
Creeping flow past a stationary sphere

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This notebook shows how to solve creeping flow past a stationary sphere (Stokes's Problem)

Reference: L. G. Leal (1992) *Laminar flow and Convective Transport Processes*, Butterworth pp 139-144.

Consider the flow past a stationary sphere where the Reynolds number is significantly less than unity. Inertia can be neglected. The far away velocity field is a constant straight flow. To make the problem as simple as possible, we will use spherical coordinates with the phi axe oriented parallel to the flow. Thus we have phi symmetry and no v-phi. This means that the v-phi equation is not needed. If we used a different coordinate system, or different orientation, the problem would be more complicated.

The governing equations are:

vr eq==

$$\begin{aligned} \text{vreq} = & \frac{\partial_r (r^2 \partial_r \text{vr}[r, \text{th}])}{r^2} + \frac{\partial_{\text{th}} (\text{Sin}[\text{th}] \partial_{\text{th}} \text{vr}[r, \text{th}])}{\text{Sin}[\text{th}] r^2} - \\ & \frac{2 \text{vr}[r, \text{th}]}{r^2} - \frac{2 \partial_{\text{th}} \text{vth}[r, \text{th}]}{r^2} - \frac{2 \text{vth}[r, \text{th}] \text{Cot}[\text{th}]}{r^2} - \partial_r p[r, \text{th}] \\ & - \frac{2 \text{vr}(r, \text{th})}{r^2} - \frac{2 \text{cot}(\text{th}) \text{vth}(r, \text{th})}{r^2} - \\ & \frac{2 \text{vth}^{(0,1)}(r, \text{th})}{r^2} + \frac{\text{csc}(\text{th}) (\cos(\text{th}) \text{vr}^{(0,1)}(r, \text{th}) + \sin(\text{th}) \text{vr}^{(0,2)}(r, \text{th}))}{r^2} - p^{(1,0)}(r, \text{th}) + \\ & \frac{\text{vr}^{(2,0)}(r, \text{th}) r^2 + 2 \text{vr}^{(1,0)}(r, \text{th}) r}{r^2} \end{aligned}$$

The v theta equation is:

$$\begin{aligned} \text{vtheq} = & \frac{\partial_r (r^2 \partial_r \text{vth}[r, \text{th}]) + \partial_{\text{th}} \frac{\partial_{\text{th}} (\text{vth}[r, \text{th}] \sin[\text{th}])}{\sin[\text{th}]} + 2 \partial_{\text{th}} \text{vr}[r, \text{th}]}{r^2} - \\ & \frac{\partial_{\text{th}} p[r, \text{th}]}{r} \\ & \frac{1}{r^2} (\text{vth}^{(2,0)}(r, \text{th}) r^2 + 2 \text{vth}^{(1,0)}(r, \text{th}) r + \\ & 2 \text{vr}^{(0,1)}(r, \text{th}) - \cot(\text{th}) \csc(\text{th}) (\cos(\text{th}) \text{vth}(r, \text{th}) + \sin(\text{th}) \text{vth}^{(0,1)}(r, \text{th})) + \\ & \csc(\text{th}) (-\sin(\text{th}) \text{vth}(r, \text{th}) + 2 \cos(\text{th}) \text{vth}^{(0,1)}(r, \text{th}) + \sin(\text{th}) \text{vth}^{(0,2)}(r, \text{th})) - \\ & \frac{p^{(0,1)}(r, \text{th})}{r} \end{aligned}$$

We will try a separation of variables technique. We will use $v[r, \text{th}] = f[r] H[\text{th}]$. The problem becomes, how to find the two functions a priori.

As mentioned by Leal, the boundary conditions often tell the form of the solution of problems in the creeping flow regime. From the boundary conditions at the far field, the following forms are suggested for the solutions

$$\text{vsr} = \text{Cos}[\text{th}] f[r];$$

$$\text{vsth} = \text{Sin}[\text{th}] g[r];$$

We don't yet know the likely th dependence of the pressure, but we can get it from the r equation. Substitute in the velocities and look at the result

$$\begin{aligned} \text{vrp} /. \{ \text{vr}^{(a1-, a2-)}[r, \text{th}] & \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{vsr}, \\ \text{vth}^{(a1-, a2-)}[r, \text{th}] & \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{vsth}, \text{vr}[r, \text{th}] \rightarrow \text{vsr}, \\ \text{vth}[r, \text{th}] & \rightarrow \text{vsth} \} \\ - \frac{4 \cos(\text{th}) f(r)}{r^2} - \frac{4 \cos(\text{th}) g(r)}{r^2} + \frac{\cos(\text{th}) f''(r) r^2 + 2 \cos(\text{th}) f'(r) r}{r^2} - p^{(1,0)}(r, \text{th}) \end{aligned}$$

$$\begin{aligned} \text{vrp} = \text{Expand} \left[\frac{\%}{\text{Cos}[\text{th}]} \right] \\ - \frac{4 f(r)}{r^2} - \frac{4 g(r)}{r^2} + \frac{2 f'(r)}{r} + f''(r) - \sec(\text{th}) p^{(1,0)}(r, \text{th}) \end{aligned}$$

This suggests that the pressure is of the form:

$$\text{press} = p0 + pp[r] \text{Cos}[\text{th}];$$

Substitute this form of the pressure into r equation to check (but of course it will work).

$$\begin{aligned} \text{vrpeq} = \text{vrp} /. \{ p^{(a1-, a2-)}[r, \text{th}] & \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{press} \} \\ - \frac{4 f(r)}{r^2} - \frac{4 g(r)}{r^2} + \frac{2 f'(r)}{r} - pp'(r) + f''(r) \end{aligned}$$

Which is the r dependence of the r momentum equation, and it matches Leal's. Now let's look at the th equation.

$$\begin{aligned} \text{vtheq} & /. \{ \text{vr}^{(a1-, a2-)} [r, th] \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{vsr}, \\ & \text{vth}^{(a1-, a2-)} [r, th] \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{vsth}, \text{vr} [r, th] \rightarrow \text{vsr}, \\ & \text{vth} [r, th] \rightarrow \text{vsth}, \text{p}^{(a1-, a2-)} [r, th] \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{press} \} \end{aligned}$$

$$\begin{aligned} & \frac{\text{pp}(r) \sin(th)}{r} + \frac{1}{r^2} (\sin(th) g''(r) r^2 + 2 \sin(th) g'(r) r - 2 \cos(th) \cot(th) g(r) - \\ & 2 f(r) \sin(th) + \csc(th) (2 \cos^2(th) g(r) - 2 g(r) \sin^2(th))) \end{aligned}$$

$$\text{vthpeq} = \text{Expand} \left[\frac{\%}{\sin[th]} \right]$$

$$-\frac{2 f(r)}{r^2} - \frac{2 g(r)}{r^2} + \frac{\text{pp}(r)}{r} + \frac{2 g'(r)}{r} + g''(r)$$

Which is the r dependence of the theta equation. Now look at continuity

$$\begin{aligned} \text{conteq} & = \frac{\partial_r (r^2 \text{vr} [r, th])}{r^2} + \frac{\partial_{th} (\sin[th] \text{vth} [r, th])}{\sin[th] r} \\ & \frac{\csc(th) (\cos(th) \text{vth}(r, th) + \sin(th) \text{vth}^{(0,1)}(r, th))}{r} + \frac{\text{vr}^{(1,0)}(r, th) r^2 + 2 \text{vr}(r, th) r}{r^2} \end{aligned}$$

Use the standard substitutions into the continuity equation

$$\begin{aligned} \text{ceq1} & = \text{conteq} /. \{ \text{vr}^{(a1-, a2-)} [r, th] \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{vsr}, \\ & \text{vth}^{(a1-, a2-)} [r, th] \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{vsth}, \text{vr} [r, th] \rightarrow \text{vsr}, \\ & \text{vth} [r, th] \rightarrow \text{vsth}, \text{p}^{(a1-, a2-)} [r, th] \Rightarrow \partial_{\{r, a1\}, \{th, a2\}} \text{press} \} \end{aligned}$$

$$\frac{2 \cos(th) g(r)}{r} + \frac{\cos(th) f'(r) r^2 + 2 \cos(th) f(r) r}{r^2}$$

$$\text{ceq2} = \text{Expand} \left[\frac{\text{ceq1}}{2 \cos[th]} \right]$$

$$\frac{f(r)}{r} + \frac{g(r)}{r} + \frac{f'(r)}{2}$$

So the three equations for g[r], f[r], pp[r] are

ceq2

$$\frac{f(r)}{r} + \frac{g(r)}{r} + \frac{f'(r)}{2}$$

vthpeq

$$-\frac{2f(r)}{r^2} - \frac{2g(r)}{r^2} + \frac{pp(r)}{r} + \frac{2g'(r)}{r} + g''(r)$$

vrpeq

$$-\frac{4f(r)}{r^2} - \frac{4g(r)}{r^2} + \frac{2f'(r)}{r} - pp'(r) + f''(r)$$

The easiest way to solve these is to eliminate variables to leave only one. This can be done by using continuity to eliminate $g[r]$ in favor of $p[r]$, then use the vth equation, (after taking one more r derivative) to eliminate $pp[r]$ from the vr equation. This will be a simple ode. (Leal has done this before.)

Solve [ceq2 == 0, g[r]]

$$\left\{ \left\{ g(r) \rightarrow \frac{1}{2} (-2f(r) - rf'(r)) \right\} \right\}$$

grsubs = g[r] /. %[[1]]

$$\frac{1}{2} (-2f(r) - rf'(r))$$

First the vr equation

vrx = vrpeq /. {g[r] → grsubs, g^(a1_)[r] ⇒ ∂_{r,a1} grsubs }

$$-\frac{4f(r)}{r^2} + \frac{2f'(r)}{r} - \frac{2(-2f(r) - rf'(r))}{r^2} - pp'(r) + f''(r)$$

Now the vth equation

vthx = vthpeq /. {g[r] → grsubs, g^(a1_)[r] ⇒ ∂_{r,a1} grsubs }

$$-\frac{2f(r)}{r^2} + \frac{pp(r)}{r} - \frac{-2f(r) - rf'(r)}{r^2} + \frac{-3f'(r) - rf''(r)}{r} + \frac{1}{2} (-4f''(r) - rf^{(3)}(r))$$

Now if we multiply by r , then take an r derivative we can add these two equations to eliminate $pp'[r]$ and leave a single equation for $f[r]$

Expand [vrx + ∂_r Expand [vthx r]]

$$-\frac{1}{2} f^{(4)}(r) r^2 - 4f^{(3)}(r) r - 4f''(r) + \frac{4f'(r)}{r}$$

$$\text{ode4} = \text{Expand} \left[-\frac{\% r^2}{4} \right]$$

$$\frac{1}{8} f^{(4)}(r) r^4 + f^{(3)}(r) r^3 + f''(r) r^2 - f'(r) r$$

We get a single ode for f[r] which is the same as Leal's eq 4-55. Amazingly, *Mathematica* knows the solution to it. This type of ode is called an Euler equation.

$$\text{DSolve} [\text{ode4} == 0, f[r], r]$$

$$\left\{ \left\{ f(r) \rightarrow c_4 r^2 + c_3 + \frac{c_2}{r} + \frac{c_1}{r^3} \right\} \right\}$$

$$\mathbf{ff} = \mathbf{f}[r] /. \%[[1]]$$

$$c_4 r^2 + c_3 + \frac{c_2}{r} + \frac{c_1}{r^3}$$

Now let's see which boundary conditions we can fit. As $r \rightarrow \text{Infinity}$, we can't have an unbounded solution so we must have $c_4 = 0$. However, it does have to match the free stream conditions which are $v_r = \text{Cos}[\text{th}]$ and $v_{\theta} = -\text{Sin}[\text{th}]$. From v_r , we set $c_3 = 1$ and hope that it works for v_{θ} . (It does). Thus the solution is

$$\mathbf{ff1} = \mathbf{ff} /. \{C[4] \rightarrow 0, C[3] \rightarrow 1\}$$

$$\frac{c_1}{r^3} + \frac{c_2}{r} + 1$$

Now we look at $r=1$. We must have $f[1]=0$ so that v_r is 0 and by looking at continuity, with $g[1]=0$, we see that $f'[1]=0$

$$\mathbf{ff2} = \partial_r \mathbf{ff1} /. r \rightarrow 1$$

$$-3 c_1 - c_2$$

$$\text{Solve} [\{\mathbf{ff2} == 0, (\mathbf{ff1} /. r \rightarrow 1) == 0\}, \{C[1], C[2]\}]$$

$$\left\{ \left\{ c_1 \rightarrow \frac{1}{2}, c_2 \rightarrow -\frac{3}{2} \right\} \right\}$$

We get the solution for f[r] to be:

$$\mathbf{ffinal} = \mathbf{ff1} /. \%[[1]]$$

$$1 - \frac{3}{2r} + \frac{1}{2r^3}$$

Which does match Leal's. Now find g[r] and pp[r]. First find g[r] from continuity.

```
Expand [ceq2 /. {f[r] → ffinal , f(a1_) [r] ⇒ ∂{r,a1} ffinal }]
```

$$\frac{g(r)}{r} + \frac{1}{r} - \frac{3}{4r^2} - \frac{1}{4r^4}$$

```
gf = Solve [% == 0 , g[r]]
```

$$\left\{ \left\{ g(r) \rightarrow -\frac{4r^3 - 3r^2 - 1}{4r^3} \right\} \right\}$$

```
gfinal = g[r] /. %[[1]]
```

$$-\frac{4r^3 - 3r^2 - 1}{4r^3}$$

Now use the vth equation to get pp[r]

```
Expand [vthpeq /. {f[r] → ffinal , f(a1_) [r] ⇒ ∂{r,a1} ffinal ,  
g[r] → gfinal , g(a1_) [r] ⇒ ∂{r,a1} gfinal }]
```

$$\frac{pp(r)}{r} + \frac{3}{2r^3}$$

```
Solve [% == 0 , pp[r]]
```

$$\left\{ \left\{ pp(r) \rightarrow -\frac{3}{2r^2} \right\} \right\}$$

```
ppfinal = pp[r] /. %[[1]]
```

$$-\frac{3}{2r^2}$$

So these can be used to obtain the dimensional pressure and velocities. It should be noted that there are no values of r and θ where the velocity is greater than the free stream value. For high Reynolds number flows, this is not the case. To check this we write

$$\mathbf{vfield} = \mathbf{vsr}^2 + \mathbf{vsth}^2$$

$$\cos^2(\theta) f(r)^2 + g(r)^2 \sin^2(\theta)$$

Now substitute for the $f[r]$ and $g[r]$

```
Expand[vfield /. {f[r] -> ffinal, f^(a1_) [r] -> D_{r,a1} ffinal,
g[r] -> gfinal, g^(a1_) [r] -> D_{r,a1} gfinal}]
```

$$-\frac{3 \cos^2(\text{th})}{r} + \frac{9 \cos^2(\text{th})}{4 r^2} + \frac{\cos^2(\text{th})}{r^3} - \frac{3 \cos^2(\text{th})}{2 r^4} + \frac{\cos^2(\text{th})}{4 r^6} + \cos^2(\text{th}) - \frac{3 \sin^2(\text{th})}{2 r} + \frac{9 \sin^2(\text{th})}{16 r^2} - \frac{\sin^2(\text{th})}{2 r^3} + \frac{3 \sin^2(\text{th})}{8 r^4} + \frac{\sin^2(\text{th})}{16 r^6} + \sin^2(\text{th})$$

```
Collect[%, r]
```

$$\cos^2(\text{th}) + \sin^2(\text{th}) + \frac{-3 \cos^2(\text{th}) - \frac{3 \sin^2(\text{th})}{2}}{r} + \frac{\cos^2(\text{th}) - \frac{\sin^2(\text{th})}{2}}{r^3} + \frac{\frac{\cos^2(\text{th})}{4} + \frac{\sin^2(\text{th})}{16}}{r^6} + \frac{\frac{3 \sin^2(\text{th})}{8} - \frac{3 \cos^2(\text{th})}{2}}{r^4} + \frac{\frac{9 \cos^2(\text{th})}{4} + \frac{9 \sin^2(\text{th})}{16}}{r^2}$$

Use the polar to Cartesian transformation:

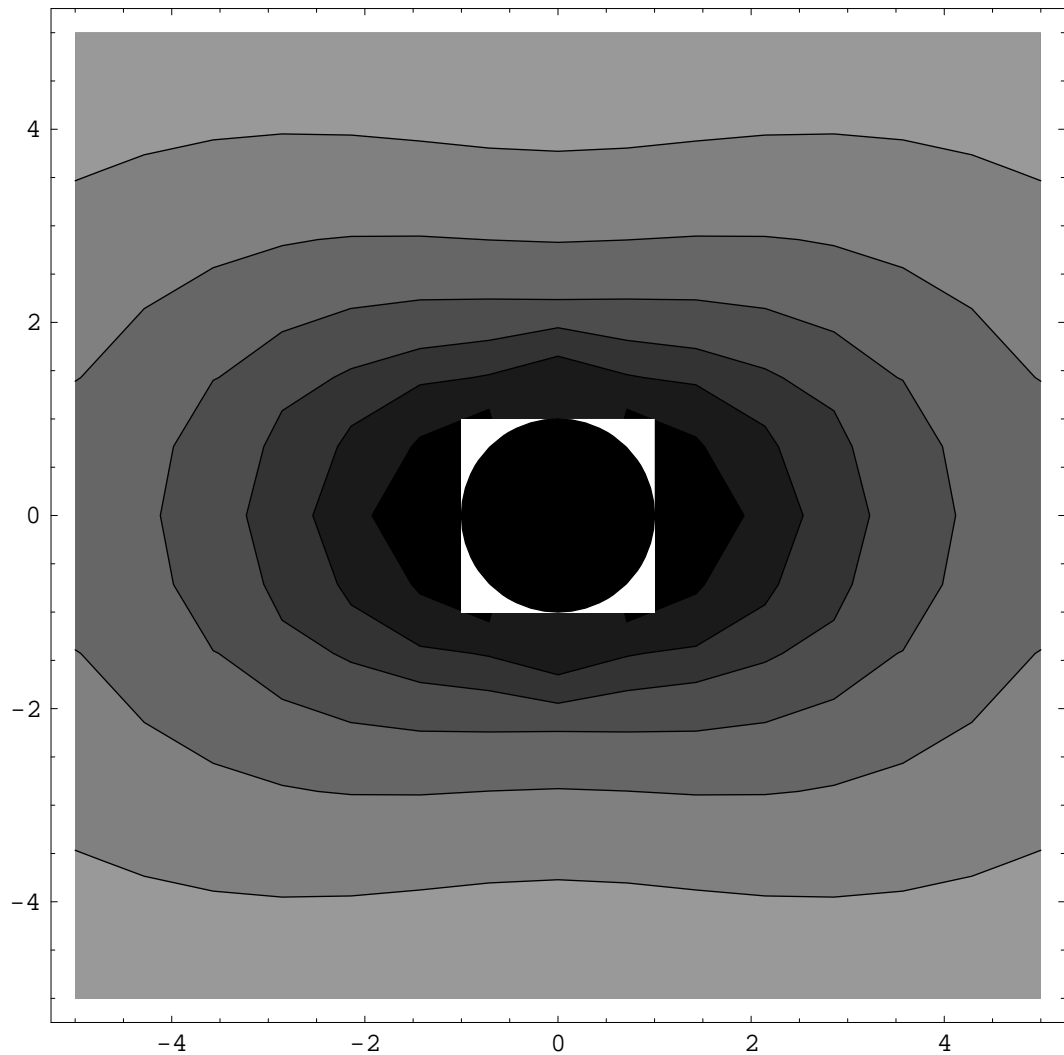
```
vcart = % /. {r -> Sqrt[x^2 + y^2], th -> ArcTan[y/x]}
```

```
plot1 = ContourPlot[vcart, {x, -5, 5}, {y, -5, 5}, Contours -> 10]
```

```
sphere = ContourPlot[x^2 + y^2, {x, -1, 1}, {y, -1, 1}, Contours -> 1]
```

The result is shown here with the sphere blacked out (and unfortunately some frame). Note that the contours smoothly change as we approach the sphere. The darker, the shading, the lower the velocity.

```
Show [plot1 , sphere ]
```



- Graphics -

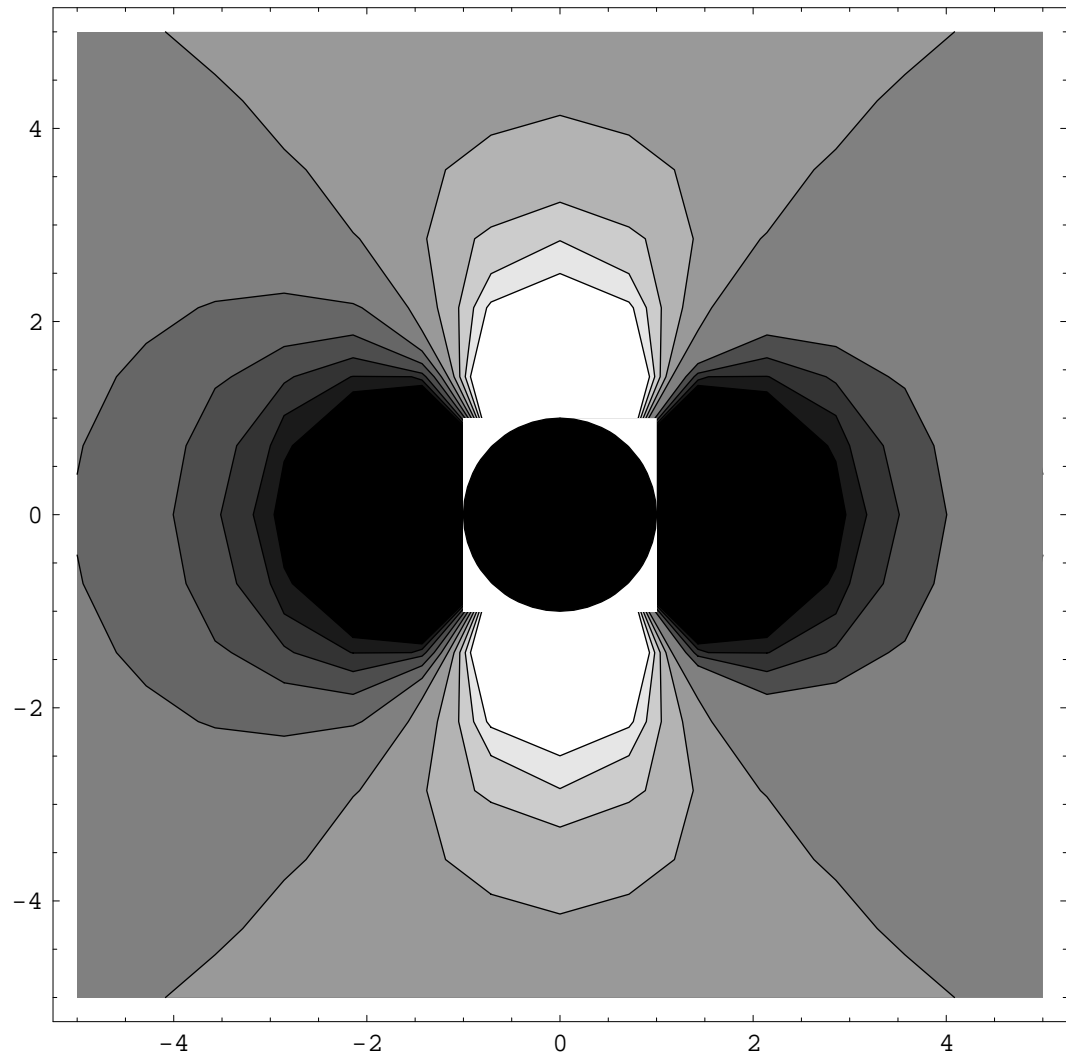
To contrast this write down the average velocity for the invscid flow case from a standard reference

$$v_{invis} = \left(\cos [th] \left(1 - \frac{1}{r^3} \right) \right)^2 + \left(-\frac{1}{2} \sin [th] \left(2 + \frac{1}{r^3} \right) \right)^2 ;$$

$$v_{cartinvis} = v_{invis} /. \{ r \rightarrow \sqrt{x^2 + y^2}, th \rightarrow \text{ArcTan} \left[\frac{y}{x} \right] \};$$

```
plot2 = ContourPlot [vcartinvis , {x, -5, 5}, {y, -5, 5}]
```

```
Show [plot2 , sphere ]
```



- Graphics -

The code is as the shading gets darker, the velocity is slower. We see that coming in the $y=0$ axis from $x=-\infty$, the velocity is decreasing to agree with the: no flow through the surface boundary condition. Looking at the $x=0$ axis, coming in from $y=\infty$, the velocity is increasing reflecting the deflection of fluid around the sphere. This results in a Bernoulli effect with local fluid acceleration. Below we check some numbers to verify these statements.

```
N[vcartinvis /. {x -> .01, y -> 1}]
```

2.24955

```
N[vcartinvis /. {x -> 1, y -> -.001}]
```

2.25×10^{-6}

```
N[vcartinvis /. {x -> 100, y -> 100}]
```

1.

The same question can be asked about the pressure field. We will find that the pressure is always at or below the far away value for $Re=0$. Substitute for the pressure field equation,

```
pfinal
```

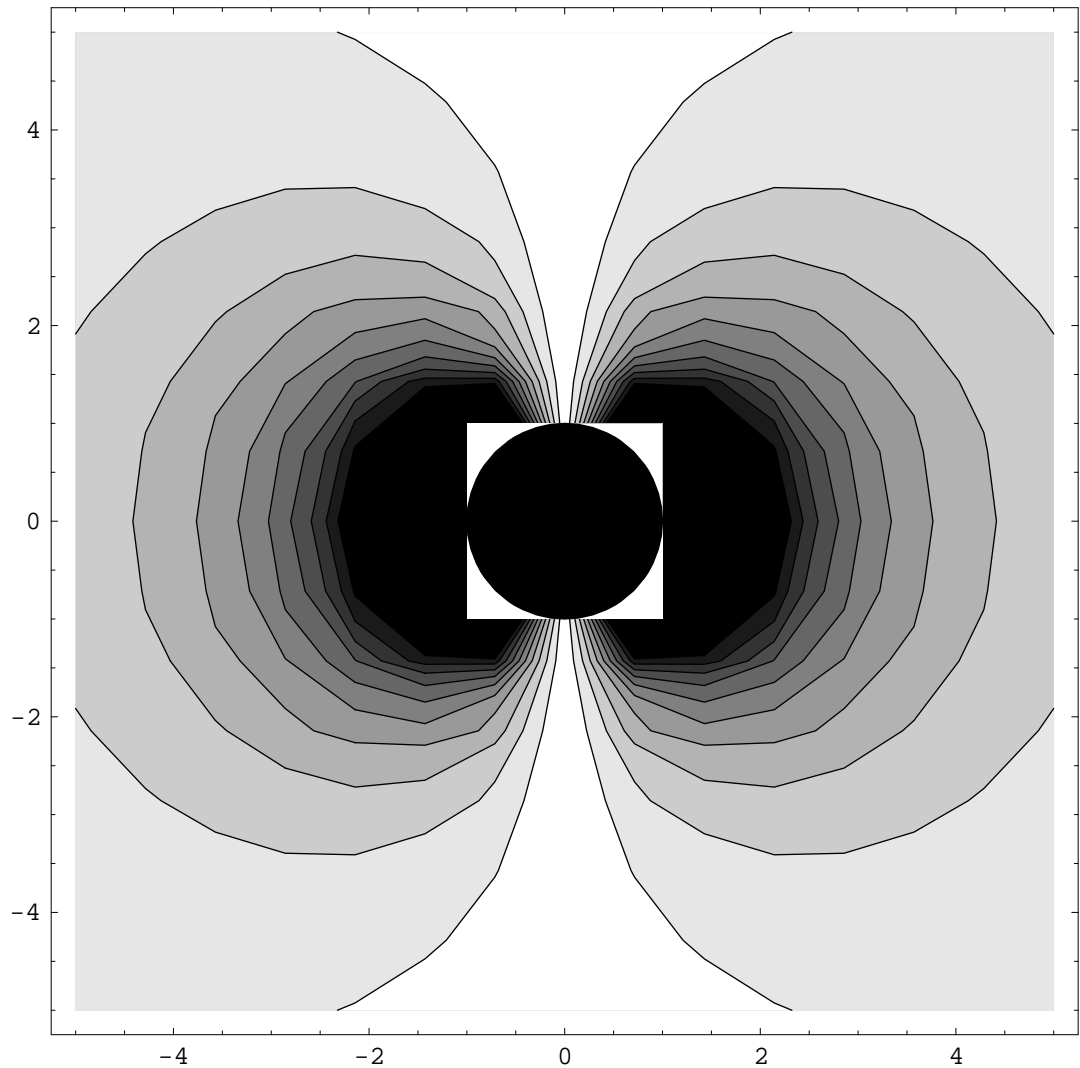
$$-\frac{3}{2r^2}$$

```
pfield = Cos[th] pfinal /. {r -> Sqrt[x^2 + y^2], th -> ArcTan[Y/x]}
```

$$-\frac{3}{2(x^2 + y^2)\sqrt{\frac{y^2}{x^2} + 1}}$$

```
plot3 = ContourPlot[pfield, {x, -5, 5}, {y, -5, 5}]
```

Show [plot3 , sphere]



- Graphics -

In contrast, if a high velocity fluid is impinging on a surface, there will be a pressure increase. For an inviscid flow, from Bernoulli's equation, we find that $dp \sim (U_0^2 - u^2)/2$, where we have already plotted u^2

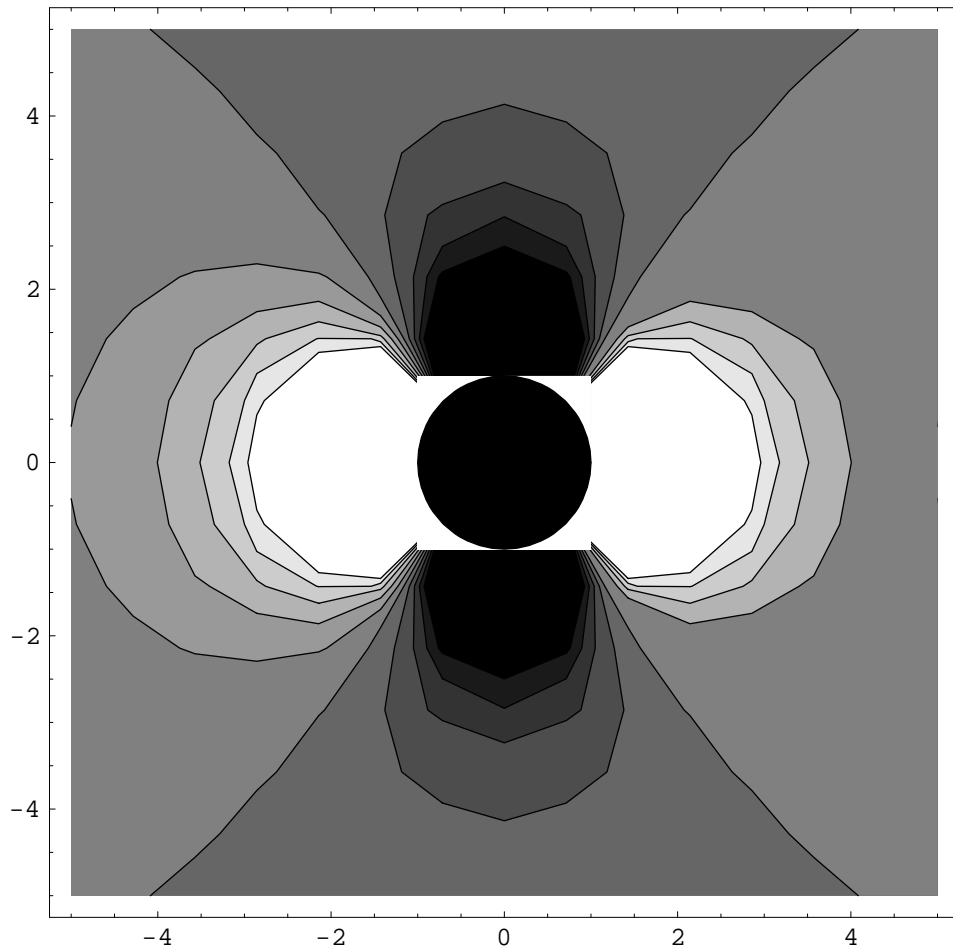
$$pinvis = \frac{1 - vcartinvis}{2}$$

— General::spell1 : Possible spelling error: new symbol name "pinvis" is similar to existing symbol "vinvis".

$$\frac{1}{2} \left(-\frac{\left(1 - \frac{1}{(x^2 + y^2)^{3/2}}\right)^2}{\frac{y^2}{x^2} + 1} - \frac{y^2 \left(2 + \frac{1}{(x^2 + y^2)^{3/2}}\right)^2}{4x^2 \left(\frac{y^2}{x^2} + 1\right)} + 1 \right)$$

plot4 = ContourPlot [pinvis , {x, -5, 5}, {y, -5, 5}]

```
Show [plot4 , sphere ]
```



- Graphics -

So that the pressure is highest at $y=0, x=1, -1$ (i.e. the stagnation point) and decreases out to infinity. The pressure is lowest, consistent with the Bernoulli effect, at $y=1, -1, x=0$, and increases out to $y=\infty$.

A stream function is another way to solve the problem. In terms of the stream function, the $Re = 0$ solution is $-\frac{1}{2} f[r] r^2 \sin[\theta]^2$

$$\frac{1}{2} f_{\text{final}} r^2 \sin[\theta]^2$$

$$\frac{1}{2} \left(1 - \frac{3}{2r} + \frac{1}{2r^3} \right) r^2 \sin^2(\theta)$$

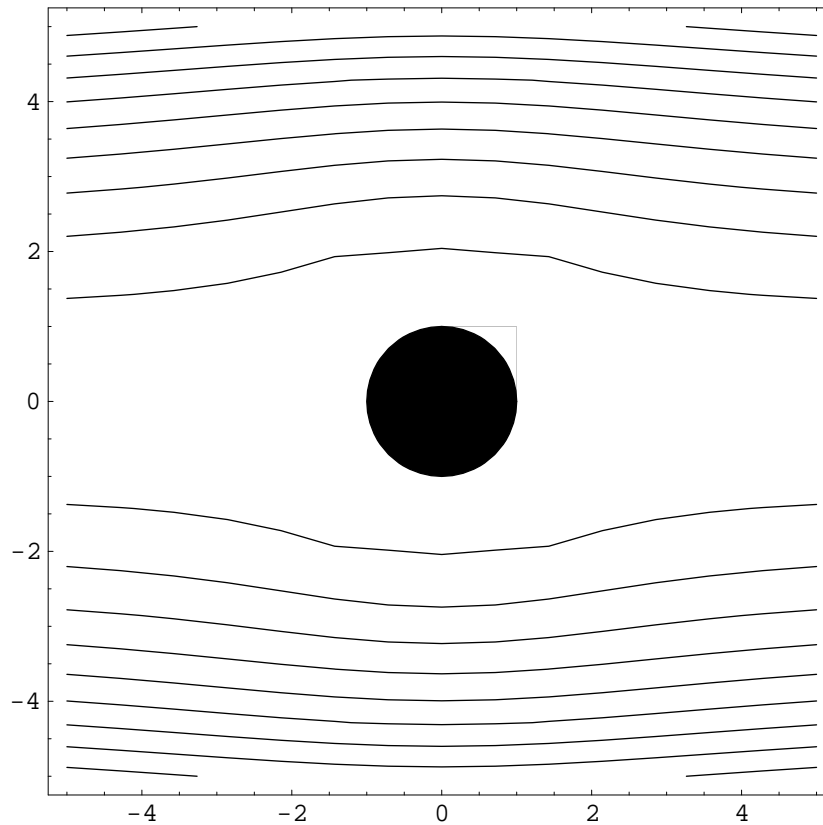
We can make a plot of this if we transform from polar to Cartesian coordinates

```
stream1 = % /. {r -> Sqrt[x^2 + y^2], th -> ArcTan[Y/x]}
```

$$\frac{y^2 (x^2 + y^2) \left(1 - \frac{3}{2\sqrt{x^2 + y^2}} + \frac{1}{2(x^2 + y^2)^{3/2}}\right)}{2x^2 \left(\frac{y^2}{x^2} + 1\right)}$$

```
plot5 = ContourPlot [stream1, {x, -5, 5}, {y, -5, 5}, Contours -> 15,
  ContourShading -> False]
```

```
Show [plot5, sphere]
```



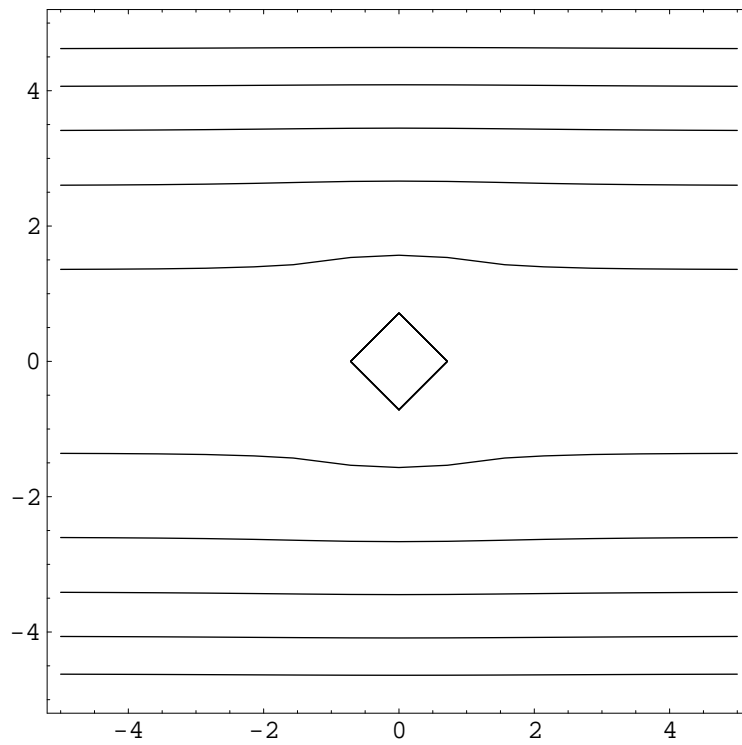
- Graphics -

Note that at $y=4$, the contours are still deformed, the effect of the sphere is felt very far away!

Now compare the Inviscid flow case:

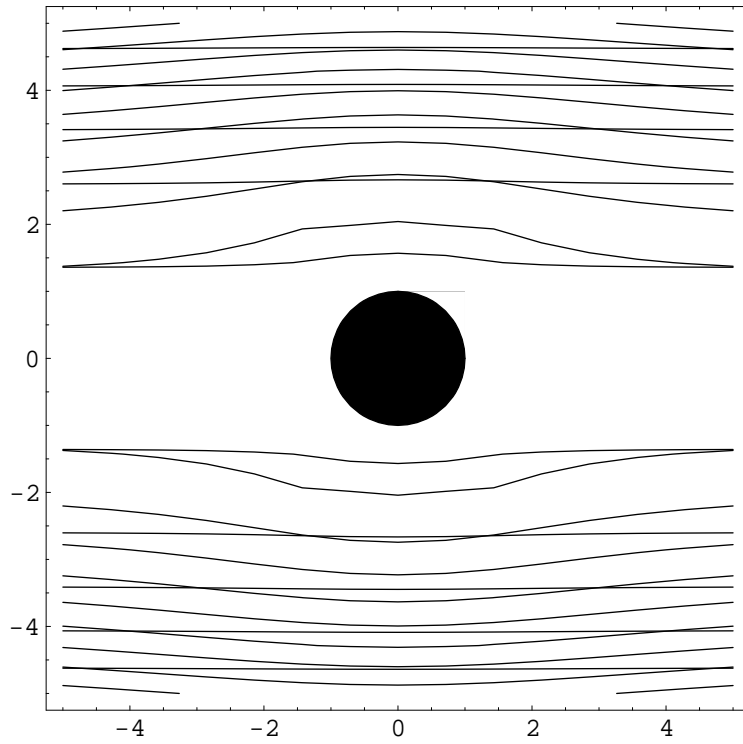
$$\text{invisstream} = \frac{1}{2} \sin[\text{th}]^2 \left(r^2 - \frac{1}{r}\right) /. \{r \rightarrow \sqrt{x^2 + y^2}, \text{th} \rightarrow \text{ArcTan}\left[\frac{y}{x}\right]\};$$

```
plot6 = ContourPlot [invisstream , {x, -5, 5}, {y, -5, 5},  
ContourShading → False]
```



- ContourGraphics -

```
Show [plot5 , plot6 , sphere ]
```



- Graphics -

This last plot is a bit messy but shows that inviscid flow is disturbed much less a few diameters from the sphere than Stokes flow.