Linear spaces and operators

If our mutlidimensional vectors "live" in some appropriate multidimensional space (See Greenberg) we need some properties

vector norms

A natural question about a vector is, how "long" is it? We might want to know this for a physical reason, say mean velocity, or for an abstract comparision.

In Cartesian space if we want the length of a vector we might define it as

```
norm[v_] := Sqrt[Sum[v[[i]]^2, {i, 1, Length[v]}]]
```

$$\text{norm}[\text{v\_}] := \sqrt{\sum_{i=1}^{\text{Length}[v]} v[\![i]\!]^2}$$

which is the Euclidean norm.

```
v = {v1, v2, v3};
```

```
norm[v]
```

$$\sqrt{v1^2 + v2^2 + v3^2}$$

```
yy = {y1, y2, y3, y4, y5};
```

```
norm[yy]
```

$$\sqrt{y1^2 + y2^2 + y3^2 + y4^2 + y5^2}$$

```
zz = {1, 3, 2, 6, 2};
```

```
norm[zz]
```

$3\sqrt{6}$

However, we might like a different measure of length for applications where the landscape is different. For example, the Taxi Cab norm or the infinity norm (take the biggest element).

Here is one definition that has no individual weightings, but does allow us to change how we count large versus small elements.

$$\texttt{generalnorm[v\_, n\_] :=} \left(\sum_{i=1}^{\texttt{Length[v]}} \texttt{Abs[v[[i]]]}^{n}\right)^{1/n}$$

```
N[generalnorm[zz, 2]]
```

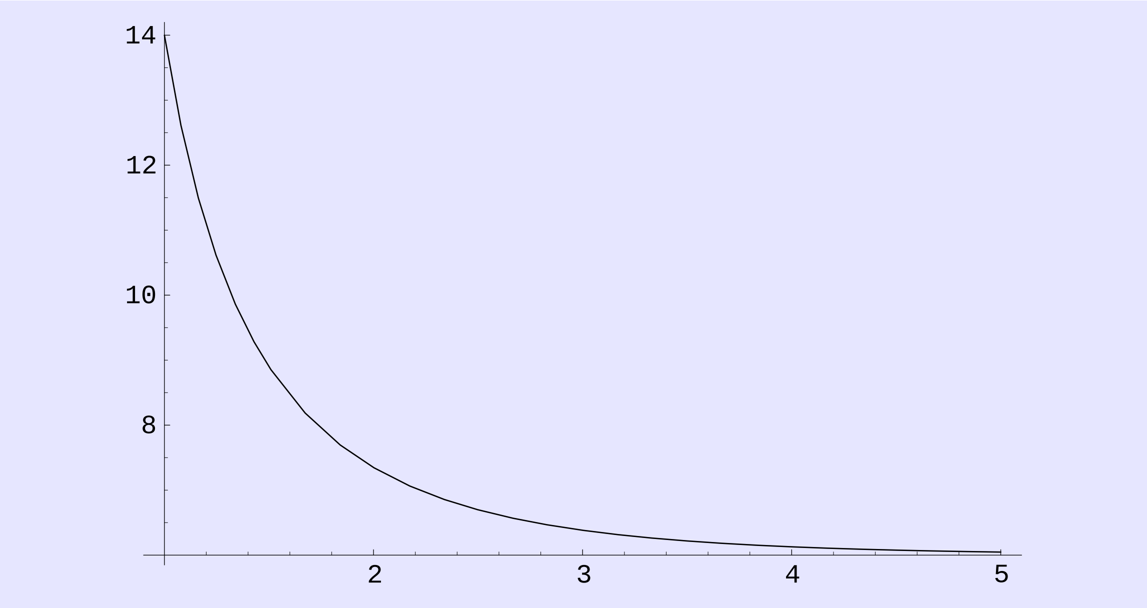
7.34847

```
N[generalnorm[zz, 10]]
```

6.00061

Here we can plot the norm as a function of n.

```
Plot[generalnorm[zz, j], {j, 1, 5}, PlotRange → All]
```



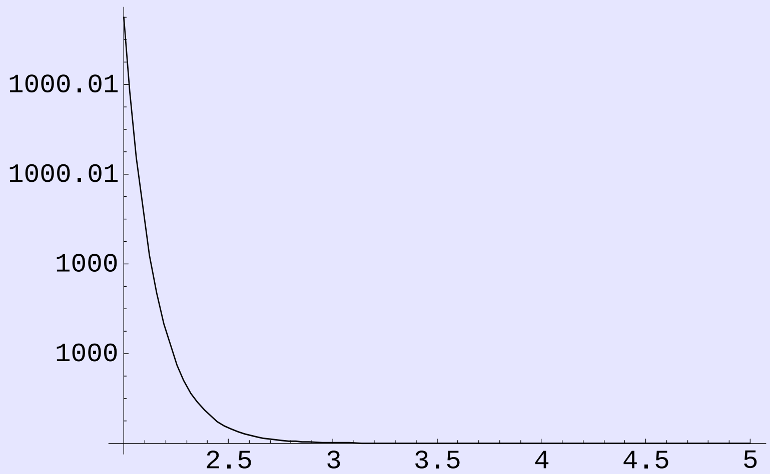

```
- Graphics -
```

We see that this "generalized" norm allows us to change the weighting from a simple sum as the length to a measure on the largest element.

If the elements are about the same size there can a be some significant % variation as n changes. However if there is only one very large element then there is not much change

```
zmax = {1, 1, 1, 2, 1000, 1, 1, 1, 3};
```

```
Plot[generalnorm[zmax, j], {j, 2, 5}, PlotRange → All]
```



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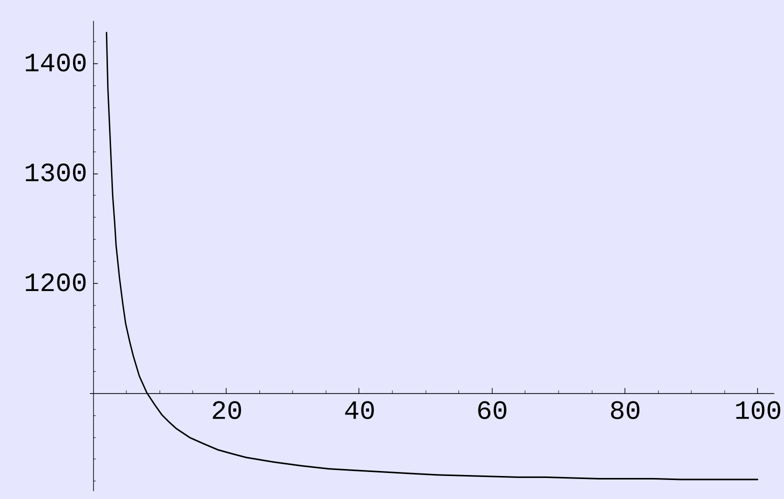
```
zmax2 = {1, 1020, 1, 2, 1000, 1, 1, 1, 3};
```

With two large elements the difference can be large

```
Plot[generalnorm[zmax2, j], {j, 2, 100}, PlotRange → All]
```



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In any case, since the norm is roughly a sum, the largest elements always contribute the most. For computationally intense problems often times a "max" norm is used where the norm is the just the magnitude of the largest element. This corresponds to n--> infinity. Of course, if you are using a max norm, do not calculate the entire sum.

- **Preserving our notion of length.**

If a norm is a length, then how do vectors behave relative to one another as we change the norm?

In the first case, the relative difference does not change much.

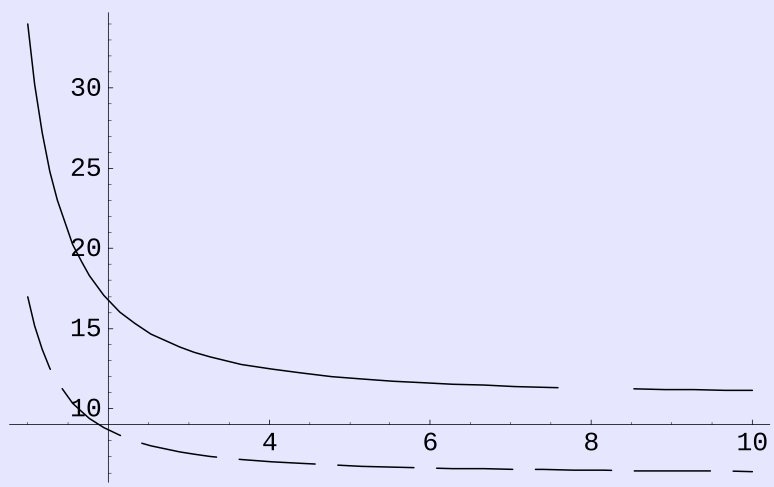
```
zz1 = {2, 5, 7, 9, 11, 0}
```

{2, 5, 7, 9, 11, 0}

```
zz2 = {1, 6, 0, 5, 3, 2}
```

{1, 6, 0, 5, 3, 2}

```
Plot[{generalnorm[zz1, j], generalnorm[zz2, j]}, {j, 1, 10},
 PlotRange → All, PlotStyle -> {Dashing[{1, 0}], Dashing[{.1, .03}]}]
```



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In the second case, the norms become identical at large values of j.

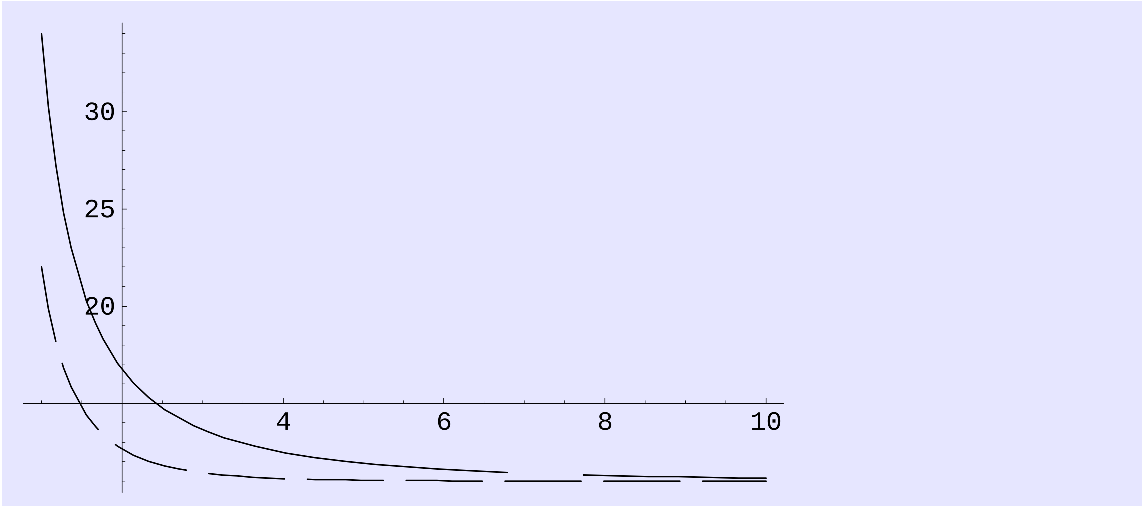
```
zz1 = {2, 5, 7, 9, 11, 0}
```

{2, 5, 7, 9, 11, 0}

```
zz2 = {1, 11, 0, 5, 3, 2}
```

{1, 11, 0, 5, 3, 2}

```
Plot[{generalnorm[zz1, j], generalnorm[zz2, j]}, {j, 1, 10},
 PlotRange → All, PlotStyle -> {Dashing[{1, 0}], Dashing[{.1, .03}]}]
```



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We can also have them cross. This does not cause a calculational problem but it does bother our sensibilities a bit. We are not used to having mathematics be subjective, (It seems like we are saying player X is a better quarterback than player Y, even though player Y has better statistics!). However, it is not subjective at all. Our rules are telling how player Z will stack up given a straightforward comparison based on our defined rules.

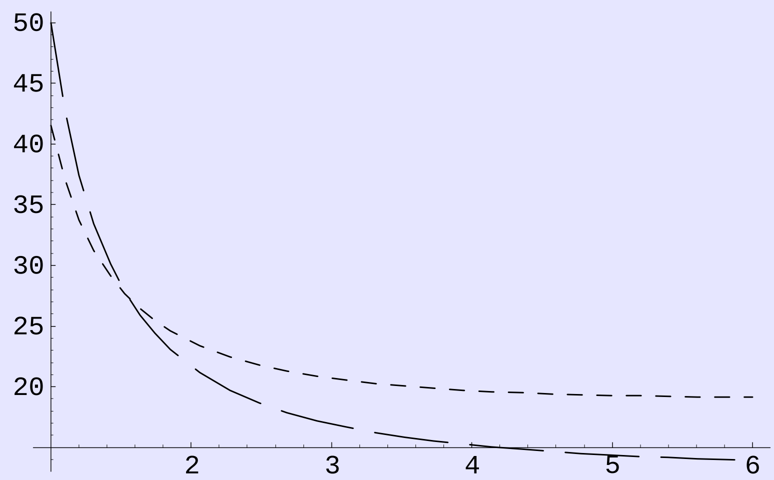
```
zz1 = {5, 5, 7, 9, 11, 13}
```

{5, 5, 7, 9, 11, 13}

```
zz2 = {10, 19, 1, 1, .5, 10}
```

{10, 19, 1, 1, 0.5, 10}

```
Plot[{generalnorm[zz1, j], generalnorm[zz2, j]},
 {j, 1, 6}, PlotRange → All,
 PlotStyle -> {Dashing[{.1, .03}], Dashing[{.02, .02}]}]
```



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Inner product

Another natural question to ask about vectors is their relative direction. Or (similarly) perhaps the difference between vectors. This is the inner product, (x,y) = inner[x,y].

Here is a definition for the inner product that corresponds to a Euclidean norm. Note however that we need the conjugate of the second argument to avoid the possibility, when we relate this back to the norm, of negative lengths.

```
inner[x_, y_] := Sum[x[[i]] Conjugate[y[[i]]], {i, 1, Length[x]}]
```

$$\text{inner}[x\_, y\_] := \sum_{i=1}^{\text{Length}[x]} x[[i]]\ \text{Conjugate}[y[[i]]]$$

— *General::spell1 : Possible spelling error: new symbol name "inner" is similar to existing symbol "Inner".*

```
vv = {v1, v2, v3, v4};
```

```
ww = {w1, w2, w3, w4};
```

```
inner[vv, ww]
```

v1 Conjugate(w1) + v2 Conjugate(w2) + v3 Conjugate(w3) + v4 Conjugate(w4)

We can use the *Mathematica* definition of a general Inner Product to construct one that is the same as ours,

```
Inner[Times, vv, Conjugate[ww], Plus]
```

v1 Conjugate(w1) + v2 Conjugate(w2) + v3 Conjugate(w3) + v4 Conjugate(w4)

```
a = {1, -3, -4 + I};
b = {-1 - I, 3, -2 - 7 I};
```

```
inner[a, b]
```

$-9 - 29\, i$

Comparing to the "dot" product, we see that they are different for complex elements.

```
a . b
```

$5 + 25\, i$

```
vv . ww
```

v1 w1 + v2 w2 + v3 w3 + v4 w4

```
inner[vv, ww]
```

v1 Conjugate(w1) + v2 Conjugate(w2) + v3 Conjugate(w3) + v4 Conjugate(w4)

We see that the inner product is not the same at a standard Dot product, which is defined only for real vectors.

If (x,y) = 0, then x and y are orthogonal in the vector space that we are considering.

- **Weighted inner product**

We will need eventually to use inner products that have some weighting factors. This makes different directions more important to the value of the norm. For example, if we were weighting test scores for high school students we might count Mathematics more than Social Studies (particularly if we were thinking of future engineers). Thus for the inner product of two student vectors, students would need similar Math scores to be "parallel" and differences in their social studies scores would matter less.

Here is a weighted Euclidean inner product.

$(x, y)_{\text{Ew}} = x_i \; W_{\text{ij}} \; y_j = \mathbf{x}^T \mathbf{W} \; \mathbf{y}$

The matrix, **W,** gives the weighting coefficients.

$$\texttt{weightedinner[x\_, y\_, wz\_] :=} \sum_{i=1}^{\texttt{Length[x]}} \sum_{j=1}^{\texttt{Length[y]}} \texttt{x[[i]] wz[[i, j]] y[[j]]}$$

```
ax = {ax1, ax2, ax3, ax4, ax5};
```

```
bx = {bx1, bx2, bx3, bx4, bx5};
```

```
wx = {{wx11, wx12, wx13, wx14, wx15},
  {wx21, wx22, wx23, wx24, wx25}, {wx31, wx32, wx33, wx34, wx35},
   {wx41, wx42, wx43, wx44, wx45}, {wx51, wx52, wx53, wx54, wx55}}
```

$$\begin{pmatrix} \text{wx11} & \text{wx12} & \text{wx13} & \text{wx14} & \text{wx15} \\ \text{wx21} & \text{wx22} & \text{wx23} & \text{wx24} & \text{wx25} \\ \text{wx31} & \text{wx32} & \text{wx33} & \text{wx34} & \text{wx35} \\ \text{wx41} & \text{wx42} & \text{wx43} & \text{wx44} & \text{wx45} \\ \text{wx51} & \text{wx52} & \text{wx53} & \text{wx54} & \text{wx55} \end{pmatrix}$$

```
test = weightedinner[ax, bx, wx]
```

ax1 bx1 wx11 + ax1 bx2 wx12 + ax1 bx3 wx13 + ax1 bx4 wx14 + ax1 bx5 wx15 +
ax2 bx1 wx21 + ax2 bx2 wx22 + ax2 bx3 wx23 + ax2 bx4 wx24 + ax2 bx5 wx25 +
ax3 bx1 wx31 + ax3 bx2 wx32 + ax3 bx3 wx33 + ax3 bx4 wx34 + ax3 bx5 wx35 +
ax4 bx1 wx41 + ax4 bx2 wx42 + ax4 bx3 wx43 + ax4 bx4 wx44 + ax4 bx5 wx45 +
ax5 bx1 wx51 + ax5 bx2 wx52 + ax5 bx3 wx53 + ax5 bx4 wx54 + ax5 bx5 wx55

```
Coefficient[test, ax1 bx4 ]
```

wx14

Often we jsut need 1 weighting function for each element so that **W** is of the form

```
wz = {{w1, 0, 0, 0, 0}, {0, w2, 0, 0, 0}, {0, 0, w3, 0, 0},
  {0, 0, 0, w4, 0}, {0, 0, 0, 0, w5}}
```

$$\begin{pmatrix} w1 & 0 & 0 & 0 & 0 \\ 0 & w2 & 0 & 0 & 0 \\ 0 & 0 & w3 & 0 & 0 \\ 0 & 0 & 0 & w4 & 0 \\ 0 & 0 & 0 & 0 & w5 \end{pmatrix}$$

```
test2 = weightedinner[ax, bx, wz]
```

ax1 bx1 w1 + ax2 bx2 w2 + ax3 bx3 w3 + ax4 bx4 w4 + ax5 bx5 w5