Creeping flow caused by a rotating sphere

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This notebook shows how to solve the flow field for a rotating sphere at low Reynolds number showing the 0 order and first order solutions. A regular perturbation expansion is used.

This notebook is intended as a supplement to
L. G. Leal (1992) *Laminar flow and Convective Transport Processes*, Butterworth.

Problem formulation

Consider an infinite quiescent fluid. A sphere of radius a is placed in the region and cause to rotate with angular velocity, Ω, such that the Reynolds number, $R = a^2 \, \Omega \, \rho \, / \, \mu$ is small compared to 1. The 0 order governing equation is just the v_ϕ equation without any inertia terms. The answer will have the same angular dependence as the velocity on the surface of the sphere (See Leal's discussion starting on Page 135) and decay with distance. Since intertia is weak, any radial flow associated with centrifugal effects will enter at higher order.

We solve the 0 order problem first. Then we look to find out how the 0 order solution produces weak inertia to cause a flow at first order.

The 0 order problem. Flow is only in ϕ direction.

A sphere is rotating in an otherwise quiescent medium. The equation for v_ϕ assuming creeping flow is:

$$\mathbf{v\phi eq} = \frac{\frac{\partial_r\left(r^2\,\partial_r \mathrm{v}\phi[r,\theta]\right)}{r^2} + \frac{\partial_\theta \frac{\partial_\theta\left(\mathrm{v}\phi[r,\theta]\,\mathrm{Sin}[\theta]\right)}{\mathrm{Sin}[\theta]}}{r^2}}{R}$$

$$\frac{1}{R}\left(\frac{1}{r^2}\left(\csc(\theta)\left(-\sin(\theta)\,\mathrm{v}\phi(r,\theta)+2\cos(\theta)\,\mathrm{v}\phi^{(0,1)}(r,\theta)+\sin(\theta)\,\mathrm{v}\phi^{(0,2)}(r,\theta)\right)-\cot(\theta)\csc(\theta)\left(\cos(\theta)\,\mathrm{v}\phi(r,\theta)+\sin(\theta)\,\mathrm{v}\phi^{(0,1)}(r,\theta)\right)\right)+\frac{\mathrm{v}\phi^{(2,0)}(r,\theta)\,r^2+2\,\mathrm{v}\phi^{(1,0)}(r,\theta)\,r}{r^2}\right)$$

We can assume a form of the solution from the boundary conditions. Note that this is simply saying the velocity is highest at the equator and 0 at the "poles".

```
vfi = f[r] Sin[θ];
```

Now substitute and see what happens

$$\mathbf{v\phi eqO01} = \mathbf{v\phi eq} \;/.\; \left\{\mathrm{v}\phi^{(a1\_,a2\_)}[r,\theta] \mapsto \partial_{\{r,a1\},\{\theta,a2\}}\,\mathrm{vfi},\ \mathrm{v}\phi[r,\theta]\to\mathrm{vfi}\right\}$$

$$\frac{\frac{\csc(\theta)\left(2\cos^2(\theta)\,f(r)-2\,f(r)\sin^2(\theta)\right)-2\cos(\theta)\cot(\theta)\,f(r)}{r^2}+\frac{\sin(\theta)\,f''(r)\,r^2+2\sin(\theta)\,f'(r)\,r}{r^2}}{R}$$

The substitution has worked and the angular dependence of the equation has simplified.

```
vϕeqO02 = Expand[vϕeqO01]
```

$$-\frac{2\,f(r)\sin(\theta)}{r^2\,R}+\frac{2\,f'(r)\sin(\theta)}{r\,R}+\frac{f''(r)\sin(\theta)}{R}$$

We can put this into a standard form, and Euler equation, by multiplying by the appropriate factor

$$\mathbf{v\phi eqO03} = \mathrm{Expand}\left[\frac{\mathrm{v\phi eqO02}\ r^2\ R}{\mathrm{Sin}[\theta]}\right]$$

$$f''(r)\,r^2+2\,f'(r)\,r-2\,f(r)$$

This is readily solved.

```
ansO01 = DSolve[{vϕeqO03 == 0}, f[r], r]
```

$$\left\{\left\{f(r)\to r\,c_1+\frac{c_2}{r^2}\right\}\right\}$$

Because we need the solution to decay away from the sphere and be 0 sufficiently far away, C[2] = 0. On the surface of the sphere, the velocity is $a\,\Omega$ Sin[θ]. The Sin[θ] is already built in so we just have that C[1] = $a^3\,\Omega$

```
ans002 = f[r] /. ans001⟦1⟧
```

$$r\,c_1+\frac{c_2}{r^2}$$

```
ans003 = ans002 /. {C[1] → 0, C[2] → a^3 Ω}
```

$$\frac{a^3\,\Omega}{r^2}$$

```
O0ans = ans003 Sin[θ]
```

$$\frac{a^3\,\Omega\,\sin(\theta)}{r^2}$$

■ The first order probem which comes from perturbation theory

Now supppose that we want to improve this by using a pertubation solution. If we examine the inertia (nonlinear) terms in the v_ϕ equation, we see that the 0 order solution for v_ϕ produces no correction terms at R^1 order. The pure rotation solution that was obtained is an exact solution to the $\mathrm{v}_\phi$equation in the absense of v_r or v_θ. Thus we need to examine the v_r and v_θ equations to find the first imrpovement term. In the v_r equation, there is a $-\mathrm{v}_\phi{}^2/r$ term. In the θ equation, there is a similar term. We are seeing that rotation with finite *R* causes a weak centrifugal force that leads to a flow in the r direction -- we suspect outward at the equator. To conserve mass, there is necessarily an inflow someplace, presumably at the point of weakest inertia, the poles. This flow at R^1requires both the v_r and v_θ equations to describe it. Let's see what we can do.

Here is the axisymetric (about the ϕ axis) v_r equation with the additional inertia term. This is produced by the expansion of the equations at R^1 order.

$$\mathbf{vr1eq} = -\frac{\mathrm{v}\phi[\mathrm{r},\theta]^2}{\mathrm{r}}-\left(\partial_\mathrm{r}\frac{\partial_\mathrm{r}(\mathrm{vr}[\mathrm{r},\theta]\,\mathrm{r}^2)}{\mathrm{r}^2}+\frac{\partial_\theta(\mathrm{Sin}[\theta]\,\partial_\theta\mathrm{vr}[\mathrm{r},\theta])}{\mathrm{r}^2\,\mathrm{Sin}[\theta]}-\frac{2\,\partial_\theta(\mathrm{v}\theta[\mathrm{r},\theta]\,\mathrm{Sin}[\theta])}{\mathrm{r}^2\,\mathrm{Sin}[\theta]}-\partial_\mathrm{r}\mathrm{p}[\mathrm{r},\theta]\right)$$

$$-\frac{\mathrm{v}\phi(r,\theta)^2}{r}+\frac{2\csc(\theta)\,(\cos(\theta)\,\mathrm{v}\theta(r,\theta)+\sin(\theta)\,\mathrm{v}\theta^{(0,1)}(r,\theta))}{r^2}-\frac{\csc(\theta)\,(\cos(\theta)\,\mathrm{vr}^{(0,1)}(r,\theta)+\sin(\theta)\,\mathrm{vr}^{(0,2)}(r,\theta))}{r^2}+p^{(1,0)}(r,\theta)+\frac{2\,(\mathrm{vr}^{(1,0)}(r,\theta)\,r^2+2\,\mathrm{vr}(r,\theta)\,r)}{r^3}-\frac{\mathrm{vr}^{(2,0)}(r,\theta)\,r^2+4\,\mathrm{vr}^{(1,0)}(r,\theta)\,r+2\,\mathrm{vr}(r,\theta)}{r^2}$$

Here is the corresponding v_θ equation.

$$\mathbf{v\theta1eq} = -\frac{\mathbf{v\phi[r,\ \theta]^2\ Cot[\theta]}}{\mathbf{r}} - \left(\frac{\partial_{\mathbf{r}}\,(\mathbf{r^2}\,\partial_{\mathbf{r}}\,\mathbf{v\theta[r,\ \theta]})}{\mathbf{r^2}} + \frac{\partial_\theta\,\frac{\partial_\theta\,(\mathbf{v\theta[r,\theta]\ Sin[\theta]})}{\mathbf{Sin[\theta]}}}{\mathbf{r^2}} + \frac{\mathbf{2}\,\partial_\theta\,\mathbf{vr[r,\ \theta]}}{\mathbf{r^2}} - \frac{\partial_\theta\,\mathbf{p[r,\ \theta]}}{\mathbf{r}}\right)$$

$$-\frac{\cot(\theta)\,\mathrm{v}\phi(r,\,\theta)^2}{r} + \frac{p^{(0,1)}(r,\,\theta)}{r} - \frac{2\,\mathrm{vr}^{(0,1)}(r,\,\theta)}{r^2} - \frac{1}{r^2}\,(\csc(\theta)\,(-\sin(\theta)\,\mathrm{v}\theta(r,\,\theta) + 2\cos(\theta)\,\mathrm{v}\theta^{(0,1)}(r,\,\theta) + \sin(\theta)\,\mathrm{v}\theta^{(0,2)}(r,\,\theta)) - \cot(\theta)\csc(\theta)\,(\cos(\theta)\,\mathrm{v}\theta(r,\,\theta) + \sin(\theta)\,\mathrm{v}\theta^{(0,1)}(r,\,\theta))) - \frac{\mathrm{v}\theta^{(2,0)}(r,\,\theta)\,r^2 + 2\,\mathrm{v}\theta^{(1,0)}(r,\,\theta)\,r}{r^2}$$

Now we can derive the equation for the stream function easily enough, or could we do something else?? (See Leal for a general method).

Let's get the stream function

$$\mathbf{eq011 = Expand}[\partial_\theta\,\mathbf{vr1eq} - \partial_{\mathbf{r}}\,(\mathbf{r\ v\theta1eq})]$$

$$-\frac{\mathrm{v}\theta(r,\,\theta)\cot^2(\theta)}{r^2} + \frac{\mathrm{vr}^{(0,1)}(r,\,\theta)\cot^2(\theta)}{r^2} - \frac{\mathrm{v}\theta^{(1,0)}(r,\,\theta)\cot^2(\theta)}{r} + \frac{\mathrm{v}\theta^{(0,1)}(r,\,\theta)\cot(\theta)}{r^2} - \frac{\mathrm{vr}^{(0,2)}(r,\,\theta)\cot(\theta)}{r^2} + 2\,\mathrm{v}\phi(r,\,\theta)\,\mathrm{v}\phi^{(1,0)}(r,\,\theta)\cot(\theta) + \frac{\mathrm{v}\theta^{(1,1)}(r,\,\theta)\cot(\theta)}{r} - \frac{\mathrm{v}\theta(r,\,\theta)}{r^2} + \frac{\mathrm{vr}^{(0,1)}(r,\,\theta)}{r^2} - \frac{2\,\mathrm{v}\phi(r,\,\theta)\,\mathrm{v}\phi^{(0,1)}(r,\,\theta)}{r} + \frac{\mathrm{v}\theta^{(0,2)}(r,\,\theta)}{r^2} - \frac{\mathrm{vr}^{(0,3)}(r,\,\theta)}{r^2} - \frac{\mathrm{v}\theta^{(1,0)}(r,\,\theta)}{r} + \frac{\mathrm{v}\theta^{(1,2)}(r,\,\theta)}{r} + 3\,\mathrm{v}\theta^{(2,0)}(r,\,\theta) - \mathrm{vr}^{(2,1)}(r,\,\theta) + r\,\mathrm{v}\theta^{(3,0)}(r,\,\theta)$$

From the continuity equation we can replace v_{r} and v_θ.

$$\mathbf{eq012 = eq011\ /.}\ \left\{\mathbf{vr}^{(\mathbf{a1\_},\mathbf{a2\_})}\mathbf{[r,\ \theta]} \mapsto \partial_{\{\mathbf{r},\mathbf{a1}\},\{\theta,\mathbf{a2}\}}\,\frac{\partial_\theta\,\psi\mathbf{[r,\ \theta]}}{\mathbf{r^2\ Sin[\theta]}},\ \mathbf{vr[r,\ \theta]} \to \frac{\partial_\theta\,\psi\mathbf{[r,\ \theta]}}{\mathbf{r^2\ Sin[\theta]}},\ \mathbf{v\theta}^{(\mathbf{a1\_},\mathbf{a2\_})}\mathbf{[r,\ \theta]} \mapsto \partial_{\{\mathbf{r},\mathbf{a1}\},\{\theta,\mathbf{a2}\}}\left(-\frac{\partial_{\mathbf{r}}\,\psi\mathbf{[r,\ \theta]}}{\mathbf{r\ Sin[\theta]}}\right),\ \mathbf{v\theta[r,\ \theta]} \to -\frac{\partial_{\mathbf{r}}\,\psi\mathbf{[r,\ \theta]}}{\mathbf{r\ Sin[\theta]}}\right\}$$

We can make it look a lot better.

```
eqO13 = Expand[eqO12 Sin[θ]]
```

$$-\frac{\psi^{(0,1)}(r,\theta)\cot^3(\theta)}{r^4}+\frac{\psi^{(2,0)}(r,\theta)\cot^2(\theta)}{r^2}+\frac{4\csc^2(\theta)\,\psi^{(0,1)}(r,\theta)\cot(\theta)}{r^4}+\frac{5\,\psi^{(0,1)}(r,\theta)\cot(\theta)}{r^4}+\frac{2\,\psi^{(0,3)}(r,\theta)\cot(\theta)}{r^4}-\frac{4\,\psi^{(1,1)}(r,\theta)\cot(\theta)}{r^3}+\frac{2\,\psi^{(2,1)}(r,\theta)\cot(\theta)}{r^2}-\frac{2\sin(\theta)\,\mathrm{v}\phi(r,\theta)\,\mathrm{v}\phi^{(0,1)}(r,\theta)}{r}-\frac{3\csc^2(\theta)\,\psi^{(0,2)}(r,\theta)}{r^4}-\frac{5\,\psi^{(0,2)}(r,\theta)}{r^4}-\frac{\psi^{(0,4)}(r,\theta)}{r^4}+2\cos(\theta)\,\mathrm{v}\phi(r,\theta)\,\mathrm{v}\phi^{(1,0)}(r,\theta)+\frac{4\,\psi^{(1,2)}(r,\theta)}{r^3}-\frac{\csc^2(\theta)\,\psi^{(2,0)}(r,\theta)}{r^2}+\frac{\psi^{(2,0)}(r,\theta)}{r^2}-\frac{2\,\psi^{(2,2)}(r,\theta)}{r^2}-\psi^{(4,0)}(r,\theta)$$

Now substitute the 0 order solution for v_ϕ

```
eqO14 = eqO13 /.
  {vϕ[r, θ] → O0ans, ∂θ vϕ[r, θ] :→ ∂θ O0ans, ∂r vϕ[r, θ] :→ ∂r O0ans}
```

$$-\frac{6\,\Omega^2\cos(\theta)\sin^2(\theta)\,a^6}{r^5}-\frac{\cot^3(\theta)\,\psi^{(0,1)}(r,\theta)}{r^4}+\frac{4\cot(\theta)\csc^2(\theta)\,\psi^{(0,1)}(r,\theta)}{r^4}+\frac{5\cot(\theta)\,\psi^{(0,1)}(r,\theta)}{r^4}-\frac{3\csc^2(\theta)\,\psi^{(0,2)}(r,\theta)}{r^4}-\frac{5\,\psi^{(0,2)}(r,\theta)}{r^4}+\frac{2\cot(\theta)\,\psi^{(0,3)}(r,\theta)}{r^4}-\frac{\psi^{(0,4)}(r,\theta)}{r^4}-\frac{4\cot(\theta)\,\psi^{(1,1)}(r,\theta)}{r^3}+\frac{4\,\psi^{(1,2)}(r,\theta)}{r^3}+\frac{\cot^2(\theta)\,\psi^{(2,0)}(r,\theta)}{r^2}-\frac{\csc^2(\theta)\,\psi^{(2,0)}(r,\theta)}{r^2}+\frac{\psi^{(2,0)}(r,\theta)}{r^2}+\frac{2\cot(\theta)\,\psi^{(2,1)}(r,\theta)}{r^2}-\frac{2\,\psi^{(2,2)}(r,\theta)}{r^2}-\psi^{(4,0)}(r,\theta)$$

Now we have a PDE that we need to solve. The problem is, how??
One way is to guess the function form for the θ dependence. The boundary relation for the 0 order problem is clear but does readily not give insight into the first order problem. Of course we could just guess and try. But what is this equation really?

It should be an axisymmetric creeping flow problem with an inhomogeneous term

$E^4\,\psi = 6\,a^6\,\Omega^2\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]^2/r^5$.

We know the general solutions to this equation. We would need a particular equation. The boundary conditions for the rotating problem are all fit at 0 order so the homogenous solution and the particular solution must cancel on the boundary for this perturbation correction to work.

Now that we have some familarity, we could translate using $\eta\equiv\cos(\theta)$ (see page 160) and look at the general solutions. A second way would be to look at the η solutions in terms of Gegenbauer polynomials for the expected form of the θ dependence(see page 163). What are the choices:
Cos[θ] + 1, which is not in the general solution.

The next one is

```
Simplify[(Cos[θ]^2 - 1) / 2]
```

$$-\frac{1}{2}\sin^2(\theta)$$

It does not look promising. Continue on with the third,

```
Simplify[Cos[θ]^3 - Cos[θ]]
```

$-\cos(\theta)\sin^2(\theta)$

This looks great!! It has the same form as the inhomogeneous term. Try this form for the θ dependence.

```
eqO15 =
 Expand[eqO14 /. {ψ^(a1_,a2_)[r, θ] :→ ∂_{r,a1},{θ,a2} (Sin[θ]^2 Cos[θ] f1[r])}]
```

$$-\frac{6\,\Omega^2\cos(\theta)\sin^2(\theta)\,a^6}{r^5}-\frac{17\cos(\theta)\,\mathrm{f1}(r)\sin^2(\theta)}{r^4}-\frac{19\cos^3(\theta)\,\mathrm{f1}(r)}{r^4}-\frac{2\cos^3(\theta)\cot^2(\theta)\,\mathrm{f1}(r)}{r^4}+\frac{2\cos(\theta)\cot^2(\theta)\,\mathrm{f1}(r)}{r^4}+\frac{17\cos(\theta)\,\mathrm{f1}(r)}{r^4}-\frac{24\cos(\theta)\sin^2(\theta)\,\mathrm{f1}'(r)}{r^3}+\frac{\cos^3(\theta)\,\mathrm{f1}''(r)}{r^2}+\frac{13\cos(\theta)\sin^2(\theta)\,\mathrm{f1}''(r)}{r^2}-\frac{\cos(\theta)\,\mathrm{f1}''(r)}{r^2}-\cos(\theta)\sin^2(\theta)\,\mathrm{f1}^{(4)}(r)$$

```
eqO16 = Simplify[eqO15]
```

$$-\frac{\cos(\theta)\sin^2(\theta)\left(6\,\Omega^2\,a^6+24\,r^2\,\mathrm{f1}'(r)-12\,r^3\,\mathrm{f1}''(r)+r^5\,\mathrm{f1}^{(4)}(r)\right)}{r^5}$$

By knowing what we want we can simplify the equation,

```
eqO17 = Expand[r^4 eqO16 / (Cos[θ] Sin[θ]^2)]
```

$$-\frac{6\,\Omega^2\,a^6}{r}-24\,r\,\mathrm{f1}'(r)+12\,r^2\,\mathrm{f1}''(r)-r^4\,\mathrm{f1}^{(4)}(r)$$

The equation is Euler's equation which is easily solved.

```
ans011 = DSolve[eqO17 == 0, f1[r], r]
```

$$\left\{\left\{\mathrm{f1}(r)\to\frac{\Omega^2\,a^6}{4\,r}+\frac{c_1}{r^2}+c_2+r^3\,c_3+r^5\,c_4\right\}\right\}$$

```
ans012 = f1[r] /. ans011[[1]]
```

$$\frac{\Omega^2\,a^6}{4\,r}+\frac{c_1}{r^2}+c_2+r^3\,c_3+r^5\,c_4$$

As mentioned earlier, we need the R^1 solution to be 0 on the boundaries. As r -->∞ this requires that C[3], C[4] = 0.

```
ans013 = ans012 /. {C[3] → 0, C[4] → 0}
```

$$\frac{\Omega^2 a^6}{4 r} + \frac{c_1}{r^2} + c_2$$

Now fit the boundary conditions on the sphere

$$\mathtt{bc1} = \frac{\mathtt{ans013}}{\mathtt{r}} \;\mathtt{/.\ r \to a}$$

$$\frac{\frac{\Omega^2 a^5}{4} + c_2 + \frac{c_1}{a^2}}{a}$$

$$\mathtt{bc2} = -\frac{\partial_{\mathtt{r}}\,\mathtt{ans013}}{\mathtt{r}^2} \;\mathtt{/.\ r \to a}$$

$$-\frac{-\frac{1}{4}\Omega^2 a^4 - \frac{2 c_1}{a^3}}{a^2}$$

Everything is 0 on the boundaries

```
consts01 = Solve[{bc1 == 0, bc2 == 0}, {C[1], C[2]}]
```

$$\left\{\left\{c_2 \to -\frac{1}{8} a^5 \Omega^2,\; c_1 \to -\frac{1}{8} a^7 \Omega^2\right\}\right\}$$

Thus, the stream function for R^1 order is

$$\mathtt{ans01last = (ans013\ /.\ consts01[\![1]\!])\ Sin[\theta]^2\ Cos[\theta]}$$

$$\left(-\frac{\Omega^2 a^7}{8 r^2} + \frac{\Omega^2 a^6}{4 r} - \frac{\Omega^2 a^5}{8}\right)\cos(\theta)\sin^2(\theta)$$

Examine the first order solution graphically.

Make the requisite polar to Cartesian transformation,

$$\mathtt{ans01Cartesian = ans01last\ /.\ \left\{r \to \sqrt{x^2 + y^2},\ \theta \to ArcTan\left[\frac{y}{x}\right]\right\}}$$

$$\frac{y^2\left(-\frac{\Omega^2 a^7}{8(x^2+y^2)} + \frac{\Omega^2 a^6}{4\sqrt{x^2+y^2}} - \frac{\Omega^2 a^5}{8}\right)}{x^2\left(\frac{y^2}{x^2}+1\right)^{3/2}}$$

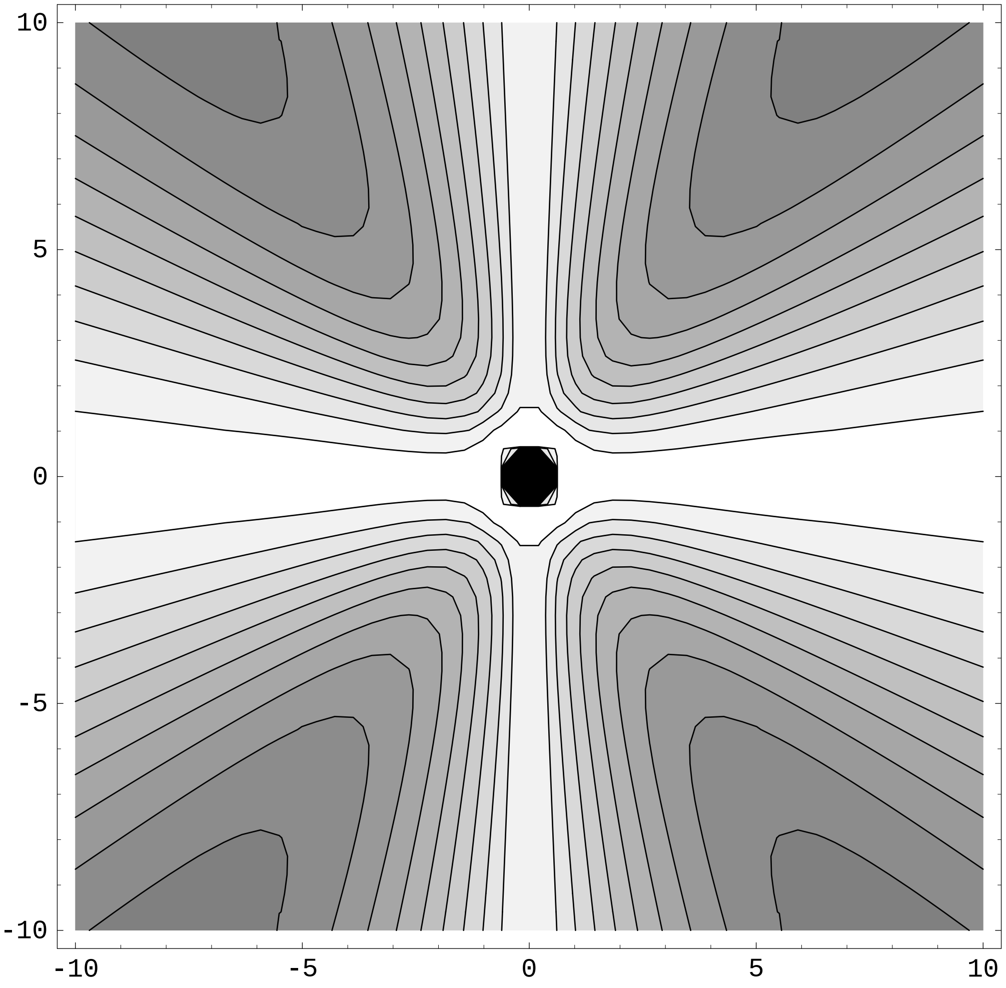
Substitute in some numbers,

```
anstoplot = ans01Cartesian /. {Ω → 5, a → 1}
```

$$\frac{y^2\left(-\frac{25}{8}+\frac{25}{4\sqrt{x^2+y^2}}-\frac{25}{8\,(x^2+y^2)}\right)}{x^2\left(\frac{y^2}{x^2}+1\right)^{3/2}}$$

The stream lines look like,

```
ContourPlot[anstoplot, {x, -10, 10}, {y, -10, 10}, Contours → 20,
 PlotPoints -> 50]
```



```
- ContourGraphics -
```

We see the circulation pattern. Flows are strongest outward along the equator and this fluid is replaced by inflows along the poles.