

Spectral based solution methods

This notebook has been written in *Mathematica* by

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Overview of the spectral approach

The basic ideal of anything "spectral" is the systematic representation/decomposition of a vector, function or solution in terms of some basis vectors, functions which are independent (in this case orthogonal/biorthogonal). The basis vectors or functions must span the space and thus form a complete set so that *any* vector, function or solution can be represented.

In symbolic form, we are interested in solving, $L[y] = f(x)$ by representing $f(x)$ in terms of our basis functions and then ultimately finding y in terms of these same functions. Thus we are not directly finding the inverse operator solution as we did in the Green's function approach, i.e., $y = L^{-1} f(x)$. We look for these expansions,

$$f(x) = \sum_{j=1}^n c_j \phi_j(x)$$

$$y(x) = \sum_{j=1}^n d_j \phi_j(x)$$

Presumably there will be a straightforward way of finding the c_j 's since $f(x)$ is a known function. (This will be to form an inner product.) We will find that we can obtain the d_j 's by substituting all that we know in the equation and forming an inner product for each term.

Once we have defined the problem as the equation, $Ly = f(x)$, and its boundary conditions, $B(y) = 0$. We need a systematic way to produce an appropriate set of basis functions. The best possible set will usually be obtained from the eigenvalue problem, $Ly = -\lambda y$ with $B(y) = 0$. For a self adjoint operator we will generate real λ 's and orthogonal eigenfunctions. If L is not self-adjoint, we can take advantage of biorthogonality.

We will define the expansion of a term in the equation by expansion in terms of eigenfunctions as a *Finite Fourier Transform* which is a linear operator and will, of course, have an inverse.

If we cannot readily solve $L[y] = -\lambda y$, we may instead use a known set of eigenfunctions from a different operator. This is effectively a *numerical* solution technique which we will discuss later.

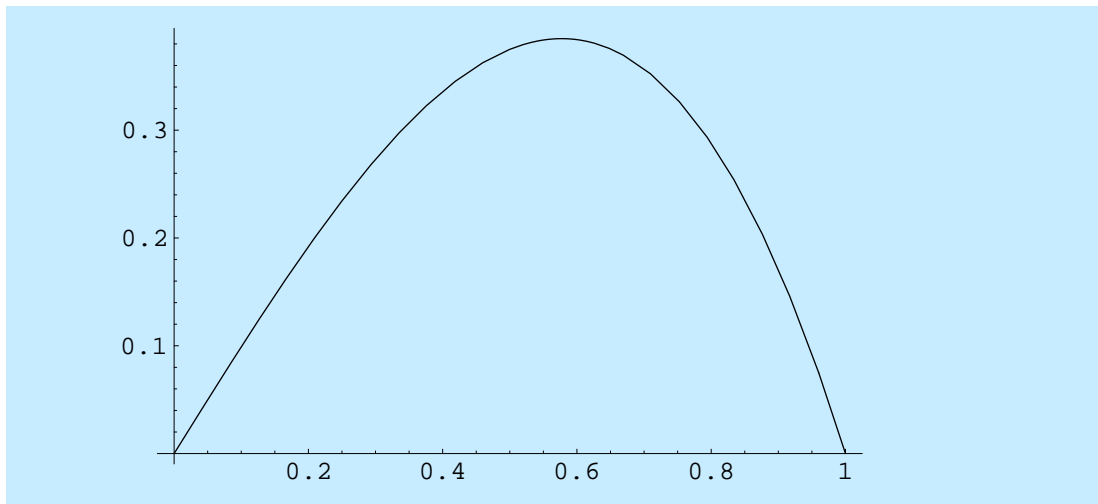
Expansion of an arbitrary function in terms of a set of eigenfunctions.

- Here are two functions that we would like to fit with our Eigenfunctions.

$$\mathbf{ff}[\mathbf{x_}] := \mathbf{x} (1 - \mathbf{x}^2)$$

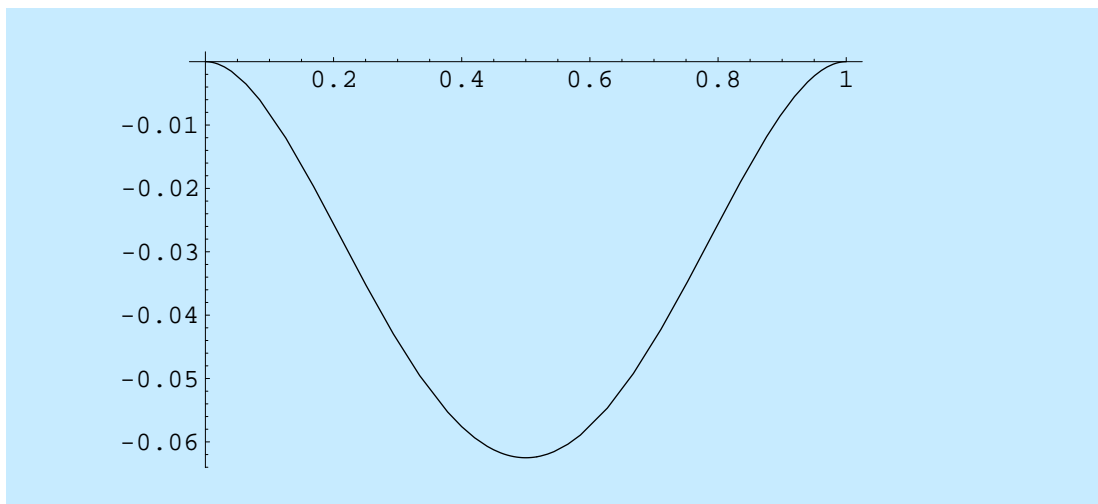
$$\mathbf{gg}[\mathbf{x_}] := -\mathbf{x} (\mathbf{x} - 2 \mathbf{x}^2 + \mathbf{x}^3)$$

```
Plot[ff[x], {x, 0, 1}]
```



- Graphics -

```
Plot[gg[x], {x, 0, 1}, PlotRange -> All]
```

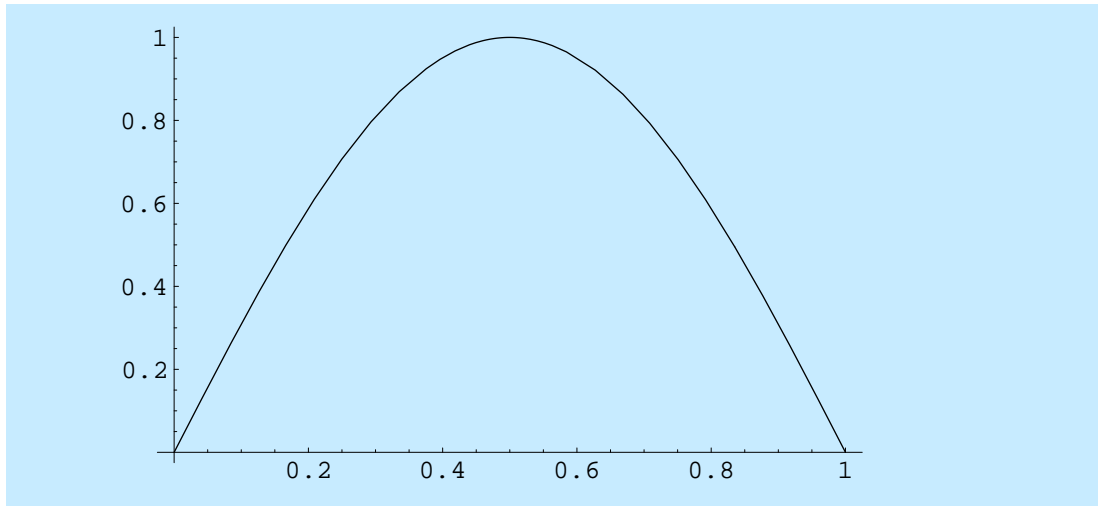


- Graphics -

■ plots of eigenfunctions

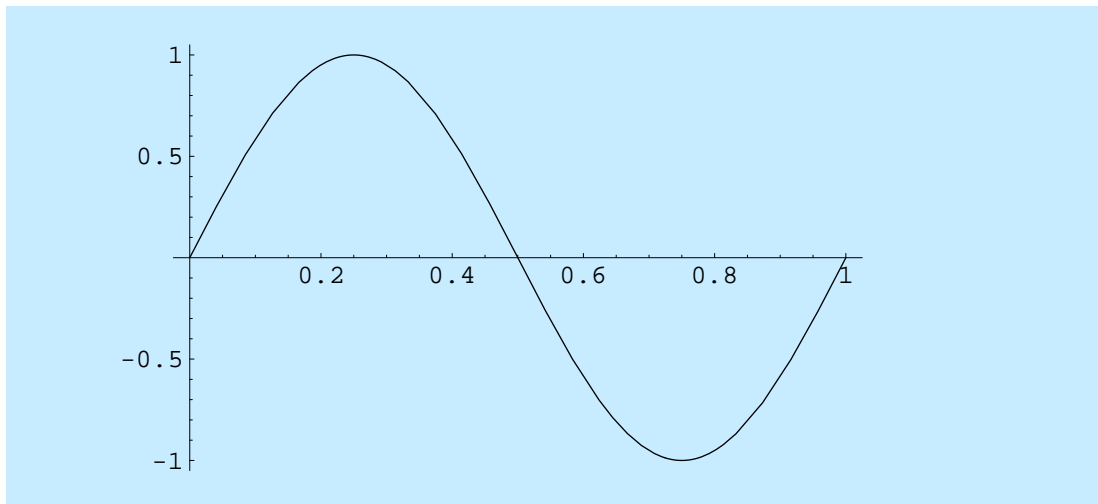
We have found a nice set of eigenfunctions that we can use to expand, $f[x]$ and $g[x]$. (Think about what would happen if the eigenfunctions did not have the same boundary conditions as the functions we want to expand.)

```
Plot[Sin[ $\pi x$ ], {x, 0, 1}]
```



- Graphics -

```
Plot[Sin[2  $\pi x$ ], {x, 0, 1}]
```



- Graphics -

As they say, etc...

- Here are our eigenfunctions

$$\phi[x_, i_] := \sqrt{2} \sin[i \pi x]$$

- Verify the orthonormality

$$\int_0^1 \text{psi}[x, 1]^2 dx$$

$$\int_0^1 \text{psi}(x, 1)^2 dx$$

$$\int_0^1 \text{psi}[x, 3]^2 dx$$

$$\int_0^1 \text{psi}(x, 3)^2 dx$$

$$\int_0^1 \text{psi}[x, 1] \text{psi}[x, 4] dx$$

$$\int_0^1 \text{psi}(x, 1) \text{psi}(x, 4) dx$$

- Now get the coefficients for ff[x]

We start with the assumed functional form for the expansion,

$$f(x) = \sum_{j=1}^n c_j \phi_j(x)$$

now form the inner product with the function $\phi_k(x)$

$$\int_0^1 \rho(x) \phi_k(x) f(x) dx = \int_0^1 \rho(x) \phi_k(x) \sum_{j=1}^n c_j \phi_j(x) dx$$

On the right side we can switch the summation and integration (assuming that everything will converge)

$$\int_0^1 \rho(x) \phi_k(x) f(x) dx = \sum_{j=1}^n \int_0^1 \rho(x) \phi_k(x) c_j \phi_j(x) dx$$

next just move the c_j outside of the integration,

$$= \sum_{j=1}^n c_j \int_0^1 \rho(x) \phi_k(x) \phi_j(x) dx$$

For properly normalized orthogonalized functions the integral just equals δ_{kj} which means that the right side is just, c_k .

$$\int_0^1 \rho(x) \phi_k(x) f(x) dx = c_k .$$

For the eigenfunctions we have chosen, $\rho(x) = 1$. This gives then for $ff[x]$, the first one and then a lot of others,

■ First expand $ff[x]$

$$\int_0^1 ff[x] \phi[x, 1] dx$$

$$\frac{6\sqrt{2}}{\pi^3}$$

```
cks = Table[Integrate[ff[x] \phi[x, i], {x, 0, 1}], {i, 1, 10}]
```

$$\left\{ \frac{6\sqrt{2}}{\pi^3}, -\frac{3}{2\sqrt{2}\pi^3}, \frac{2\sqrt{2}}{9\pi^3}, -\frac{3}{16\sqrt{2}\pi^3}, \frac{6\sqrt{2}}{125\pi^3}, -\frac{1}{18\sqrt{2}\pi^3}, \frac{6\sqrt{2}}{343\pi^3}, \right. \\ \left. -\frac{3}{128\sqrt{2}\pi^3}, \frac{2\sqrt{2}}{243\pi^3}, -\frac{3}{250\sqrt{2}\pi^3} \right\}$$

If you like numbers we have

```
cksN = Table[NIntegrate[ff[x]  $\phi$ [x, i], {x, 0, 1}], {i, 1, 10}]
```

— General::spell1 : Possible spelling error: new symbol name "cksN" is similar to existing symbol "cks".

```
{0.273663, -0.0342079, 0.0101357, -0.00427599, 0.00218931, -0.00126696,
0.000797852, -0.000534499, 0.000375396, -0.000273663}
```

Make some partial sums for the function

$$\text{onex} = \sum_{i=1}^1 \phi[x, i] \text{cks}[[i]]$$

$$\frac{12 \sin(\pi x)}{\pi^3}$$

$$\text{twox} = \sum_{i=1}^2 \phi[x, i] \text{cks}[[i]]$$

$$\frac{12 \sin(\pi x)}{\pi^3} - \frac{3 \sin(2\pi x)}{2\pi^3}$$

$$\text{threex} = \sum_{i=1}^3 \phi[x, i] \text{cks}[[i]]$$

$$\frac{12 \sin(\pi x)}{\pi^3} - \frac{3 \sin(2\pi x)}{2\pi^3} + \frac{4 \sin(3\pi x)}{9\pi^3}$$

$$\text{sixx} = \sum_{i=1}^6 \phi[x, i] \text{cks}[[i]]$$

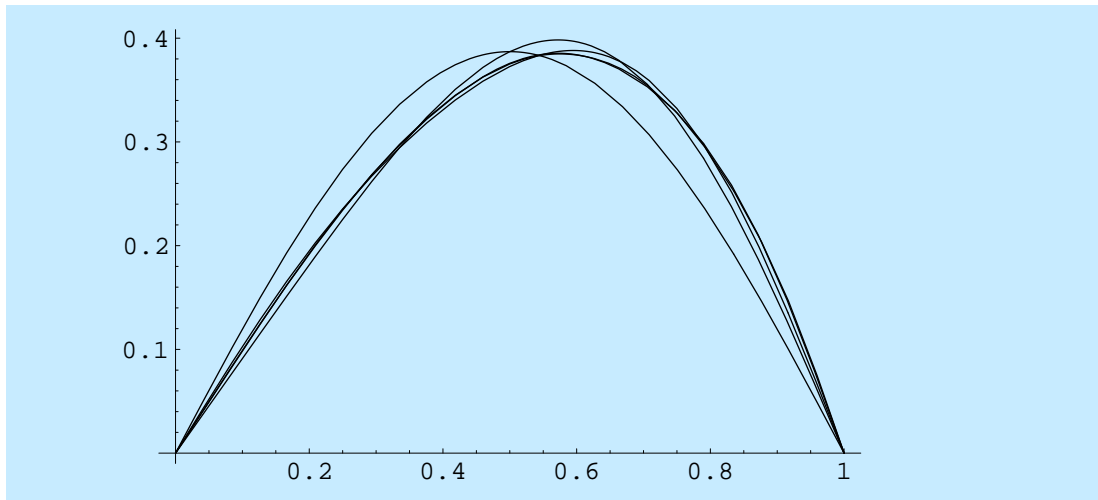
$$\frac{12 \sin(\pi x)}{\pi^3} - \frac{3 \sin(2\pi x)}{2\pi^3} + \frac{4 \sin(3\pi x)}{9\pi^3} - \frac{3 \sin(4\pi x)}{16\pi^3} + \frac{12 \sin(5\pi x)}{125\pi^3} - \frac{\sin(6\pi x)}{18\pi^3}$$

$$\text{tenx} = \sum_{i=1}^{10} \phi[\mathbf{x}, i] \text{cks}[[i]]$$

$$\begin{aligned} & \frac{12 \sin(\pi x)}{\pi^3} - \frac{3 \sin(2\pi x)}{2\pi^3} + \frac{4 \sin(3\pi x)}{9\pi^3} - \frac{3 \sin(4\pi x)}{16\pi^3} + \frac{12 \sin(5\pi x)}{125\pi^3} - \frac{\sin(6\pi x)}{18\pi^3} + \\ & \frac{12 \sin(7\pi x)}{343\pi^3} - \frac{3 \sin(8\pi x)}{128\pi^3} + \frac{4 \sin(9\pi x)}{243\pi^3} - \frac{3 \sin(10\pi x)}{250\pi^3} \end{aligned}$$

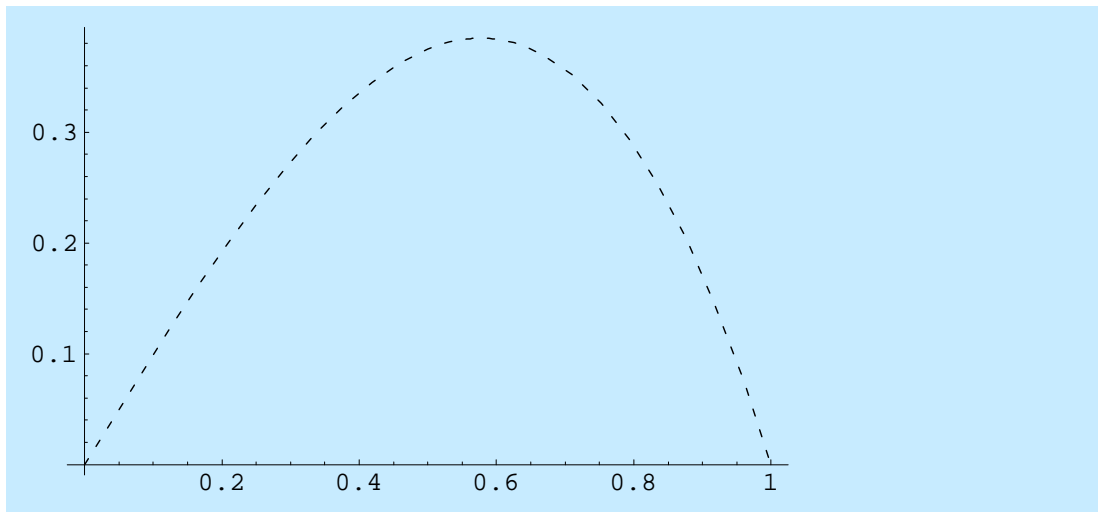
We plot to see that it takes only about 3 terms

```
plt1 = Plot[{onex, twox, threex, sixx, tenx}, {x, 0, 1}]
```



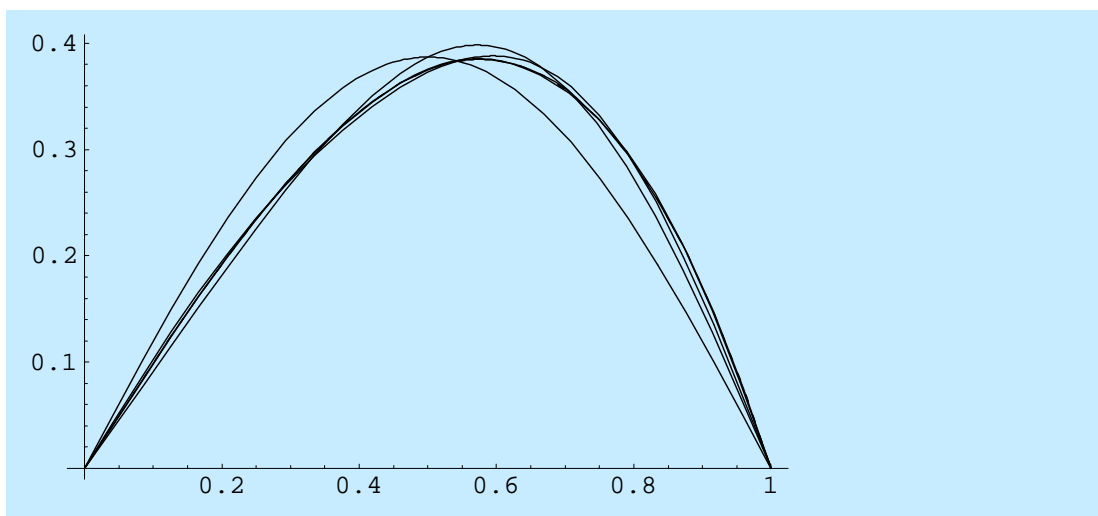
- Graphics -

```
plt2 = Plot[ff[x], {x, 0, 1}, PlotStyle -> Dashing[ {.01, .02}]]
```



- Graphics -

```
Show[plt1, plt2]
```



- Graphics -

■ Now do gg

$$\int_0^1 gg[x] \phi[x, 1] dx$$

$$\frac{4\sqrt{2}(-12 + \pi^2)}{\pi^5}$$

```
ckss = Table[Integrate
  [gg[x] φ[x, i], {x, 0, 1}], {i, 1, 10}]
```

— *General::spell1* : Possible spelling error: new symbol name "ckss" is similar to existing symbol "cks".

$$\left\{ \frac{4\sqrt{2}(-12 + \pi^2)}{\pi^5}, 0, \frac{4\sqrt{2}(-4 + 3\pi^2)}{81\pi^5}, 0, -\frac{\sqrt{2}(24 - 50\pi^2)}{3125\pi^5} + \frac{2\sqrt{2}(-12 + 25\pi^2)}{3125\pi^5}, \right. \\ \left. 0, \frac{4\sqrt{2}(-12 + 49\pi^2)}{16807\pi^5}, 0, \frac{4\sqrt{2}(-4 + 27\pi^2)}{19683\pi^5}, 0 \right\}$$

We see that because gg is symmetric, the odd functions do not contribute.

Again qwe can check the numerical values.

```
ckssN = Table[NIntegrate
  [gg[x] φ[x, i], {x, 0, 1}], {i, 1, 10}]
```

- *General::spell* : Possible spelling error: new symbol name "ckssN" is similar to existing symbols {cksN, ckss}.
- *NIntegrate::ploss* : Numerical integration stopping due to loss of precision. Achieved neither the requested PrecisionGoal nor AccuracyGoal; suspect highly oscillatory integrand, or the true value of the integral is 0. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate.
- *NIntegrate::ploss* : Numerical integration stopping due to loss of precision. Achieved neither the requested PrecisionGoal nor AccuracyGoal; suspect highly oscillatory integrand, or the true value of the integral is 0. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate.
- *NIntegrate::ploss* : Numerical integration stopping due to loss of precision. Achieved neither the requested PrecisionGoal nor AccuracyGoal; suspect highly oscillatory integrand, or the true value of the integral is 0. If your integrand is oscillatory try using the option Method->Oscillatory in NIntegrate.
- *General::stop* : Further output of NIntegrate::ploss will be suppressed during this calculation.

$$\{-0.0393809, -2.29038 \times 10^{-18}, 0.00584427, -2.59785 \times 10^{-18}, 0.00138855, \\ -5.69206 \times 10^{-19}, 0.000518703, -1.48216 \times 10^{-18}, 0.000246507, 5.42101 \times 10^{-19}\}$$

$$\text{oney} = \sum_{i=1}^1 \phi[x, i] \text{ckss}[i]$$

$$\frac{8(-12 + \pi^2) \sin(\pi x)}{\pi^5}$$

$$\mathbf{twoy} = \sum_{i=1}^2 \phi[\mathbf{x}, i] \mathbf{ckss}[i]$$

— *General::spell1* : Possible spelling error: new symbol name "twoy" is similar to existing symbol "twox".

$$\frac{8(-12 + \pi^2) \sin(\pi x)}{\pi^5}$$

$$\mathbf{sixy} = \sum_{i=1}^6 \phi[\mathbf{x}, i] \mathbf{ckss}[i]$$

— *General::spell1* : Possible spelling error: new symbol name "sixy" is similar to existing symbol "sixx".

$$\frac{8(-12 + \pi^2) \sin(\pi x)}{\pi^5} + \frac{8(-4 + 3\pi^2) \sin(3\pi x)}{81\pi^5} + \sqrt{2} \left(-\frac{\sqrt{2}(24 - 50\pi^2)}{3125\pi^5} + \frac{2\sqrt{2}(-12 + 25\pi^2)}{3125\pi^5} \right) \sin(5\pi x)$$

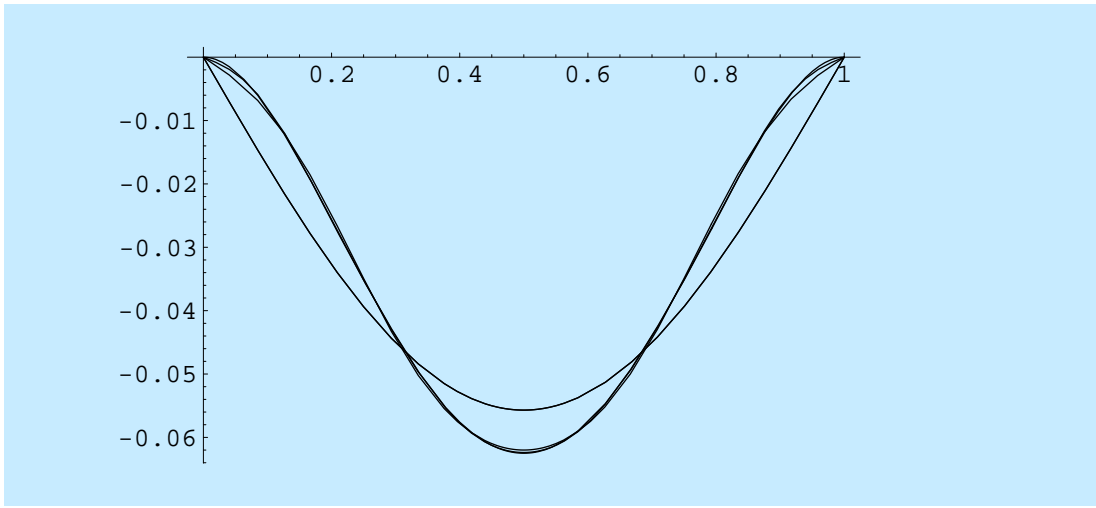
$$\mathbf{teny} = \sum_{i=1}^{10} \phi[\mathbf{x}, i] \mathbf{ckss}[i]$$

— *General::spell1* : Possible spelling error: new symbol name "teny" is similar to existing symbol "tenx".

$$\frac{8(-12 + \pi^2) \sin(\pi x)}{\pi^5} + \frac{8(-4 + 3\pi^2) \sin(3\pi x)}{81\pi^5} + \sqrt{2} \left(-\frac{\sqrt{2}(24 - 50\pi^2)}{3125\pi^5} + \frac{2\sqrt{2}(-12 + 25\pi^2)}{3125\pi^5} \right) \sin(5\pi x) + \frac{8(-12 + 49\pi^2) \sin(7\pi x)}{16807\pi^5} + \frac{8(-4 + 27\pi^2) \sin(9\pi x)}{19683\pi^5}$$

This one converges real fast also.

```
Plot[{gg[x], oney, twoy, sixy, teny}, {x, 0, 1}]
```



- Graphics -

- Now do one with non homogeneous boundary conditions:

$$\int_0^1 (1 - ff[x]) \phi[x, 1] dx$$

$$-\frac{6\sqrt{2}}{\pi^3} + \frac{2\sqrt{2}}{\pi}$$

```
ckss = Table[Integrate
  [(1 - ff[x]) \phi[x, i], {x, 0, 1}], {i, 1, 10}]
```

$$\left\{ -\frac{6\sqrt{2}}{\pi^3} + \frac{2\sqrt{2}}{\pi}, \frac{3}{2\sqrt{2}\pi^3}, -\frac{2\sqrt{2}}{9\pi^3} + \frac{2\sqrt{2}}{3\pi}, \frac{3}{16\sqrt{2}\pi^3}, -\frac{6\sqrt{2}}{125\pi^3} + \frac{2\sqrt{2}}{5\pi}, \right. \\ \left. \frac{1}{18\sqrt{2}\pi^3}, -\frac{6\sqrt{2}}{343\pi^3} + \frac{2\sqrt{2}}{7\pi}, \frac{3}{128\sqrt{2}\pi^3}, -\frac{2\sqrt{2}}{243\pi^3} + \frac{2\sqrt{2}}{9\pi}, \frac{3}{250\sqrt{2}\pi^3} \right\}$$

Again qwe can check the numerical values.

```
ckssN = Table[NIntegrate
  [(1 - ff[x]) φ[x, i], {x, 0, 1}], {i, 1, 10}]
```

— General::spell1 : Possible spelling error: new symbol name "ckssN" is similar to existing symbol "ckss".

```
{0.626653, 0.0342079, 0.28997, 0.00427599, 0.177874, 0.00126696, 0.127819,
  0.000534499, 0.0996598, 0.000273663}
```

$$\text{onez} = \sum_{i=1}^1 \phi[\mathbf{x}, i] \text{ckss}[[i]]$$

$$\sqrt{2} \left(-\frac{6\sqrt{2}}{\pi^3} + \frac{2\sqrt{2}}{\pi} \right) \sin(\pi x)$$

$$\text{twoz} = \sum_{i=1}^2 \phi[\mathbf{x}, i] \text{ckss}[[i]]$$

$$\sqrt{2} \left(-\frac{6\sqrt{2}}{\pi^3} + \frac{2\sqrt{2}}{\pi} \right) \sin(\pi x) + \frac{3 \sin(2\pi x)}{2\pi^3}$$

$$\text{sixz} = \sum_{i=1}^6 \phi[\mathbf{x}, i] \text{ckss}[[i]]$$

$$\begin{aligned} & \sqrt{2} \left(-\frac{6\sqrt{2}}{\pi^3} + \frac{2\sqrt{2}}{\pi} \right) \sin(\pi x) + \frac{3 \sin(2\pi x)}{2\pi^3} + \sqrt{2} \left(-\frac{2\sqrt{2}}{9\pi^3} + \frac{2\sqrt{2}}{3\pi} \right) \sin(3\pi x) + \\ & \frac{3 \sin(4\pi x)}{16\pi^3} + \sqrt{2} \left(-\frac{6\sqrt{2}}{125\pi^3} + \frac{2\sqrt{2}}{5\pi} \right) \sin(5\pi x) + \frac{\sin(6\pi x)}{18\pi^3} \end{aligned}$$

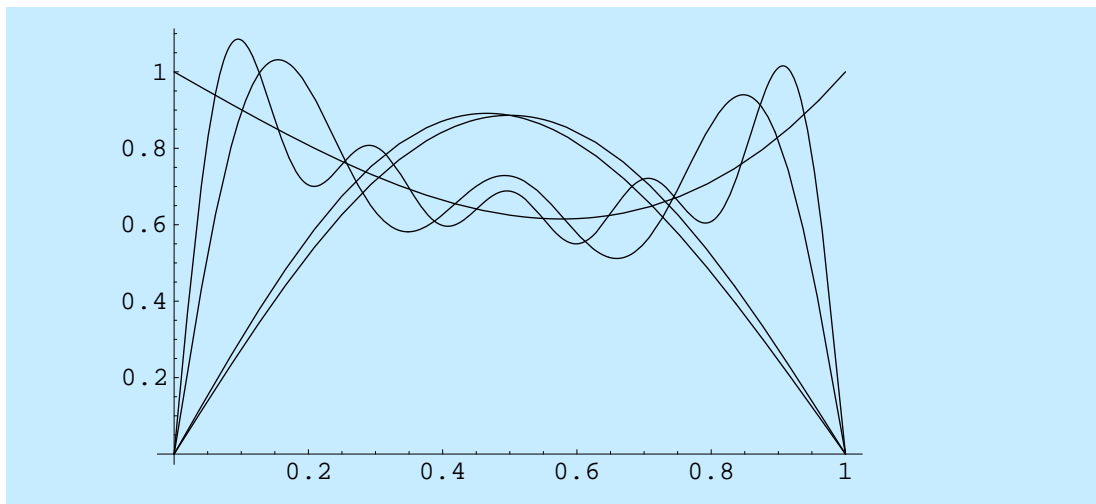
$$\mathbf{tenz} = \sum_{i=1}^{10} \phi[\mathbf{x}, i] \mathbf{ckss}[i]$$

$$\begin{aligned} & \sqrt{2} \left(-\frac{6\sqrt{2}}{\pi^3} + \frac{2\sqrt{2}}{\pi} \right) \sin(\pi x) + \frac{3 \sin(2\pi x)}{2\pi^3} + \sqrt{2} \left(-\frac{2\sqrt{2}}{9\pi^3} + \frac{2\sqrt{2}}{3\pi} \right) \sin(3\pi x) + \\ & \frac{3 \sin(4\pi x)}{16\pi^3} + \sqrt{2} \left(-\frac{6\sqrt{2}}{125\pi^3} + \frac{2\sqrt{2}}{5\pi} \right) \sin(5\pi x) + \frac{\sin(6\pi x)}{18\pi^3} + \\ & \sqrt{2} \left(-\frac{6\sqrt{2}}{343\pi^3} + \frac{2\sqrt{2}}{7\pi} \right) \sin(7\pi x) + \frac{3 \sin(8\pi x)}{128\pi^3} + \sqrt{2} \left(-\frac{2\sqrt{2}}{243\pi^3} + \frac{2\sqrt{2}}{9\pi} \right) \sin(9\pi x) + \\ & \frac{3 \sin(10\pi x)}{250\pi^3} \end{aligned}$$

This one does not converge very well because the boundary conditions cannot be met. We get the "ringing" that is shown in figure 3.7 of V&M.

Think of the consequences of this observation for our problem below.

```
Plot[{(1 - ff[x]), onez, twoz, sixz, tenz}, {x, 0, 1}]
```



- Graphics -

■ Summary of function expansion

To recap, our function is known (at least symbolically) and we choose a set of known eigenfunctions from a convenient operator. The expansion is easily accomplished by forming the inner product which directly gives values of the expansion coefficients for, say,

$$f(x) = \sum_{j=1}^n c_j \phi_j(x).$$

Finite Fourier transform

Now we are ready to solve an inhomogenous ODE with a spectral expansion in terms of the eigenfunctions for $y(x)$ and $f(x)$.

A simple example equation (3.19.1) in V&M is

$$\frac{d^2 y}{dx^2} - U^2 y = -f(x), \quad x \in (0, L)$$

with $y(0) = y(L) = 0$.

The obvious eigenfunctions for this problem come from the eigenvalue problem,

$$\frac{d^2 \phi}{dx^2} = -\lambda \phi, \quad \text{with } \phi(0) = \phi(L) = 0, \quad \text{which we have already solved.}$$

The differential operator, $L[\bullet]$, is then $\frac{d^2 \bullet}{dx^2}$ although as V&M show, it could also be $\frac{d^2 \bullet}{dx^2} - U^2 \bullet$ with no real problem.

We choose, $\frac{d^2 \phi}{dx^2} = -\lambda \phi$ and know that the normalized eigenfunctions are,

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots$$

****Our presumption is that the answer will be: $y[x] = \mathcal{F}^{-1}[y_n] = \sum_{n=1}^{\infty} y_n \phi_n$ *****

Here we go starting with:

$L[y] - U^2 y = -f(x)$, we form the inner product of both sides of the equation with $\phi_n(x)$ by multiplying and integrating

$$\int_0^L \phi_n L[y] dx - U^2 \int_0^L \phi_n y dx = - \int_0^L \phi_n f(x) dx \quad (\text{eq1})$$

We will define an operator called the finite Fourier Transform, $\mathcal{F}[\bullet]$ such that:

$$\mathcal{F}[\bullet] = \int_0^L \rho(x) \phi_n \bullet dx. \quad \text{In this problem, } \rho(x) = 1.$$

From this definition and what we know about expanding an arbitray function and also knowing that $L[\phi] = -\lambda \phi$,

We rearrange eq1 into

$$U^2 \int_0^L \phi_n y dx = U^2 \mathcal{F}[y(x)] = U^2 y_n$$

$$- \int_0^L \phi_n f(x) dx = - \mathcal{F}[f(x)] = - f_n$$

finally we have the term, $\int_0^L \phi_n(x) L[y(x)] dx$.

Since the complete problem is self adjoint, $(\phi_n(x), L[y(x)]) = (y, L[\phi_n(x)])$, we have

$$\int_0^L \phi_n(x) L[y(x)] dx = \int_0^L y(x) L[\phi_n(x)] dx$$

Now we use $L[\phi] = -\lambda\phi$ for the right side of the equation to get rid of the $L[\phi_n(x)]$

****This a common and key step in all of these problems as is forming the inner product****

$$\int_0^L \phi_n(x) L[y(x)] dx = \int_0^L y(x) (-\lambda_n \phi_n(x)) dx = -\lambda_n \int_0^L y(x) (\phi_n(x)) dx = -\lambda_n \mathcal{F}[y(x)] = -\lambda_n y_n.$$

Thus eq1 becomes:

$$-(\lambda_n + U^2) y_n = -f_n.$$

We rearrange it as

$$y_n = \frac{f_n}{(\lambda_n + U^2)}$$

which gives the final answer as:

$$y(x) = \mathcal{F}^{-1}[y_n] = \sum_{n=1}^{\infty} \frac{f_n}{(\lambda_n + U^2)} \phi_n$$

or if we substitute everything,

$$\frac{2}{L} \sum_{n=1}^{\infty} \frac{\left(\int_0^L f(t) \sin\left[\frac{n\pi t}{L}\right] dt \right) \sin\left[\frac{n\pi x}{L}\right]}{U^2 + \frac{n^2 \pi^2}{L^2}}$$

■ Here are some plots of our answer

Here is the solution to the example problem in V&M, p. 275. I need to limit this to a finite number of terms, say 8.

$$\text{ans1} = \frac{2 \sum_{n=1}^8 \frac{\left(\int_0^L f(t) \sin\left[\frac{n\pi t}{L}\right] dt \right) \sin\left[\frac{n\pi x}{L}\right]}{U^2 + \frac{n^2 \pi^2}{L^2}}}{L}$$

$$\frac{1}{L} \left(2 \left(\frac{\left(\int_0^L f(t) \sin\left(\frac{\pi t}{L}\right) dt \right) \sin\left(\frac{\pi x}{L}\right)}{U^2 + \frac{\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{2\pi t}{L}\right) dt \right) \sin\left(\frac{2\pi x}{L}\right)}{U^2 + \frac{4\pi^2}{L^2}} + \right. \right. \\ \frac{\left(\int_0^L f(t) \sin\left(\frac{3\pi t}{L}\right) dt \right) \sin\left(\frac{3\pi x}{L}\right)}{U^2 + \frac{9\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{4\pi t}{L}\right) dt \right) \sin\left(\frac{4\pi x}{L}\right)}{U^2 + \frac{16\pi^2}{L^2}} + \\ \frac{\left(\int_0^L f(t) \sin\left(\frac{5\pi t}{L}\right) dt \right) \sin\left(\frac{5\pi x}{L}\right)}{U^2 + \frac{25\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{6\pi t}{L}\right) dt \right) \sin\left(\frac{6\pi x}{L}\right)}{U^2 + \frac{36\pi^2}{L^2}} + \\ \left. \left. \frac{\left(\int_0^L f(t) \sin\left(\frac{7\pi t}{L}\right) dt \right) \sin\left(\frac{7\pi x}{L}\right)}{U^2 + \frac{49\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{8\pi t}{L}\right) dt \right) \sin\left(\frac{8\pi x}{L}\right)}{U^2 + \frac{64\pi^2}{L^2}} \right) \right)$$

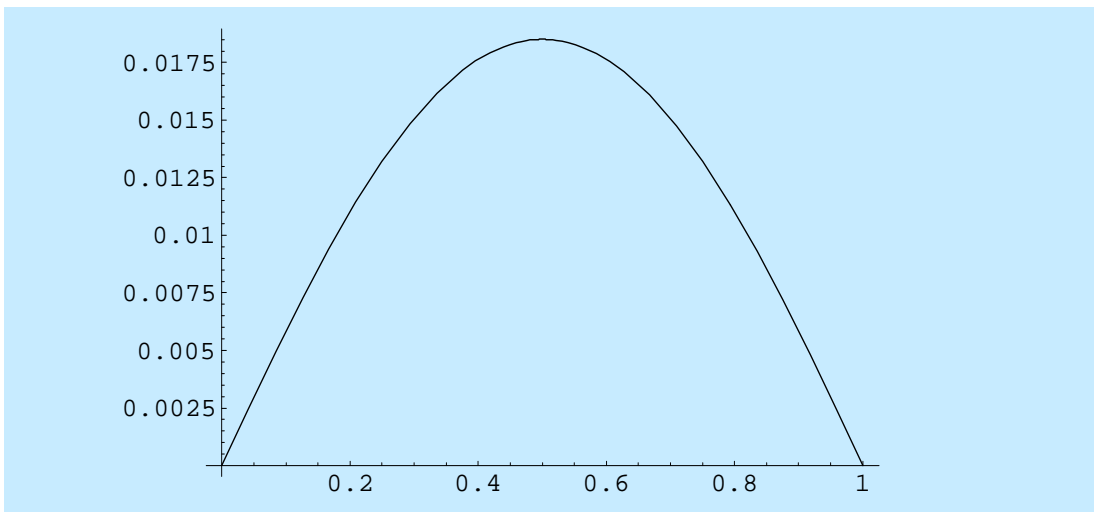
Choose a length of 1 and order 1 external cooling,

`ans2 = ans1 /. {L -> 1, U -> 2}`

$$2 \left(\frac{\left(\int_0^1 f(t) \sin(\pi t) dt \right) \sin(\pi x)}{4 + \pi^2} + \frac{\left(\int_0^1 f(t) \sin(2\pi t) dt \right) \sin(2\pi x)}{4 + 4\pi^2} + \right. \\ \frac{\left(\int_0^1 f(t) \sin(3\pi t) dt \right) \sin(3\pi x)}{4 + 9\pi^2} + \frac{\left(\int_0^1 f(t) \sin(4\pi t) dt \right) \sin(4\pi x)}{4 + 16\pi^2} + \\ \frac{\left(\int_0^1 f(t) \sin(5\pi t) dt \right) \sin(5\pi x)}{4 + 25\pi^2} + \frac{\left(\int_0^1 f(t) \sin(6\pi t) dt \right) \sin(6\pi x)}{4 + 36\pi^2} + \\ \left. \frac{\left(\int_0^1 f(t) \sin(7\pi t) dt \right) \sin(7\pi x)}{4 + 49\pi^2} + \frac{\left(\int_0^1 f(t) \sin(8\pi t) dt \right) \sin(8\pi x)}{4 + 64\pi^2} \right)$$

We choose heating of the form $f[t]$ from above

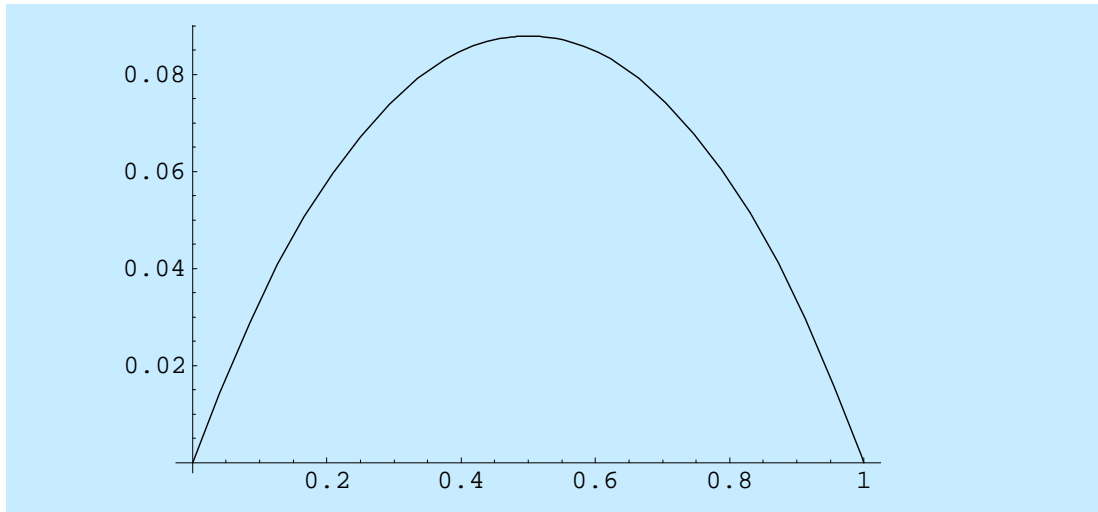
`Plot[ans2 /. {f[t] -> t (1 - t)}, {x, 0, 1}]`



- Graphics -

We see that it is hottest in the middle where the heat generation is largest. But this is not an insightful answer because the ends are fixed at $T=0$ so it will always be hottest in the middle. For example consider uniform heating

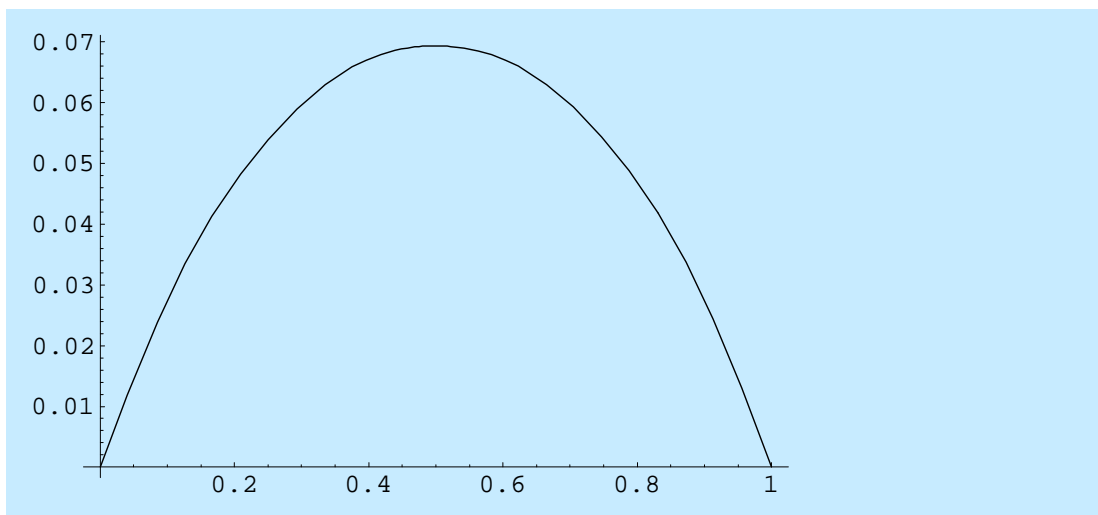
```
Plot[ans2 /. {f[t] -> 1}, {x, 0, 1}]
```



- Graphics -

Let's try to mess it up a bit with heating that is strongest at the ends,

```
Plot[ans2 /. {f[t] -> 1 - t (1 - t)}, {x, 0, 1}]
```



- Graphics -

Let's see if this last one changes much if the number of terms is reduced.

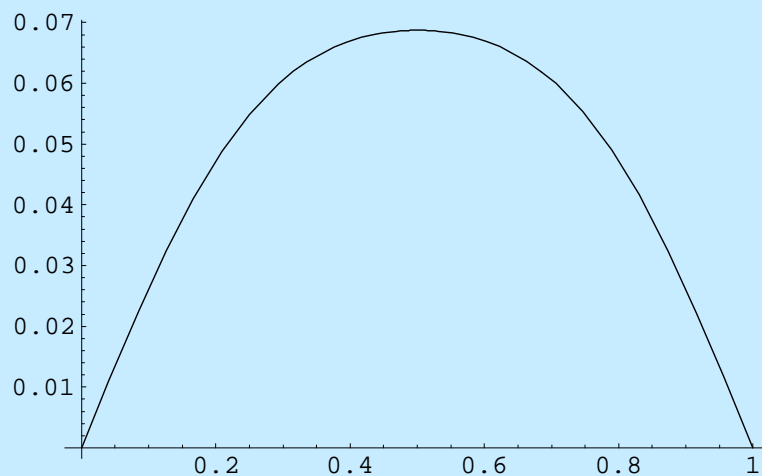
$$\text{ans3} = \frac{2 \sum_{n=1}^4 \frac{\left(\int_0^1 f[t] \sin\left[\frac{n\pi t}{L}\right] dt\right) \sin\left[\frac{n\pi x}{L}\right]}{U^2 + \frac{n^2 \pi^2}{L^2}}}{L}$$

$$\frac{1}{L} \left(2 \left(\frac{\left(\int_0^L f(t) \sin\left(\frac{\pi t}{L}\right) dt\right) \sin\left(\frac{\pi x}{L}\right)}{U^2 + \frac{\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{2\pi t}{L}\right) dt\right) \sin\left(\frac{2\pi x}{L}\right)}{U^2 + \frac{4\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{3\pi t}{L}\right) dt\right) \sin\left(\frac{3\pi x}{L}\right)}{U^2 + \frac{9\pi^2}{L^2}} + \frac{\left(\int_0^L f(t) \sin\left(\frac{4\pi t}{L}\right) dt\right) \sin\left(\frac{4\pi x}{L}\right)}{U^2 + \frac{16\pi^2}{L^2}} \right) \right)$$

$$\text{ans4} = \text{ans3} /. \{\mathbf{L} \rightarrow 1, \mathbf{U} \rightarrow 2\}$$

$$2 \left(\frac{\left(\int_0^1 f(t) \sin(\pi t) dt\right) \sin(\pi x)}{4 + \pi^2} + \frac{\left(\int_0^1 f(t) \sin(2\pi t) dt\right) \sin(2\pi x)}{4 + 4\pi^2} + \frac{\left(\int_0^1 f(t) \sin(3\pi t) dt\right) \sin(3\pi x)}{4 + 9\pi^2} + \frac{\left(\int_0^1 f(t) \sin(4\pi t) dt\right) \sin(4\pi x)}{4 + 16\pi^2} \right)$$

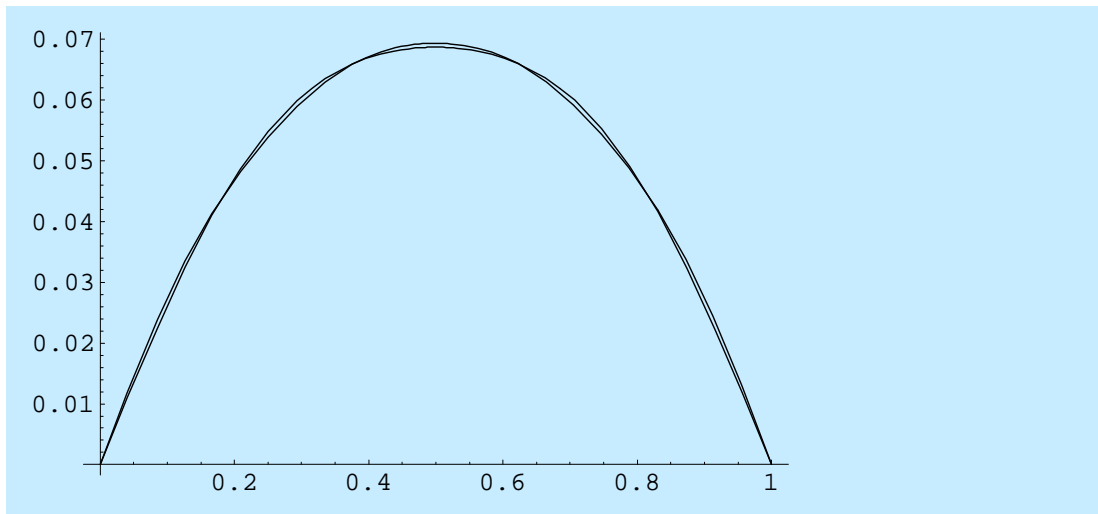
$$\text{Plot}[\text{ans4} /. \{\mathbf{f}[t] \rightarrow 1 - t(1 - t)\}, \{\mathbf{x}, 0, 1\}]$$



- Graphics -

Here we compare the solution with $f[t] = 1-t(1-t)$ for 4 terms and 8 terms. There is not much difference perhaps because the end regions, where the fit to $f(t)$ is not easy to do, does not contribute too much. You may wish to check this further.

Show [%32, %]



- Graphics -

■ Summary of Finite Fourier transform technique

From the form of the ode, $L[y] = -f(x)$ (Note that original equation could also be $L[y] + \beta y = -f(x)$ because it leads to the same eigenvalue problem.) and boundary conditions we formulate $L[\phi] = -\lambda\phi$ and solve (using the same boundary conditions) to get the basis functions (A complete set of eigenfunctions that will be able to represent any finite function or solution).

We then presume a spectral solution form of $y[x] = \mathcal{F}^{-1}[y_n] = \sum_{n=1}^{\infty} y_n \phi_n$.

We take the Finite Fourier transform of each term in the equation. This means form the inner product: (ODE, ϕ_n) .

- Any term that contains a factor, say β , multiplying y will just yield $\beta \mathcal{F}[y(x)] = \beta y_n$
- Any term with a prescribed inhomogeneous function becomes, $\mathcal{F}[f(x)] = f_n$ (and we know how to expand any function).
- The one term that requires manipulation is, $\int_0^L \phi_n(x) L[y(x)] dx$, which now came from $(\phi_n(x), L[y(x)])$ and thus must be equal, for our self adjoint equation to, $(y(x), L[\phi_n(x)])$. This is just, $(y(x), -\lambda_n \phi_n(x))$ or in terms of the definition of $\mathcal{F}[y(x)]$ it becomes, $-\lambda_n y_n$
- Just solve this equation for y_n and then transform back to get $y[x]$,

$$y(x) = \mathcal{F}^{-1}[y_n] = \sum_{n=1}^{\infty} y_n \phi_n$$

- Substitute what you need.