
Solution of the Heat Equation for transient conduction by LaPlace Transform

This notebook has been written in *Mathematica* by

Mark J. McCready
Professor and Chair of Chemical Engineering
University of Notre Dame
Notre Dame IN 46556
USA

Mark.J.McCready.1@nd.edu
<http://www.nd.edu/~mjm/>

It is copyrighted to the extent allowed by whatever laws pertain to the World Wide Web and the Internet.

I would hope that as a professional courtesy, that if you use it, that this notice remain visible to other users.
There is no charge for copying and dissemination

Version: 3/17/98

This notebook shows how to solve transient heat conduction in a semi-infinite slab. It is intended as a supplement to

L. G. Leal (1992) *Laminar flow and Convective Transport Processes*, Butterworth pp 139-144.

Leal mentions the possible use of *linear transform techniques* but does not give examples. Many students are not familiar with these. This notebook is intended to illustrate the use of the LaPlace transform to solve a simple PDE, and to show how it is implemented in *Mathematica*.

This problem is the heat transfer analog to the "Rayleigh" problem that starts on page 91.

■ Problem formulation

Consider a semi-infinite slab where the distance variable, y , goes from 0 to ∞ . The temperature is initially uniform within the slab and we can consider it to be 0. At $t=0$, the temperature at $y=0$ is suddenly increased to 1. We would like to calculate the temperature as a function of time, t , within the slab.

This problem is commonly solved by a *similarity variable* technique that arises because the absence of a physical length scale. In this notebook we use the *Laplace Transform*, which is an integral transform that effectively converts (linear) PDEs, (if we wish to use it on say *time*), to a set of ODEs in *frequency* space. I say a "set" of problems because the transform creates an explicit parameter, s , which is effectively a *frequency* and the equations need to be solved for all positive s .

■ Equation

$$\text{heateq} = \partial_t \theta[t, y] - \alpha \partial_{\{y, 2\}} \theta[t, y]$$

$$\theta^{(1,0)}(t, y) - \alpha \theta^{(0,2)}(t, y)$$

■ boundary conditions

the boundary conditions are

$$\theta(t, y) = 0 \quad t < 0,$$

$$\theta(t, y) = 0 \quad \text{as } y \rightarrow \infty,$$

$$\theta(t, y=0) = 1 \quad \text{for } t > 0.$$

■ initialization of *Mathematica*, load a package

```
Needs["Calculus`LaplaceTransform`"]
```

■ Here is the analysis

If we use a transform technique, we intend to simplify the problem by transforming the pde to an ode (or an algebraic equation from an ode). Once the transform is done, we will need to evaluate the integral that arises at the boundaries. So the boundary conditions and the domain of the problem must be in a form conducive to this. The Laplace transform is defined from 0 to ∞ . In this problem both of the domains are from 0 to ∞ , however first try to do the transform in *time*. In *Mathematica* this command is `LaplaceTransform[heateq,t,s]` and the new parameter is *s*.

```
LaplaceTransform [heateq , t , s]
```

```
s LaplaceTransform ( $\theta(t, y), t, s$ ) -  $\alpha$  LaplaceTransform ( $\theta^{(0,2)}(t, y), t, s$ ) -  $\theta(0, y)$ 
```

The first two terms make an ode in the transformed $\theta(t,y)$. Let's call this $\hat{\theta}$. The last term, which arose from integration by parts of the $\partial\theta(t,y)/\partial t$ term, must be evaluated from the boundary conditions. The temperature for all *y* at *t=0* is zero, thus this term is 0. Note that the $t \rightarrow \infty$ boundary term is usually zero (we don't see it in this calculation) because it is multiplied by $\text{Exp}[-s t]$ (and thus *Mathematica* automatically makes it 0). Thus we have an ode in $\hat{\theta}(y)$, and have generated an explicit parameter, *s*. The `LaplaceTransform[t,s]` does not affect any *y* derivatives. We can write

```
eq1 = s  $\hat{\theta}$  [y] -  $\alpha$   $\partial_{\{y,2\}}$   $\hat{\theta}$  [y]
```

```
s  $\hat{\theta}(y)$  -  $\alpha$   $\hat{\theta}''(y)$ 
```

This is easily solved by doing

```
ans1 = DSolve [eq1 == 0 ,  $\hat{\theta}$  [y] , y]
```

```
{{ $\hat{\theta}(y) \rightarrow e^{-\frac{\sqrt{s} y}{\alpha}} c_1 + e^{\frac{\sqrt{s} y}{\alpha}} c_2$ }}
```

Now get the solution out of the `{{}}`'s.

```
soln =  $\hat{\theta}$  [y] /. ans1 [[1]]
```

```
 $e^{-\frac{\sqrt{s} y}{\alpha}} c_1 + e^{\frac{\sqrt{s} y}{\alpha}} c_2$ 
```

We see that we cannot stand an exponentially increasing part so that $c_2=0$. Now, we need to evaluate $\hat{\theta}(y,s)$ at $y=0$ or at some place that we know it. Well, we know $\theta[t,y]$ @ $y=0$ (=1). Thus, we can transform this to get a value for $\theta[y=0,s]$.

```
bc0 = LaplaceTransform [1, t , s]
```

```
 $\frac{1}{s}$ 
```

Now find the value of `C[1]` after setting `C[2] = 0` and evaluating the expression at $y=0$.

```
bc0 == (soln /. {y → 0, C[2] → 0})
```

$$\frac{1}{s} == c_1$$

Thus our solution in transformed space is

```
expression = soln /. {C[1] →  $\frac{1}{s}$ , C[2] → 0}
```

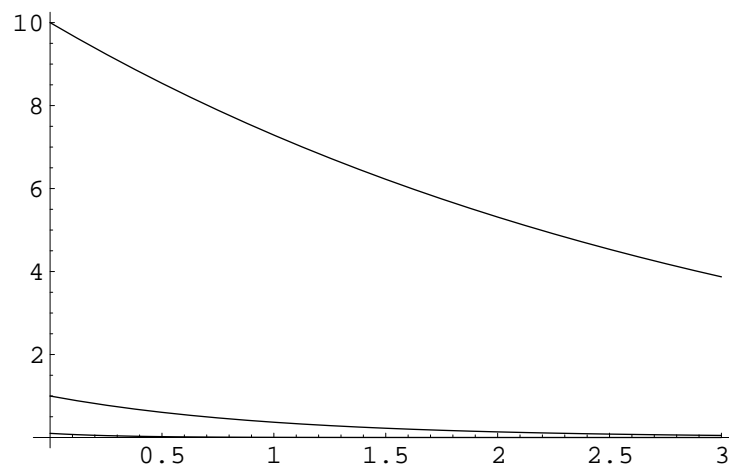
— General::spell1 : Possible spelling error: new symbol name "expression" is similar to existing symbol "Expression".

$$\frac{e^{-\frac{\sqrt{s} y}}{\sqrt{\alpha}}}{s}$$

■ Plot solution in frequency space to see what s does

What does this look like ??

```
p1 = Plot [expression /. {α → 1, s → 1}, {y, 0, 3}]
p2 = Plot [expression /. {α → 1, s → 10}, {y, 0, 3}]
p3 = Plot [expression /. {α → 1, s → .1}, {y, 0, 3}]
Show [p1, p2, p3]
```



- Graphics -

The top plot is for $s = 10$.

It is a difficult to tell for sure, but the different values of s represent different modes. The term s is roughly a frequency so that higher s 's decay faster!!

■ Examine the issue of time to "frequency"

We can see more of what s means by transforming $t^n \rightarrow s$

```
LaplaceTransform [tn, t, s]
```

$$s^{-n-1} \Gamma(n+1)$$

```
LaplaceTransform [t, t, s]
```

$$\frac{1}{s^2}$$

```
LaplaceTransform [\sqrt{t}, t, s]
```

$$\frac{\sqrt{\pi}}{2 s^{3/2}}$$

```
LaplaceTransform [1, t, s]
```

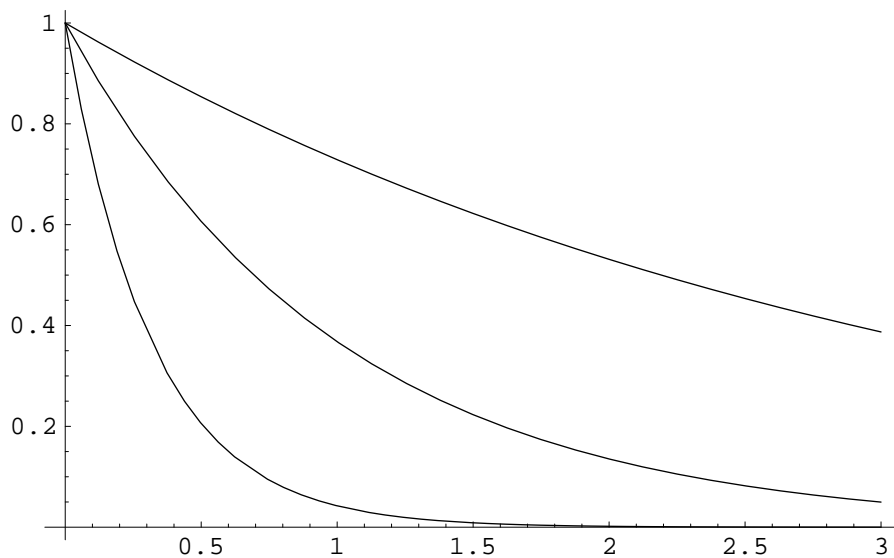
$$\frac{1}{s}$$

So we see the inverse relation of t and s -- which that is a little justification for calling s the "frequency".

■ Plot solution in frequency space to see what α does

Now if we make plots for different values of α we get:

```
p4 = Plot [expression /. {\alpha -> .1, s -> 1}, {y, 0, 3}]
p5 = Plot [expression /. {\alpha -> 10, s -> 1}, {y, 0, 3}]
p6 = Plot [expression /. {\alpha -> 1, s -> 1}, {y, 0, 3}]
Show [p4, p5, p6]
```



- Graphics -

The top curve is for $\alpha = 10$. It is seen that the value of α gives the rate of "diffusion" which we also expect once the solution is transformed back.

■ Transform back to time

Now we must transform back to get the solution in physical space. There are 3 ways. One is to get a table of transforms and inverse transforms. A good table is in Spiegel's math handbook (M. R. Spiegel, *Mathematical Handbook*, Schaum's Outline Series, McGraw-Hill, 1968). The second way is to use *Mathematica* or Maple, the problem would be, do you believe the answer. The third way is to do the complex integration yourself, the problem is, would you believe the answer ?? Anyway *Mathematica* agrees with the tables

expression

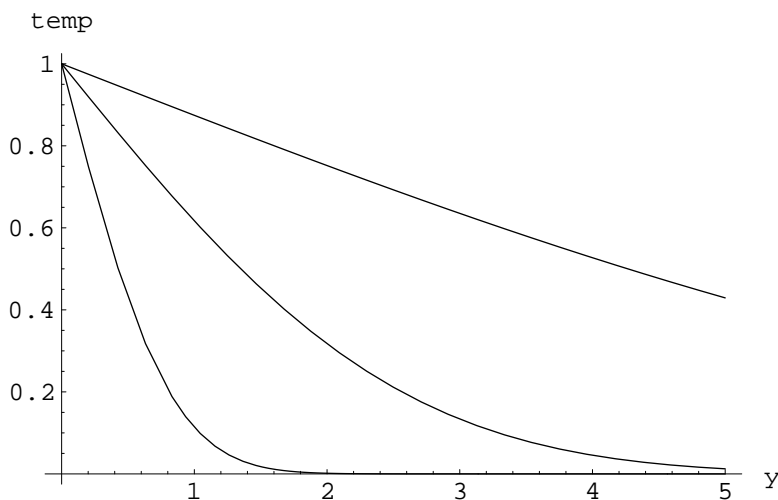
$$\frac{e^{-\frac{\sqrt{s} y}{\sqrt{\alpha}}}}{s}$$

ans = InverseLaplaceTransform [expression , s, t]

$$\text{erfc}\left(\frac{y}{2\sqrt{t}\sqrt{\alpha}}\right)$$

■ Plot the solution in physical space

```
p7 = Plot [ans /. {α → 10, t → .2}, {y, 0, 5}]
p8 = Plot [ans /. {α → 10, t → .02}, {y, 0, 5}]
p9 = Plot [ans /. {α → 10, t → 2}, {y, 0, 5}]
Show [p7, p8, p9, AxesLabel → {y, temp}]
```



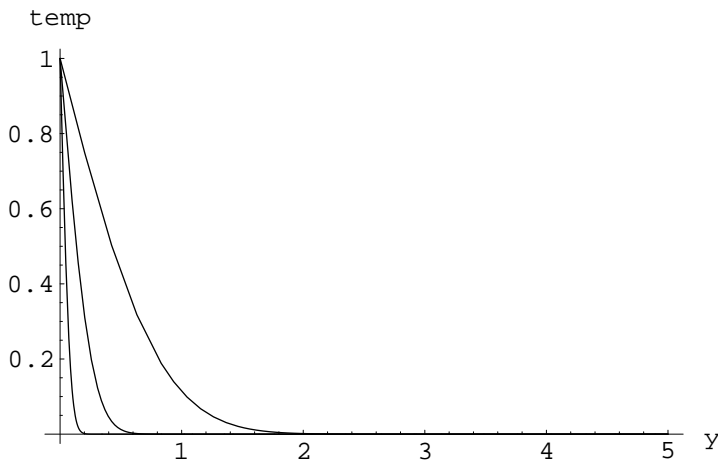
- Graphics -

Note that the bottom curve is for $t = .02$

```

p10 = Plot [ans /. {α → .1, t → .2}, {y, 0, 5}]
p11 = Plot [ans /. {α → .1, t → 2}, {y, 0, 5}]
p12 = Plot [ans /. {α → .1, t → .02}, {y, 0, 5}]
Show [p10, p11, p12, PlotRange → All, AxesLabel → {y, temp}]

```



- Graphics -

Again the bottom curve is for $t = 0.02$. By comparing the two plots for different α , we can see that the value of the diffusivity controls how fast heat transfer occurs.

■ Why did we transform the time variable instead of the space variable?

Now suppose that we had initially transformed the y variable

$$\text{LaplaceTransform} [\partial_t \theta[t, y] - \alpha \partial_{\{y, 2\}} \theta[t, y], y, s]$$

$$\text{LaplaceTransform} (\theta^{(1,0)}(t, y), y, s) - \alpha (\text{LaplaceTransform} (\theta(t, y), y, s) s^2 - \theta(t, 0) s - \theta^{(0,1)}(t, 0))$$

Now we need to evaluate the boundary terms. The first one $s \theta[t, 0] = s$ for all t of interest. However, the second one presents a bit of a problem. We don't really know the heat *flux* at the boundary so we don't know the derivative. Consequently, transforming y to s does not help solve the problem!!