Laminar channel flow over a (long)wavy surface

This notebook has been written in *Mathematica* by

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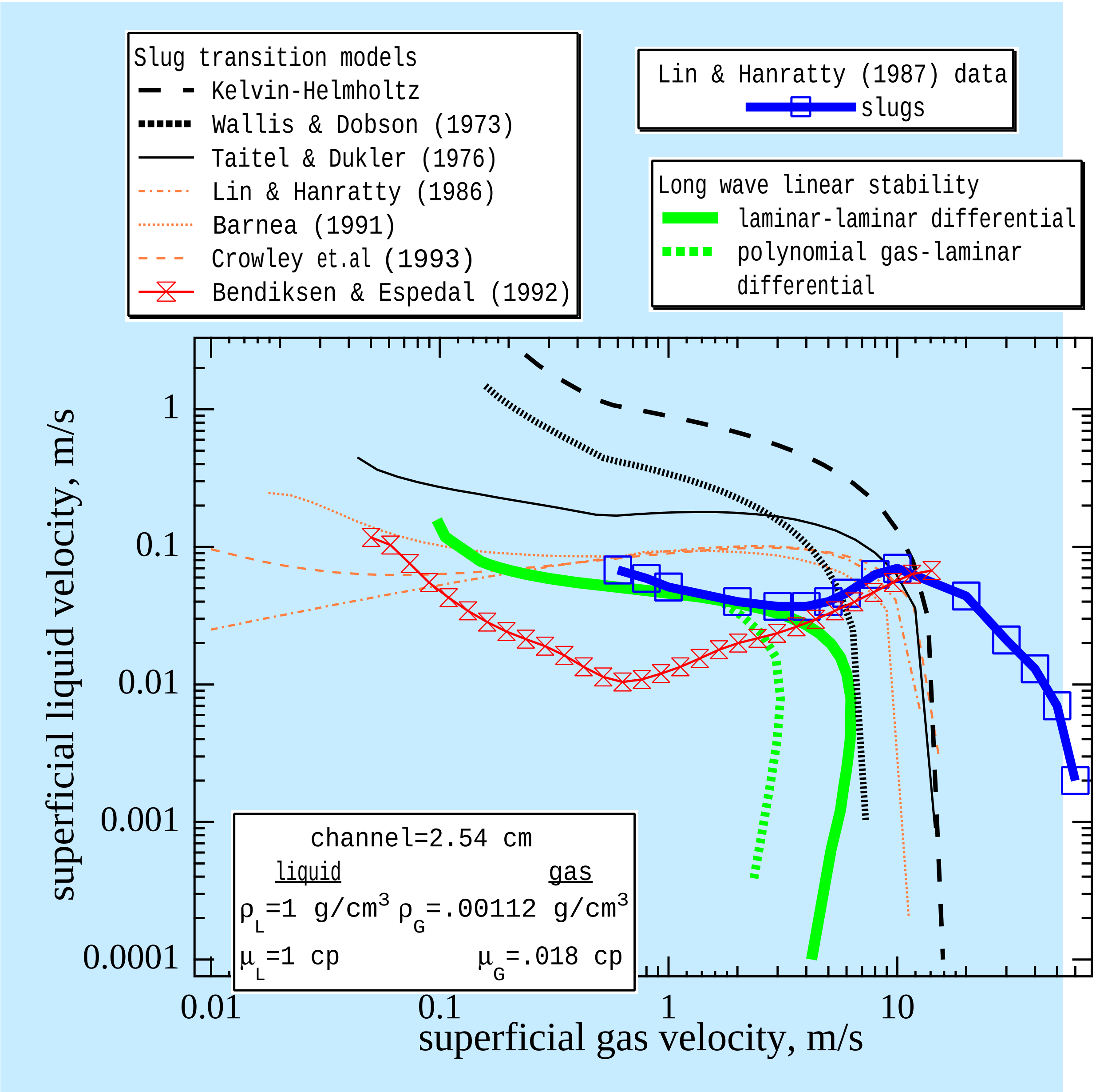
More recent versions of this notebook should be available at the web site:
http://www.nd.edu/~mjm/pressure.shear.fluctuations.nb

Motivation for this notebook



A number of studies have suggested that regime transitions from stratified to slug and/or annular flow occur when then stratified flow becomes unstable to long waves. The plot above shows a number of different models, most based on linear stability theory, to predict these flow regime transitions. They do not agree. Further, the integral equation models, Lin & Hanratty, Barnea, Crowley et al., use an integrated form of the momentum equations -- as opposed to the differential equations, and these results do not agree (compare orange versus green lines). This lack of agreement is troubling and there are two major issues.

First we need to determine the inherent error in using differential as opposed to an integral form of the momentum equations. We cannot expect these two to be the same.

The second issue concerns the effect of turbulence. While for a laminar flow the differential equations can be solved for all conditions with the appropriate numerical or possibly analytical approach, a turbulent flow requires a model for the turbulence which is necessarily of somewhat limited accuracy. (Note that it is effectively of unknown accuracy because turbulence models have been "tuned" for various single phase flows but not for the region near the interface of two-phase flows). Given this question of accuracy, it might be reasonable to consider using an integral model for the turbulent case. However, it is still important to know the inherent accuracy of integral models.

In this notebook, we attempt to get some understanding of both of these issues by solving the much simpler problem of flow over a wavy surface and looking at the fluctuations in pressure and shear stress that result. We consider only the long wave region, $H/\lambda \ll 1$ (amplitude/H and amplitude/λ are also small). Hanratty (1983) (in *Waves on fluid interfaces,* p221-259. Academic Press) emphasizes how important shear and pressure variations are in the wave formation problem for gas-liquid systems. The pressure and shear stress components are be compared for different classes of problems to see the quantitative and qualitiative differences. The integral momentum model of Lin and Hanratty (1986) *IJMF* is solved along with the differential laminar equations and a differential model that uses a polynomial model.

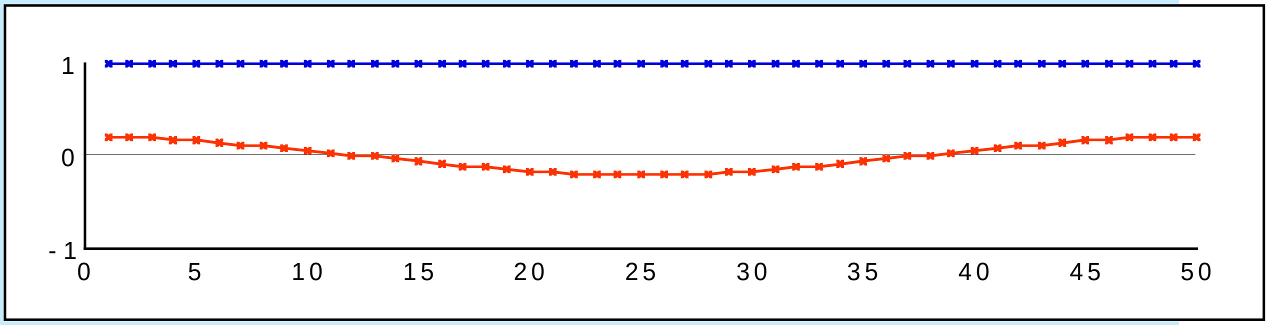
The wave is chosen to be *long* because it is the formation and growth of the longest waves that could cause regime transitions. Further, the integral equations make sense only when the waves are long.

We hope that a comparison for laminar flow between the integral and differential models gives an estimate of the inherent accuracy of the integral model at predicing linear stability. Further, the comparison of the polynomial differential and laminar differential models should give some idea how much difference turbulence might be in the stability problem.

■ Description and physical situation

This notebook solves the problem of laminar flow over a wavy solid surface. The linearized equations are used. A long wave expansion perturbation technique is employed. The major point of this notebook is to calculate the pressure and shear stress components at the wave surface. These provide insight into the wave growth process. A comparison of the longwave perturbation and boundary-layer approximations are given.

The total average channel height is H and we nondimensionalize with it so that the top wall is at y=1. Also, we use the average velocity, $\overline{U}$, to nondimensionalize the velocities.



Mathematica aside

How to navigate this notebook

I should be possible to run the "Derivations" section which will fill in answers for all of the stress and pressure variations. The polynomial profile section will take several minutes to run on a Mac G3. You could also make changes or correct my mistakes by altering these sections. Alternatively, you can just use my answers by clicking on the "Results" section.

Preview of conclusions

A comparison of the laminar differential and integral models suggests that the first terms match exactly which for the stability problem control the wave speed. However there is about a 20% difference between the pressure at second order and the integral model does not produce a second term in the shear stress. The second term controls the wave growth and while the shear stress seems to be less important than pressure for wave growth, we expect at least this 20% difference to show up. Thus we must expect a significant difference between differential and integral stability analyses.

The comparison of the laminar and polynomial differential models shows consistent significant differences. Thus we can expect a difference in the stability boundaries. This is, of course, expected. The problem is that we can't tell exactly how to normalize the effects.

Derivations of pressure and shear variations using all of the models

Pressure and shear stress variations from the linearized *differential* equation for a laminar flow.

Pressure and shear variation caclulated from the linearized *integral* momentum equation for a laminar flow.

Pressure and shear stress variations from the linearized *differential* equation for a polynomial profile (i.e. turbulent) flow.

Pressure and shear variation calculated from the linearized *integral* momentum equation for a turbulent flow.

Results

The analytical expressions

You can just click on this section and load in all of the results without rederiving them above. Of course, this means that if you change something above, you do not want to click on the same result below because it will get changed to the old result!!

laminar stress

$$\texttt{lamavestress} = \frac{6}{R}$$

$$\frac{6}{R}$$

$$\texttt{lamstress} = -\frac{i\,(17\,R^3 - 53900\,R)\,\alpha^2}{943250\,R} + \frac{4\,(R^2 + 1078)\,\alpha}{2695\,R} + \frac{6\,i\,\alpha}{35} + \frac{12}{R};$$

Back to conclusions

- **integral laminar stress**

```
stressintlam = 12 / R;
```

Back to conclusions.

- **laminar pressure**

```
lampressure
```

$$\frac{3\,(411\,R^3 - 1331330\,R)\,\alpha^2}{12262250\,R} + \left(-\frac{3\,i\,(-39\,R^2 - 59290)}{13475\,R} - \frac{6\,i}{R}\right)\alpha - \frac{54}{35} + \frac{36\,i}{R\,\alpha}$$

```
lampressure =
```

$$\frac{3\,(411\,R^3 - 1331330\,R)\,\alpha^2}{12262250\,R} + \left(-\frac{3\,i\,(-39\,R^2 - 59290)}{13475\,R} - \frac{6\,i}{R}\right)\alpha - \frac{54}{35} + \frac{36\,i}{R\,\alpha}$$

$$\frac{3\,(411\,R^3 - 1331330\,R)\,\alpha^2}{12262250\,R} + \left(-\frac{3\,i\,(-39\,R^2 - 59290)}{13475\,R} - \frac{6\,i}{R}\right)\alpha - \frac{54}{35} + \frac{36\,i}{R\,\alpha}$$

Back to conclusions

- **integral laminar pressure**

$$\texttt{pressureintlam} = F\sin(\theta) + \frac{36\,i}{R\,\alpha} - \frac{6}{5};$$

Back to conclusions

- **polynomial stress**

```
turbavestress
```

turbavestress

$$\texttt{turbavestress} = -\frac{6\,(m-1)\,(2\,m+1)\,\left(\frac{4\,(-1)^{2\,m-1}\,m\,(1-s)}{m-1}-\frac{4\,(s-m)}{m-1}\right)}{R\,\left(-8\,m^2-4\,(s-2)\,m-3\,(-1)^{2\,m}+3\,(-1)^{2\,m}\,s+s+3\right)};$$

This means nothing to us in this form but...

$$\begin{aligned}
&\texttt{totalturbstress} = \\
&\Big(12\,(2\,m+1)\,\Big(-280\,\left(-8\,m^2-4\,(s-2)\,m-3\,(-1)^{2\,m}+3\,(-1)^{2\,m}\,s+s+3\right) \\
&\quad\left(-(-1)^{2\,m}\,m+(-1)^{2\,m}\,s\,m+m-s\right)+ \\
&\quad 140\,\left(-8\,m^2-4\,(s-2)\,m-3\,(-1)^{2\,m}+3\,(-1)^{2\,m}\,s+s+3\right) \\
&\quad\left(2\,(-1)^{2\,m}\,(s-1)\,m^2+\left(-(-1)^{2\,m}\,s+(-1)^{2\,m}+1\right)\,m-s\right)+ \\
&\quad\Big(i\,(2\,m+1)\,R\,\Big(32\,\left((-1)^{2\,m}\,s-(-1)^{2\,m}+1\right)\,m^6+ \\
&\qquad 32\,s\,\left(6\,(-1)^{2\,m}\,s-6\,(-1)^{2\,m}+5\right)\,m^5- \\
&\qquad 8\,\big(3\,\left(8-56\,(-1)^{2\,m}+35\,(-1)^{4\,m}\right)\,s^2+ \\
&\qquad\quad \left(-168+398\,(-1)^{2\,m}-210\,(-1)^{4\,m}\right)\,s+ \\
&\qquad\quad 5\,\left(25-46\,(-1)^{2\,m}+21\,(-1)^{4\,m}\right)\big)\,m^4- \\
&\qquad 8\,\Big(420\,\left(-1+(-1)^{2\,m}\right)^2+3\,\left(56-177\,(-1)^{2\,m}+140\,(-1)^{4\,m}\right)\,s^2+ \\
&\qquad\quad \left(-551+1371\,(-1)^{2\,m}-840\,(-1)^{4\,m}\right)\,s\Big)\,m^3+ \\
&\qquad \big(-3\,\left(1136-2177\,(-1)^{2\,m}+1225\,(-1)^{4\,m}\right)\,s^2+ \\
&\qquad\quad \left(6531-13753\,(-1)^{2\,m}+7350\,(-1)^{4\,m}\right)\,s+ \\
&\qquad\quad 7222\,(-1)^{2\,m}-3675\,(-1)^{4\,m}-3547\big)\,m^2+ \\
&\qquad \Big(-840\,\left(-1+(-1)^{2\,m}\right)^2-3\,\left(1057-1425\,(-1)^{2\,m}+280\,(-1)^{4\,m}\right)\,s^2+ \\
&\qquad\quad \left(4147-5955\,(-1)^{2\,m}+1680\,(-1)^{4\,m}\right)\,s\Big)\,m+ \\
&\qquad 120\,s\,\left(\left(-5+7\,(-1)^{2\,m}\right)\,s-7\,(-1)^{2\,m}+7\right)\Big)\,\alpha\Big)\Big/ \\
&\quad \left((m+1)\,(m+2)\,(2\,m+3)\,(2\,m+5)\right)\Big)\Big)\Big/ \\
&\Big(35\,R\,\left(-8\,m^2-4\,(s-2)\,m-3\,(-1)^{2\,m}+3\,(-1)^{2\,m}\,s+s+3\right)^2\Big);
\end{aligned}$$

- **polynomial pressure**

- **integral turbulent stress**

- **Integral turbulent pressure**

Comparison of laminar differential and integral cases.

The first term of the pressure and shear stress for each of the two cases are identical. The second term in the pressure fluctuation is about 20% different.

```
N[(54 / 35 - 6 / 5) / (54 / 35)]
```

0.222222

There is no second term in the shear stress for the integral model. A normalization of the second term of the second term is 6/35 α/(6/R) = fluctation/average stress. It could be that this is less important that the pressure in causing wave growth, but this difference could be important.

```
6 / 35 I α / (6 / R)
```

$$\frac{i R \alpha}{35}$$

Back to conclusions

Here are some comparisons of the two differential models

Here are the expressions for s and m

```
mm = Round[-.617 + .008211 R^.786] ;
ss = .585 + .003172 R^.833;
```

```
<< Graphics`Graphics`
```

Shear stress

The real part of the polynomial shear stress normalized with the average shear stress.

The imaginary part of the polynomial shear stress

The real part of the laminar shear stress normalized with the average stress.

The normalized imaginary part of the laminar shear stress.

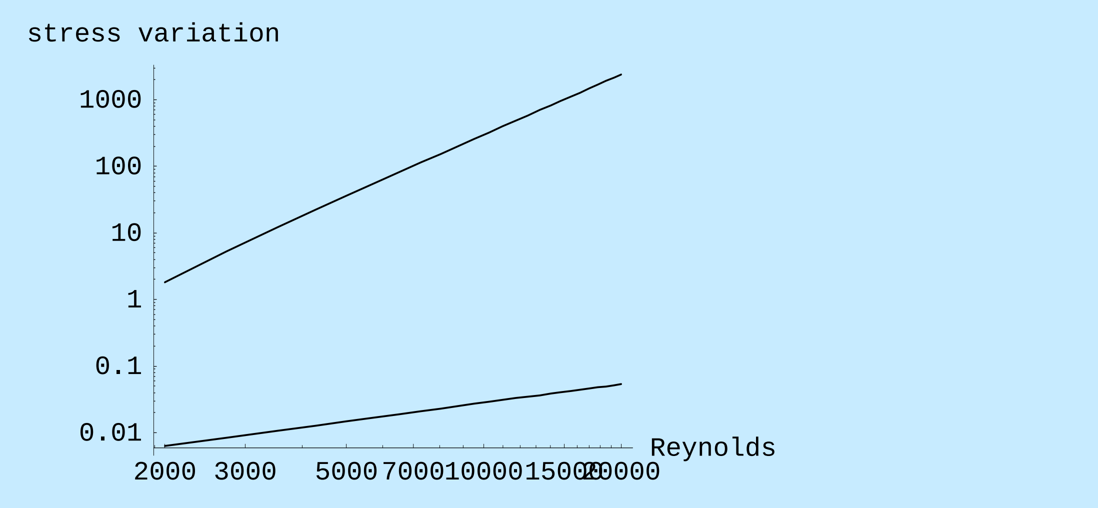
Stress comparisons

Here is the Real part of the shear stress. The laminar model has a much larger stress variation.

Back to conclusions

Here is the imaginary part of the shear stress. The laminar model has a much larger stress variation.

```
Show[turbTRplotIm, lamTRplotIm, PlotRange -> All,
AxesLabel -> {"Reynolds", "stress variation"}]
```



- Graphics -

Pressure

The real part of the polynomial pressure

The imaginary part of the polynomial shear stress

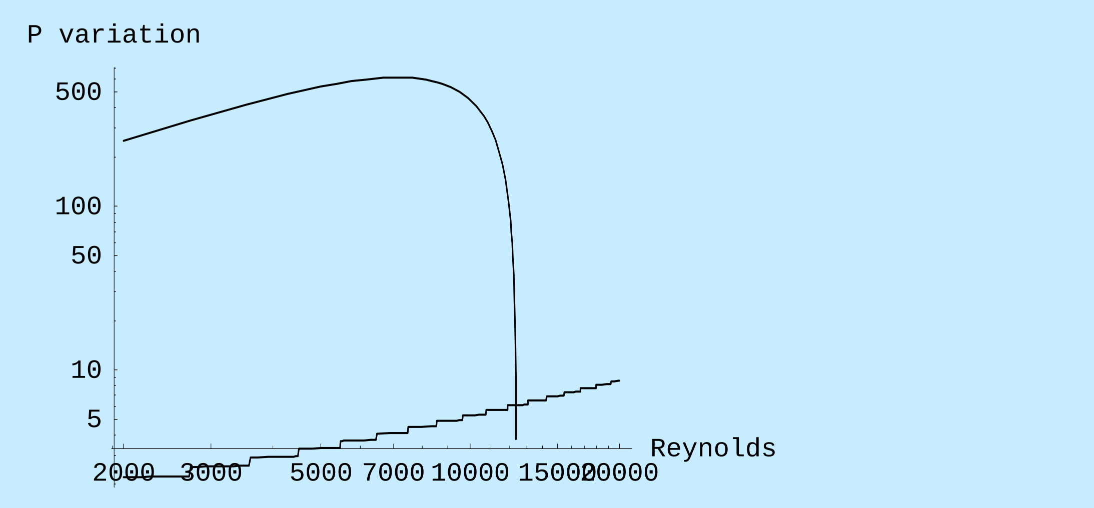
The real part of laminar pressure. Note that it changes signs so we won't worry about the highest Re range

There is no imaginary part of the laminar shear stress

Pressure comparisons

Here is the Real part of the pressure The agreement is not real good since they head off in opposite directions.

```
Show[lamPRplotRe, turbPRplotRe, PlotRange -> All, AxesLabel ->
{"Reynolds", "P variation"}]
```



- Graphics -

Back to conclusions

Here is the imaginary part of the pressure. The quantitative agreement starts off by about 50% then the polynomial model does not change. This could be fixed by patiently solving for the next term. Please let me know how it comes out!!

Here are some comparisons of the turbulent models

Shear stress

The real part of the polynomial shear stress

The imaginary part of the polynomial shear stress

The real part of the integral shear stress

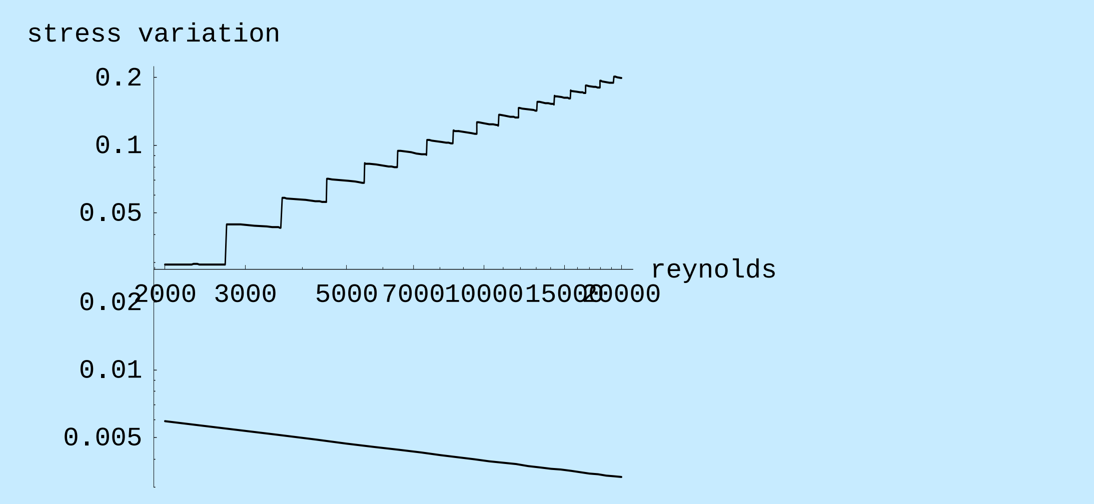
There is no imaginary part of the integral shear stress

Stress comparisons

Here is the Real part of the shear stress, which is the first term (the only term for the integral model). The agreement is not very good.

Back to conclusions

```
Show[turbTRplotRe, intturbTRplotRe, PlotRange -> All,
AxesLabel -> {"Reynolds", "stress variation"}]
```



- Graphics -

Pressure

The real part of the polynomial pressure

The imaginary part of the polynomial pressure

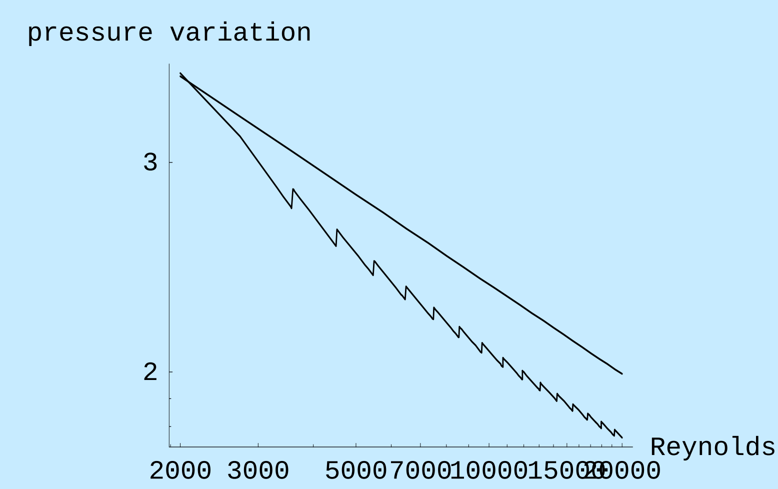
The real part of the integral pressure

The imaginary part of the integral pressure.

Pressure comparisons

Here is the out-of-phase (imaginary) pressure comparison, which is the lowest order term. The agreement is getting somewhat worse at higher Re with about a 20% difference.

```
Show[turbPRplotIm, intturbPRplotIm, PlotRange -> All,
AxesLabel -> {"Reynolds", "pressure variation"}]
```

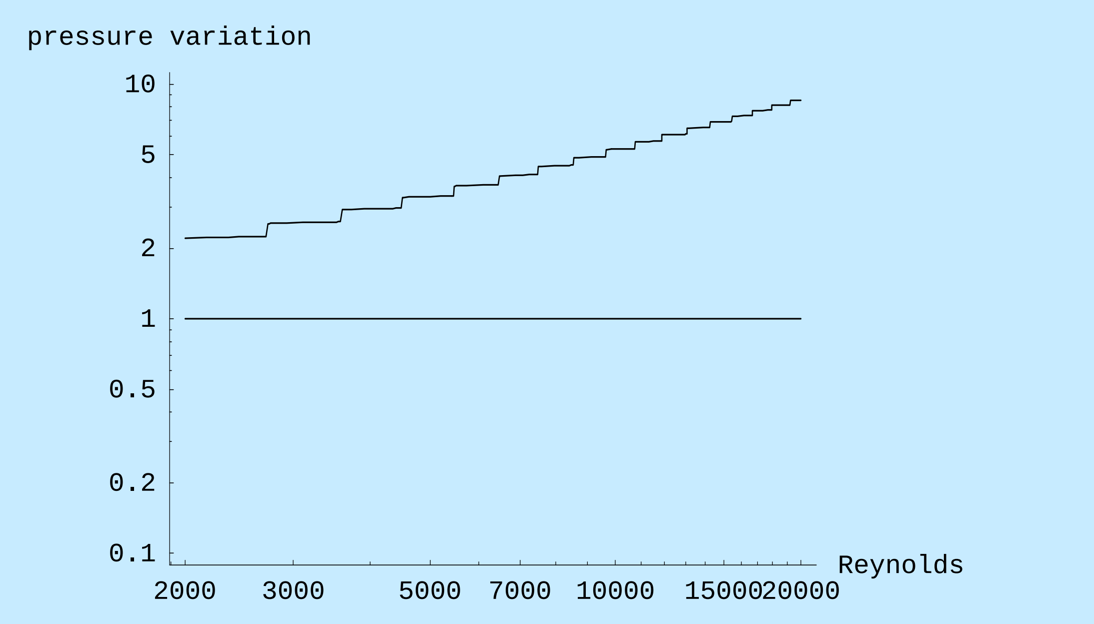


- Graphics -

Here is the in-phase pressure comparison which is the second term. This never really agrees at all with a factor of 2 at the beginning and then a steady worsening.

Back to conclusions

```
Show[intturbPRplotRe, turbPRplotRe,
AxesLabel -> {"Reynolds", "pressure variation"}]
```



- Graphics -

Summary and conclusions

This notebook attempts to provide insight into how different models for stability will predict the formation of waves and by implication, the transition from stratified flow to another regime. It is expected for two fluids, where one is much more viscous and dense, and the other has a Reynolds number large compared to unity, that the pressure shear variations over a linear wave strongly control the wave growth. See Hanratty (1983) in *Waves on fluid interfaces*, p221-259. Academic Press, Kuru, Sangalli, Uphold and McCready (1995) *IJMF*, **21**, p733-753.

1. Comparision of laminar flow differential pressure, different stress, integral pressure and integral stress, shows that the integral model agrees to some extent, but that there can be important differences. The 20% difference in pressure variation and the arbitrary differences for the shear component that cause wave growth could be significant.

2. The laminar and polynomial differential results have more significant qualitative differences in both the pressure and shear stress. Note that these comparisons are made after normalization by the average stress-- which is different for laminar versus turbulent flow.

3. The polynomial and turbulent integral models give very different answers for the pressure and shear stress. This is troubling because even though the polynomial model is not necessarily correct, it produces the same average pressure drop as the integral model. In the absence of a better differential model that reliably produces the correct basestates and has been verified to produce reasonable growth information, we cannot completely discount either of these two. Thus, at the present time, there is a great deal of uncertainty in the prediction of the stability behavior for turbulent flows -- the only case that matters for process conditions.

4. Finally, we have shown above how to solve a difficult ordinary differential equation by a regular perturbation method. Note that presence of inhomogeneities in the boundary conditions at the lowest order from d*omain perturbation* that allows us to evaluate the interfacial conditions (very much more conveniently) at y=0, instead of the exact position of the interface.