1}(X) \setminus X(VF) \rightarrow \rho_V^{-1}(Y) \setminus Y(VF)$. This bijection gives rise to an identity in $K(Var)$, from which the theorem is deduced.

15 minutes

5. Ingredients of proof

The key to all the results is a detailed analysis of the definable sets in $ACVF$, and the relations between them. Some of the key ideas underlying the analysis are reviewed below.

6 minutes

5.1. **Elimination of Imaginaries.** As mentioned above, the theory is considered in the language with two sorts, VF and RV. Nevertheless, we keep referring, e.g., to the value group Γ. The reason is that Γ is canonically determined by the data of the theory in the given language. Γ is thus an example of an *imaginary sort.* In general, Given a definable set X in an arbitrary theory, and a definable equivalence relation E on it, the quotient is determined canonically, and, together with the projection from X is termed an imaginary sort in the theory. The definable subsets of imaginary sorts are those induced by definable sets on the pre-images of the projections. Thus the original theory extends canonically to a theory in the extended language, with equivalent categories of models. A theory (in a prescribed language) is said to eliminate imaginaries, if any imaginary sort is definably isomorphic to one in the prescribed language.

Thus, $ACVF$ consists of some additional sort which are missing from the explicit definition. It is essential to describe at least some of them in order to understand the definable sets. Here are some of them: Given two elements $c \neq u \in VF$, let $B(c, u)$ be the smallest closed ball containing both c and u.Define $(c_1, u_1)E(c_2, u_2)$ if $B(c_1, u_1) = B(c_2, u_2)$. This is a definable equivalence relation, and the quotient is the set of closed balls in VF. Restricting to $c = 0$, we get the set of closed ball around 0, which is identified with the value group. Similarly, it is possible to defined the set of open balls, the set of their quotients (a family of vector spaces over the residue field), and their higher dimensional analogues. In algebraic terms, these correspond to $\mathcal{O}$-lattices and their torsors, and the reductions to the residue field. It is the main result of [DHM] that with these sorts, $ACVF$ eliminates imaginaries.

7 minutes

5.2. **Minimality.** Let $T_0 \subseteq T$ be a restriction of T to a sub-language. T is said to be T_0-minimal if every unary definable subset in T, possibly with parameters, is definable in T_0. When T_0 is simple enough, this property may have very strong consequences. In fact, most of the results in the paper are proved more generally for a certain kind of minimal theories. Here are some of the main examples:

5.2.1. *Strongly minimal theories.* are obtained by taking T_0 to be the language of equality. This means that every unary definable set is finite or co-finite. Thus, for example, ACF is strongly minimal. Some properties of ACF, for example, dimension of definable sets, generalise to general strongly minimal theories. In $ACVF$ the residue field, and the rest of the sets in Res are strongly minimal.

5.2.2. *o-minimal theories.* are theories with an order $<$, and are obtained by taking T_0 the theory of dense linear ordering. Thus, any unary set is a finite union of intervals. The theory RCF of real closed fields is an example of such a theory. As in the strongly minimal case, there is a notion of dimension for definable sets in such theories.

5.2.3. *C-minimal theories.* These are obtained by taking T_0 to be a theory with two sorts, VF and Γ_∞, whose axioms describe the "non-algebraic" relationship between VF and Γ in $ACVF$. Thus, there is a function symbol $v : VF^2 \to \Gamma_\infty$

(corresponding to $val(x-y)$), and a dense linear order $<$ on Γ_∞, with greatest element ∞, such that $v(x,y)=\infty$ if and only if $x=y$, $v(x,y)\geq\alpha$ (hence $v(x,y)>\alpha$) is an equivalence relation, whose classes are called closed (open) α balls, and each closed ball contains infinitely many open balls of the same radius.

It follows that Γ is *o*-minimal, and, given a closed ball B, the set of maximal open sub-balls of it is strongly minimal. If 0 is a distinguished element of VF, sets of the form $v(x,0)\geq\alpha$ are called closed balls around 0. The set RV can be formed as the collection of maximal open sub balls of closed balls around 0. Thus, geometrically, RV is a family of balls over Γ, where over a given $\alpha\in\Gamma$ is the set of open balls (not containing 0) of radius α in the closed ball of radius α around 0. In the context of $ACVF$, the fibre over $\alpha=0$ is thus the residue field, and other fibres are vector spaces of dimension 1 over it. Much of the theory in the paper is done in the context of C-minimal theories.

2 minutes

5.3. **Orthogonality.** Two definable sets are said to be *orthogonal* if any definable subset of $X^n\times Y^m$ is a finite union of products of corresponding subsets of X^n, Y^m. This means that there is no relation between X and Y. In particular, there are no definable bijections between subsets of X^n and Y^m. This is obviously important for understanding Grothendieck groups. In fact, if X and Y are orthogonal, then $K(X\times Y)=K(X)\otimes K(Y)$.

It is easy to show that any definable function from a strongly minimal set to an O-minimal one has finite image. It follows that any strongly minimal set is orthogonal to any *o*-minimal one. In particular, the residue field is orthogonal to the value group.

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