Green's Functions

This notebook has been written in *Mathematica* by

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Summary of important points

This notebook is intended to give some insight into *what* is happening when the Green's function approach is used to solve inhomogeneous ODE's. I think that you will see (or have seen) the following:

1. No matter how the original problem is motivated, you end up with a problem of the form, L[g] = δ[x-t]
2. Solving this problem amounts to solving L[u] = 0, it produces functions that look like the solution of L[u] = 0, but there is a "kink" at $x=t$ that results from the singular δ function at this point.
3. To match the kink, you may produce a function that you are not real familiar with.
4. Since you have already solved L[u] = 0, resolving with δ[x-t] on the right side should be easier than solving the nonhomgenous problem (at least this is nice claim!). Then you just need to evaluate an integral to get the answer. This integral also will involve some of these new "noncommon" functions. Note that this could be a book-keeping problem, but fortunately *Mathematica* takes care of it for you.

Initialization

This notebook is a bit rough in format (you need to do it order as some of the names will get written over) but it is an exploration of the Green's function and its use to solve inhomogeneous ODE's. You will have to work through it to get much benefit.

We will need this add-on package which defines the delta function and the Heaviside function -- which is called the UnitStep function

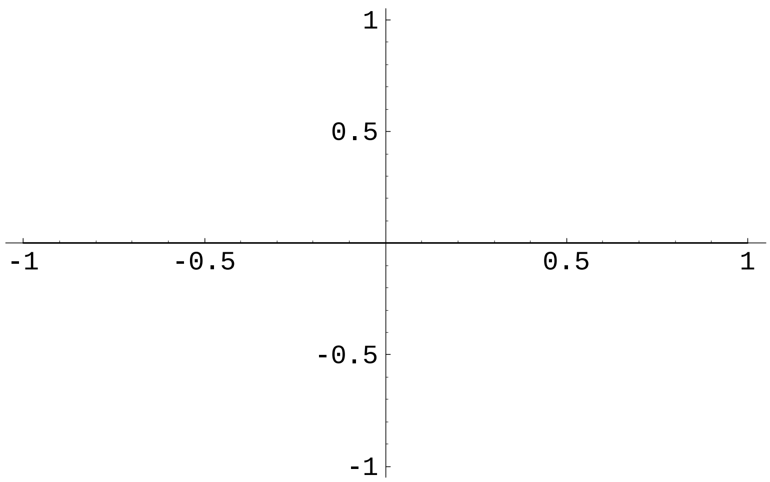
```
<< Calculus`DiracDelta`
```

On some setups, you may need to use a "Needs" command.

Needs["Calculus``DiracDelta` "]

Delta function and Heaviside (Unit-Step) function

```
Plot[DiracDelta[x], {x, -1, 1}]
```

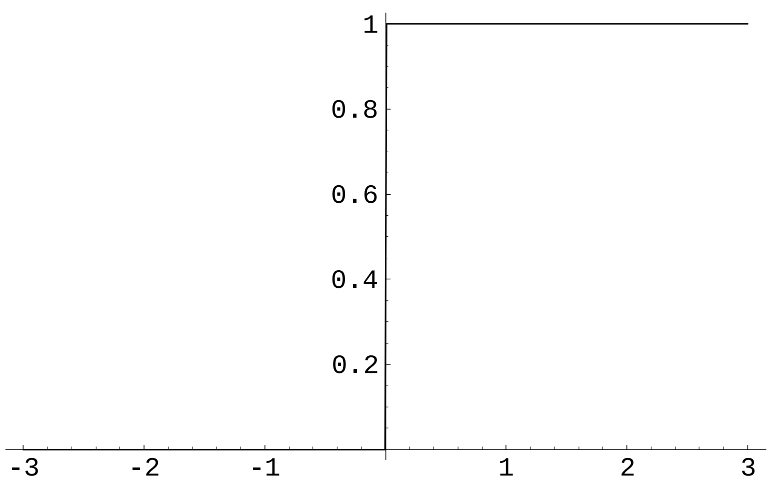


- Graphics -

```
Table[DiracDelta[i], {i, -3, 3}]
```

{0, 0, 0, DiracDelta(0), 0, 0, 0}

```
Plot[UnitStep[x], {x, -3, 3}]
```



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Find the Green's function for $\frac{\partial^2}{\partial x^2}$.

Now we can solve the problem where the operator is $\frac{\partial^2}{\partial x^2}$ and the boundary conditions are homogeneous. Note we leave off the second variable in the argument of g[t] because *Mathematica* does not solve ode's with more than one argument.

```
eq1 = D[g[t], {t, 2}] - DiracDelta[x - t]
```

$g''(t) - \text{DiracDelta}(x - t)$

```
ans1 = DSolve[{eq1 == 0, g[0] == 0, g[1] == 0}, g[t], t]
```

$\{\{g(t) \to (t - x)\,\text{UnitStep}(t - x) + x\,\text{UnitStep}(-x) + t\,(x\,\text{UnitStep}(1 - x) - \text{UnitStep}(1 - x) - x\,\text{UnitStep}(-x))\}\}$

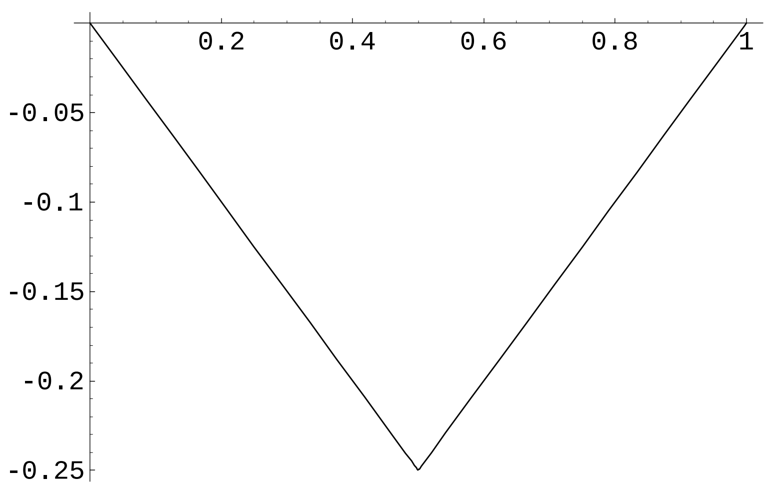
At this point, g[ ] is effectively now g[x,t] as we would like.

```
greenfunc1 = g[t] /. ans1[[1]]
```

$(t - x)\,\text{UnitStep}(t - x) + x\,\text{UnitStep}(-x) + t\,(x\,\text{UnitStep}(1 - x) - \text{UnitStep}(1 - x) - x\,\text{UnitStep}(-x))$

- **Now plot the Green's function at some values.**

```
Plot[greenfunc1 /. x -> .5, {t, 0, 1}]
```



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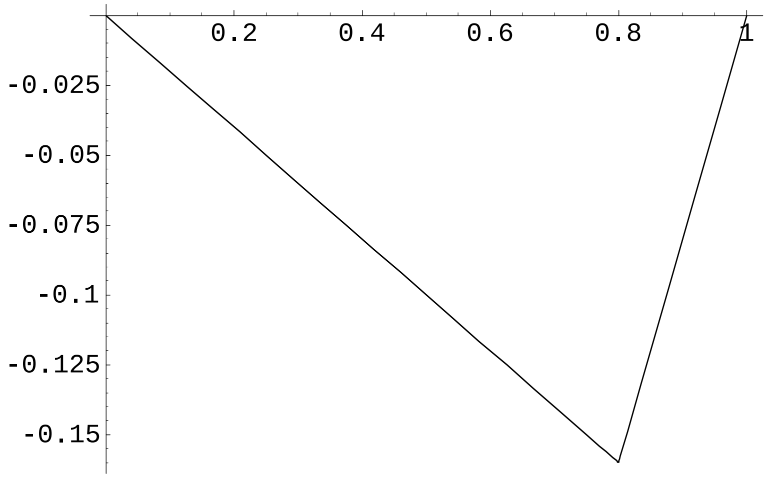
```
Plot[greenfunc1 /. x -> .8, {t, 0, 1}]
```



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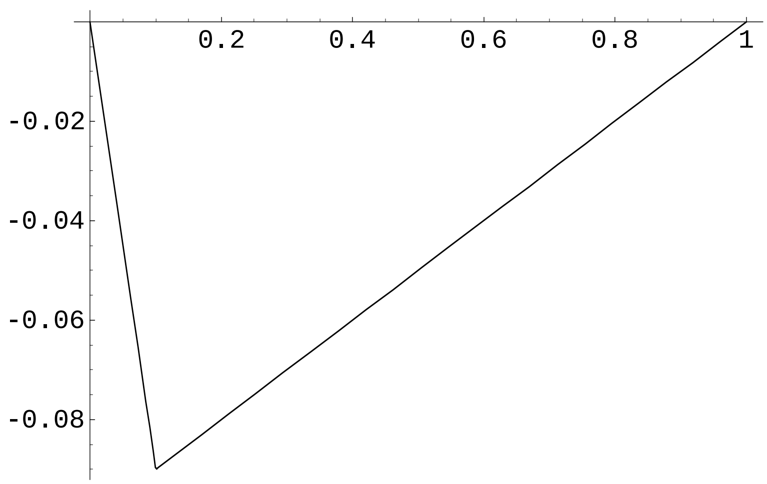
```
Plot[greenfunc1 /. x -> .1, {t, 0, 1}]
```



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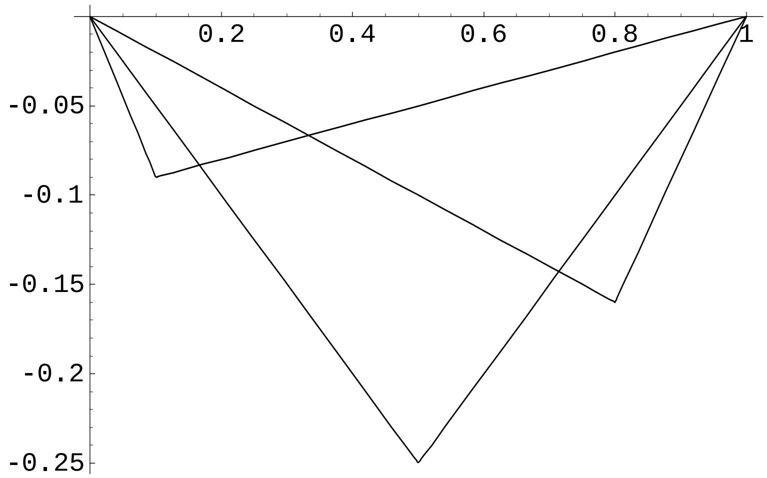
```
Show[%, %%, %%%]
```



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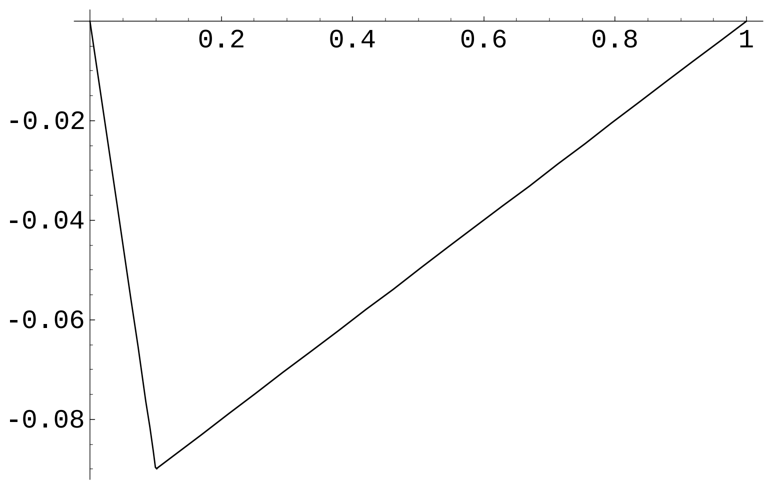
- **We think, what else could g[x,t] look like since it has have a singular 2nd derivative and a constant 1st derivative. Now switch x and t.**

```
Plot[greenfunc1 /. t -> .1, {x, 0, 1}]
```



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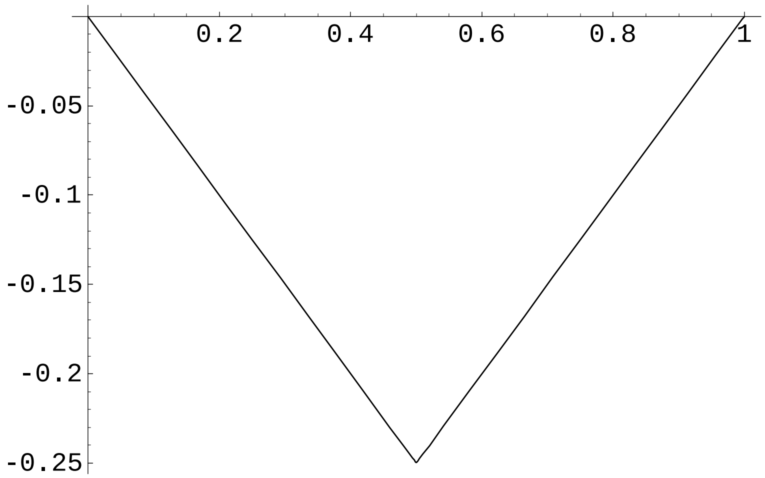
```
Plot[greenfunc1 /. t -> .5, {x, 0, 1}]
```



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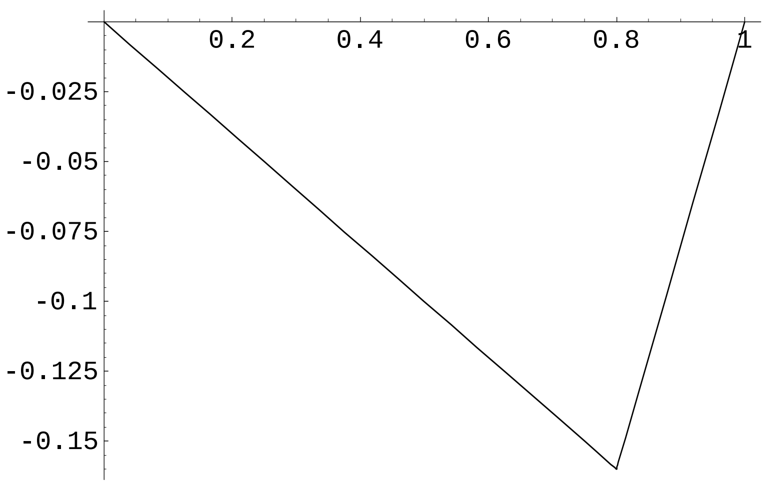
```
Plot[greenfunc1 /. t -> .8, {x, 0, 1}]
```



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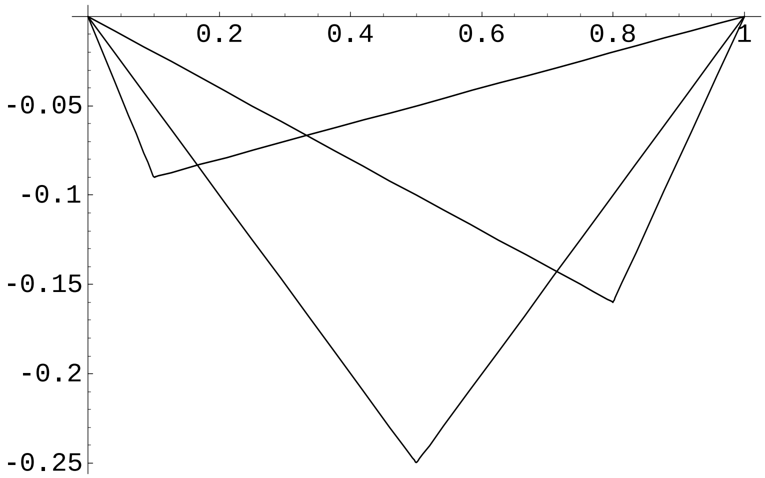
```
Show[%, %%, %%%]
```



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- **Interesting... we see that greenfunc1 is symmetric in x and t. This is likely to always be the case if the boundary conditions are homogenous.**

Now solve the inhomogeneous equation for different powers of x as the inhomogeneities. First x^0

```
ans10 = Integrate[1 greenfunc1, {x, 0, 1}]
```

$$\frac{1}{2}\,\text{UnitStep}(t)\,t^2 - \frac{t}{2} + \left(-\frac{t^2}{2} + t - \frac{1}{2}\right)\text{UnitStep}(t-1)$$

It does not change much when simplified, but not that t^2 is the leading term.

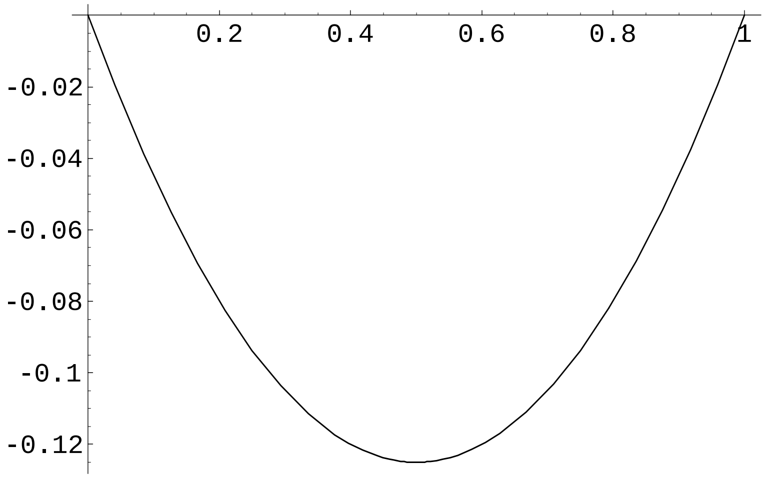
```
ans10x = FullSimplify[ans10]
```

— *General::spell1 : Possible spelling error: new symbol name "ans10x" is similar to existing symbol "ans10".*

$$\frac{1}{2}\,(t\,(t\,\text{UnitStep}(t) - 1) - (t-1)^2\,\text{UnitStep}(t-1))$$

greenans1

```
plot1 = Plot[ans10x, {t, 0, 1}]
```



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Down to DSolve answer

```
ans11 = Integrate[x greenfunc1, {x, 0, 1}]
```

$$\frac{1}{6}\text{UnitStep}(t)\,t^3-\frac{t}{6}+\left(-\frac{t^3}{6}+\frac{t}{2}-\frac{1}{3}\right)\text{UnitStep}(t-1)$$

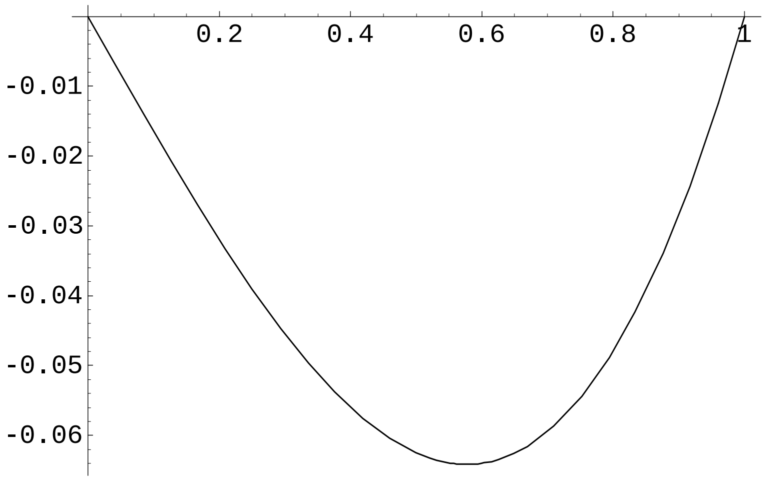
We see that t^3 is the leading term.

```
ans11x = FullSimplify[ans11]
```

— *General::spell1 : Possible spelling error: new symbol name "ans11x" is similar to existing symbol "ans11".*

$$\frac{1}{6}\left(\text{UnitStep}(t)\,t^3-t-(t^3-3\,t+2)\,\text{UnitStep}(t-1)\right)$$

```
plot2 = Plot[ans11x, {t, 0, 1}]
```



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```
ans12 = Integrate[x^2 greenfunc1, {x, 0, 1}]
```

$$\frac{1}{12}\,\text{UnitStep}(t)\,t^4 - \frac{t}{12} + \left(-\frac{t^4}{12} + \frac{t}{3} - \frac{1}{4}\right)\text{UnitStep}(t-1)$$

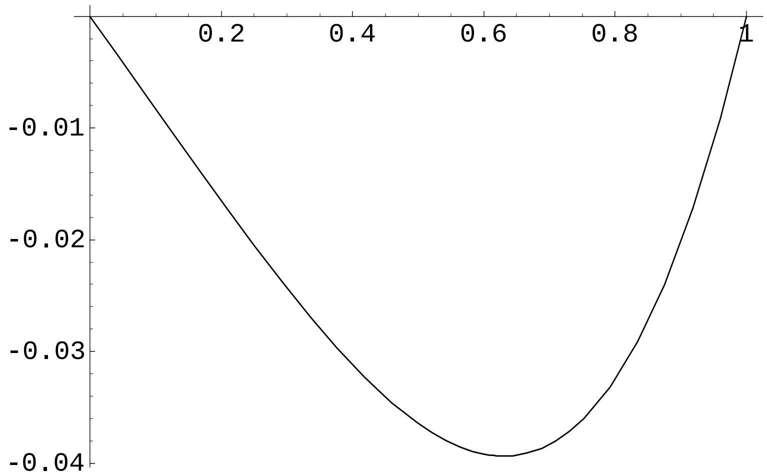
We see that t^4 is the leading term

```
ans12x = FullSimplify[ans12]
```

— *General::spell1 : Possible spelling error: new symbol name "ans12x" is similar to existing symbol "ans12".*

$$\frac{1}{12}\left(\text{UnitStep}(t)\,t^4 - t - (t^4 - 4\,t + 3)\,\text{UnitStep}(t-1)\right)$$

```
plot3 = Plot[ans12x, {t, 0, 1}]
```



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```
ans16 = Integrate[x^6 greenfunc1, {x, 0, 1}]
```

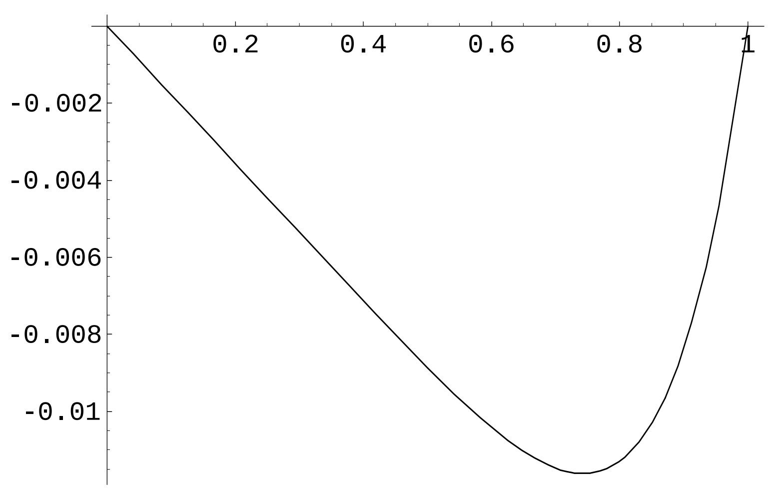
$$\frac{1}{56}\,\text{UnitStep}(t)\,t^8 - \frac{t}{56} + \left(-\frac{t^8}{56} + \frac{t}{7} - \frac{1}{8}\right)\text{UnitStep}(t-1)$$

```
ans16x = FullSimplify[ans16]
```

— *General::spell1 : Possible spelling error: new symbol name "ans16x" is similar to existing symbol "ans16".*

$$\frac{1}{56}\,(t\,(t^7\,\text{UnitStep}(t) - 1) - (t^8 - 8\,t + 7)\,\text{UnitStep}(t-1))$$

```
plot4 = Plot[ans16x, {t, 0, 1}]
```



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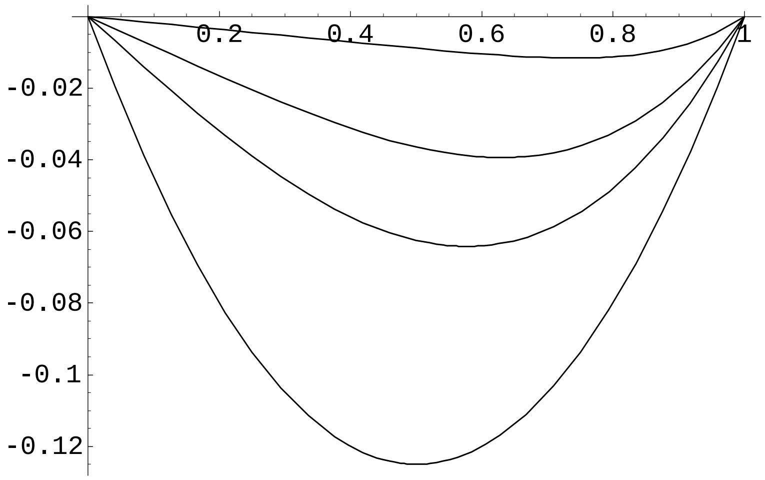
```
plotsx = Show[plot1, plot2, plot3, plot4]
```



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We see that uniform cooling gives the greatest deviation and is, not surprisingly, symmetric. A higher power of x is more asymmetric but the effect is less (is this obvious?).

Find the Green's function for $\frac{\partial^2}{\partial x^2} + k^2$.

■ Now we can solve the problem where the operator is $\frac{\partial^2}{(\partial x)^2} + k^2$ and the boundary conditions are homogeneous.

```
eq2 = D[g[t], {t, 2}] + k^2 g[t] - DiracDelta[x - t]
```

$g(t)\, k^2 - \text{DiracDelta}(x - t) + g''(t)$

```
ans2 = DSolve[{eq2 == 0, g[0] == 0, g[1] == 0}, g[t], t]
```

— *General::spell1 : Possible spelling error: new symbol name "ans2" is similar to existing symbol "ans12".*

$$\left\{\left\{g(t) \to \frac{1}{k} (\sin(k\, t - k\, x)\, \text{UnitStep}(t - x) + \cos(k\, t)\, \sin(k\, x)\, \text{UnitStep}(-x) - \sin(k\, t)\, (\csc(k)\, \sin(k - k\, x)\, \text{UnitStep}(1 - x) + \cot(k)\, \sin(k\, x)\, \text{UnitStep}(-x)))\right\}\right\}$$

At this point, g[ ] is effectively now g[x,t] as we would like.

Note that (as expected), the Green's function consists of *sin/cos* type terms as required by the operator.

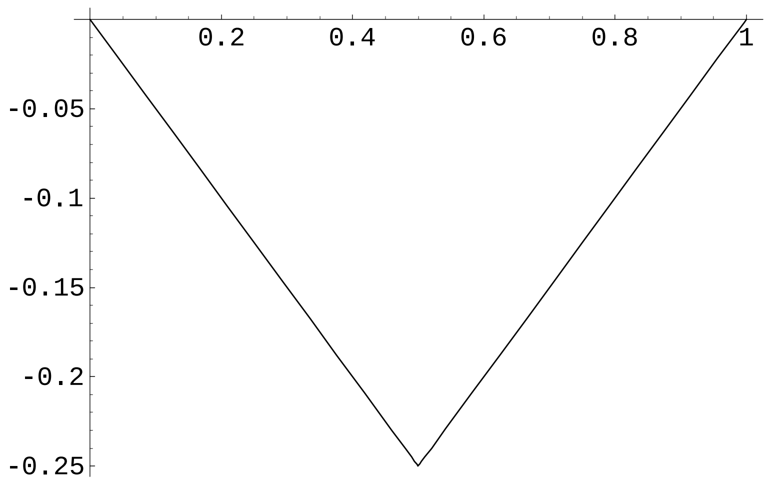
```
greenfunc2 = FullSimplify[g[t] /. ans2[[1]]]
```

$$\frac{1}{k}\left(-\csc(k)\sin(k\,t)\sin(k-k\,x)\,\mathrm{UnitStep}(1-x)+\sin(k\,(t-x))\,\mathrm{UnitStep}(t-x)+\csc(k)\sin(k-k\,t)\sin(k\,x)\,\mathrm{UnitStep}(-x)\right)$$

■ Plots of the Green's function for $\frac{\partial}{\partial x^2} + k^2$.

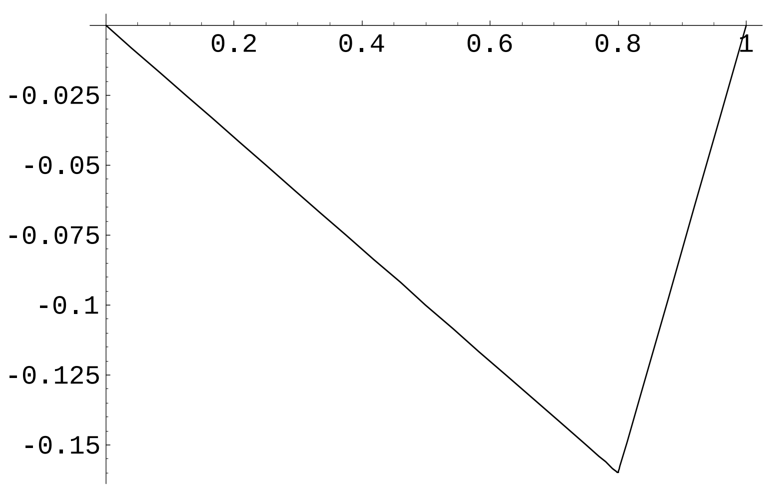
- **Now plot the Green's function at some values. First choose *k* to be small to get the answers from the example above.**

```
pl21 = Plot[greenfunc2 /. {x -> .5, k -> 0.001}, {t, 0, 1}]
```



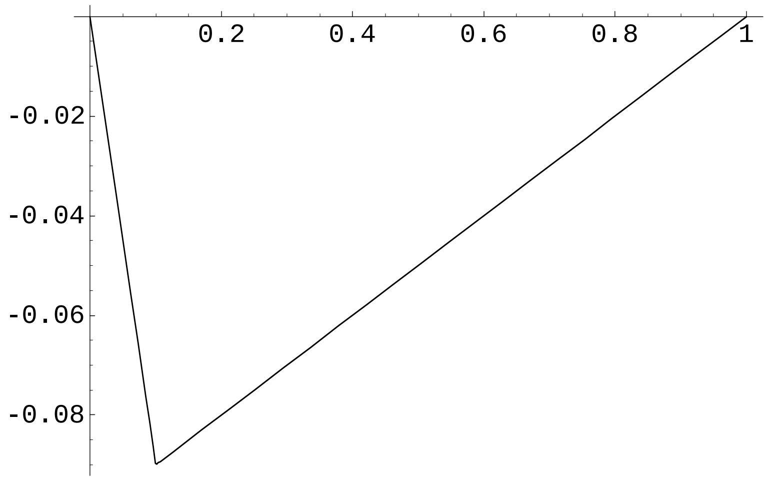
- Graphics -

```
pl22 = Plot[greenfunc2 /. {x -> .8, k -> .001}, {t, 0, 1}]
```



- Graphics -

```
pl23 = Plot[greenfunc2 /. {x -> .1, k -> .001}, {t, 0, 1}]
```



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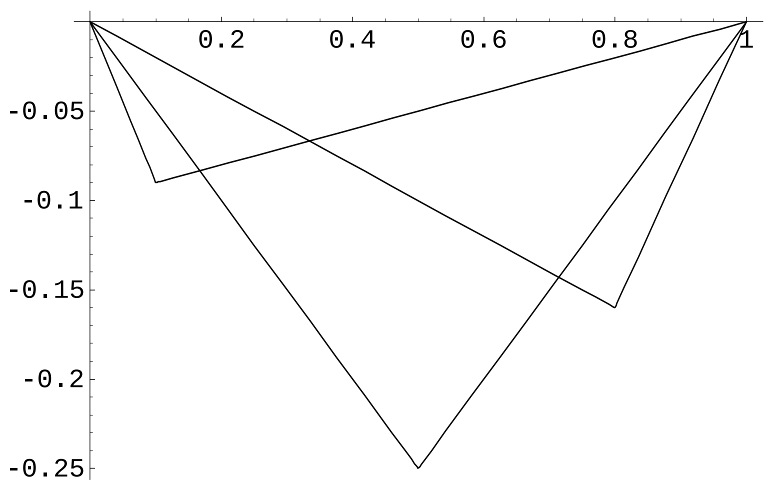
```
Show[pl21, pl22, pl23]
```



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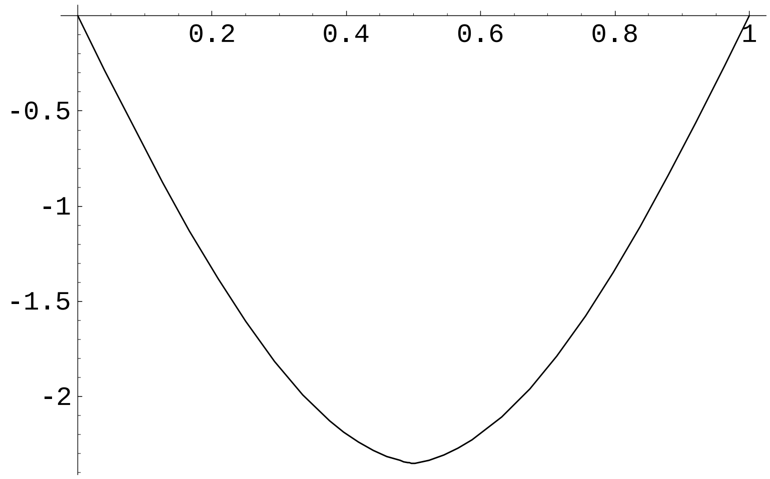
- **Now change k to a larger value, k=3**

```
pl31 = Plot[greenfunc2 /. {x -> .5, k -> 3}, {t, 0, 1}]
```



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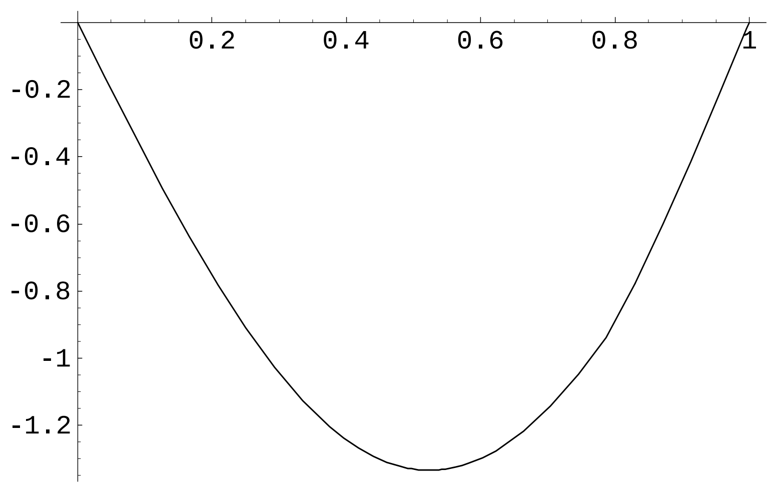
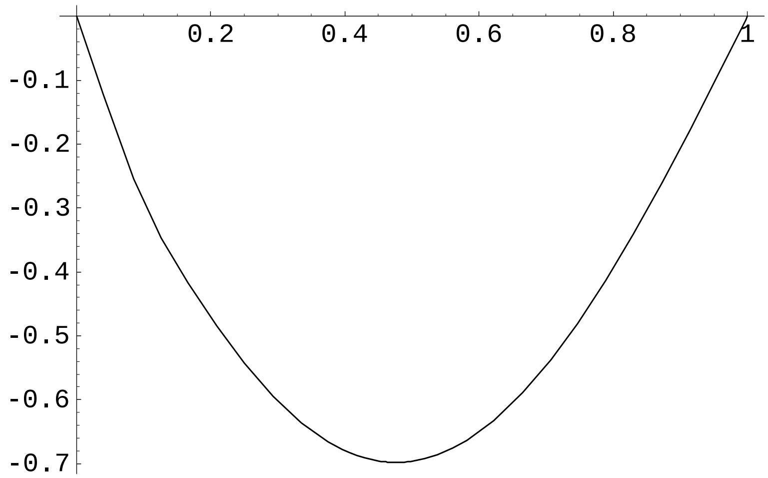
```
pl32 = Plot[greenfunc2 /. {x -> .8, k -> 3}, {t, 0, 1}]
```

— *General::spell1 : Possible spelling error: new symbol name "pl32" is similar to existing symbol "pl23".*



- Graphics -

```
pl33 = Plot[greenfunc2 /. {x -> .1, k -> 3}, {t, 0, 1}]
```



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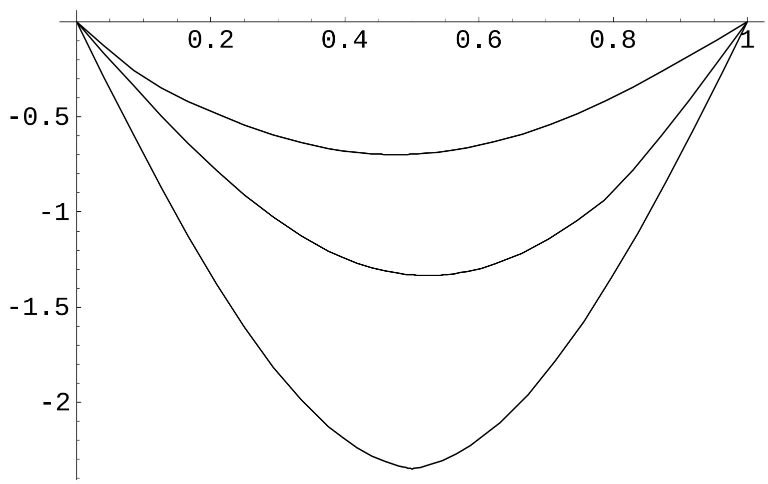
```
Show[pl31, pl32, pl33]
```



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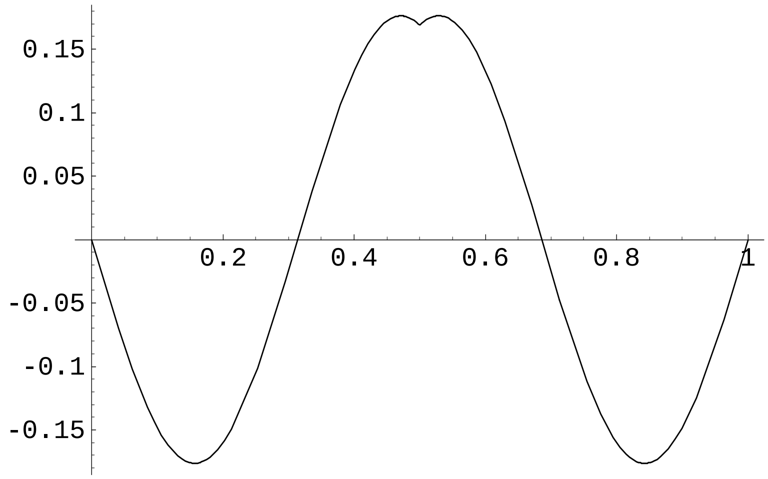
- **If we increase k still further, we can see the expected oscillation in g[x,t]**
- **Again increase k**

```
pl51 = Plot[greenfunc2 /. {x -> .5, k -> 10}, {t, 0, 1}]
```



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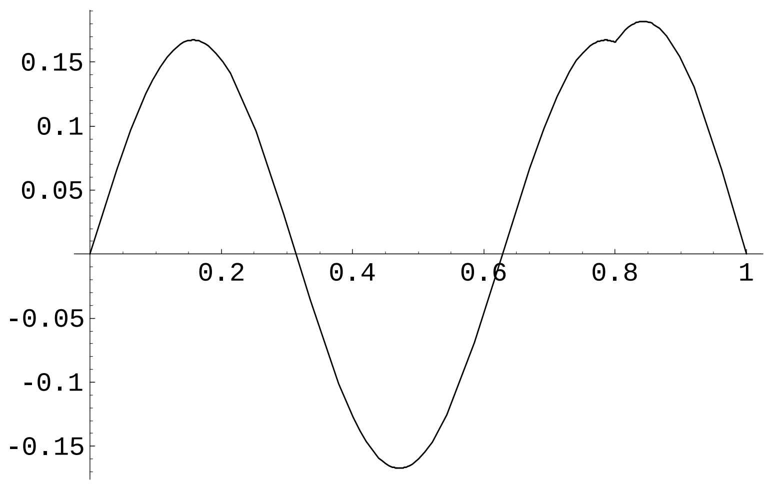
```
pl52 = Plot[greenfunc2 /. {x -> .8, k -> 10}, {t, 0, 1}]
```



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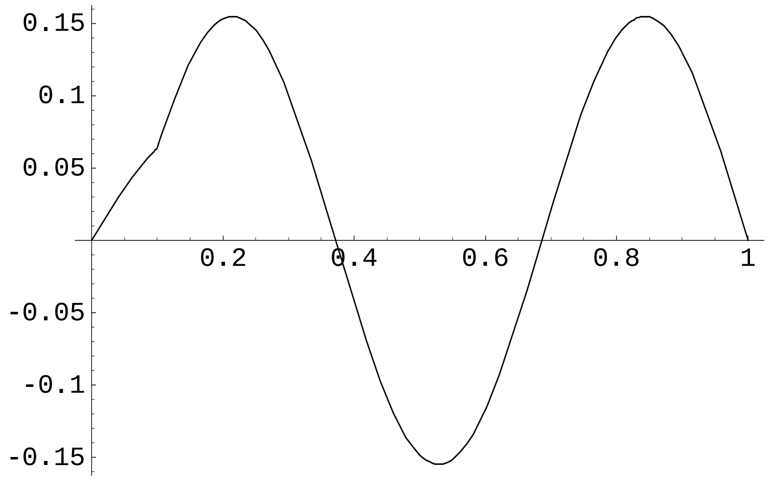
```
pl53 = Plot[greenfunc2 /. {x -> .1, k -> 10}, {t, 0, 1}]
```



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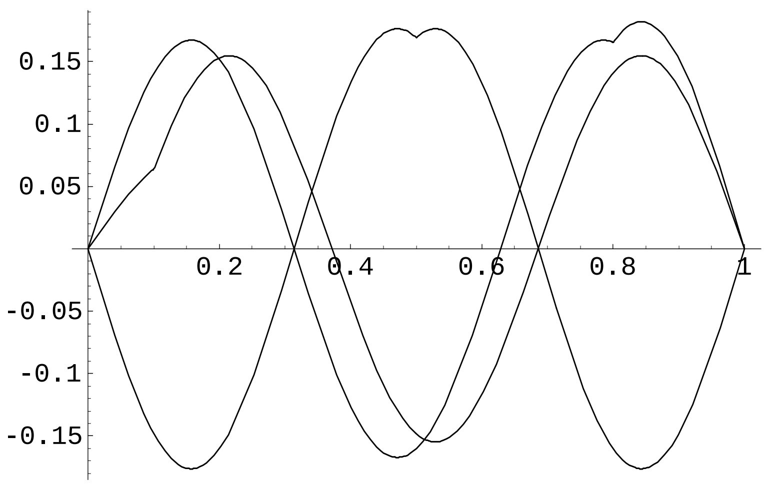
```
Show[pl51, pl52, pl53]
```



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Perhaps some insight into the Green's function is emerging. It seems to be the obvious function for the homogeneous (e.g., L [g] ==0) operator but with a kink at t=x.

■ The new operator is an oscillator and there is inhomogeneous forcing.

First choose f[x]=1.

```
ans31 = Integrate[1 greenfunc2, {x, 0, 1}]
```

— *General::spell1 : Possible spelling error: new symbol name "ans31" is similar to existing symbol "ans1".*

$$-\frac{\sin(k\,t)\tan(\frac{k}{2})}{k^2}+\frac{(\cos(k\,(t-1))-1)\,\text{UnitStep}(t-1)}{k^2}-\frac{(\cos(k\,t)-1)\,\text{UnitStep}(t)}{k^2}$$

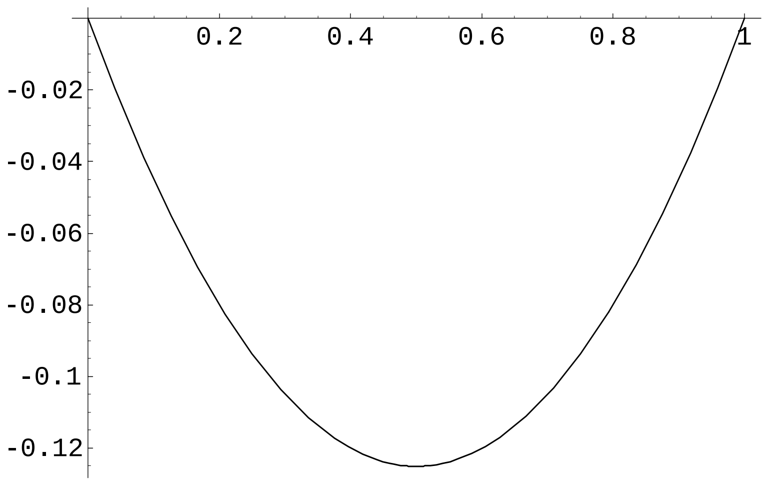
We see the required *sin(kt), cos(kt)* behavior.

```
ans31x = FullSimplify[ans31]
```

— *General::spell1 : Possible spelling error: new symbol name "ans31x" is similar to existing symbol "ans31".*

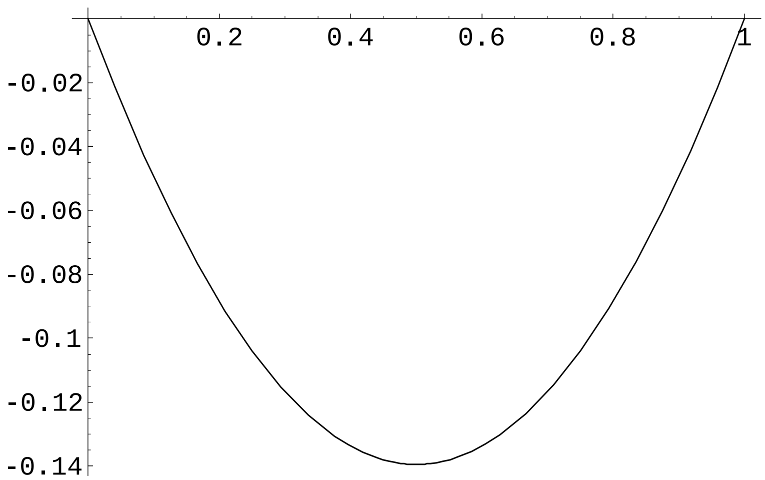
$$\frac{-\sin(k\,t)\tan(\frac{k}{2})+(\cos(k\,(t-1))-1)\,\text{UnitStep}(t-1)-\cos(k\,t)\,\text{UnitStep}(t)+\text{UnitStep}(t)}{k^2}$$

```
pl61 = Plot[ans31x /. k -> .1, {t, 0, 1}]
```



- Graphics -

```
pl62 = Plot[ans31x /. k -> 1, {t, 0, 1}]
```



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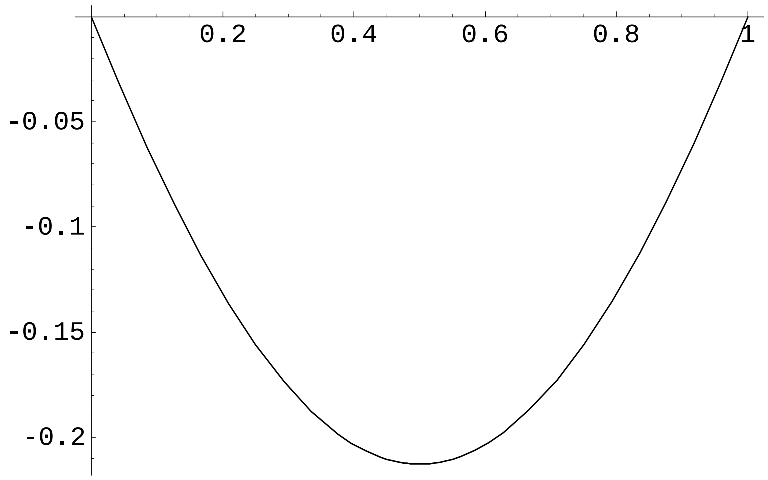
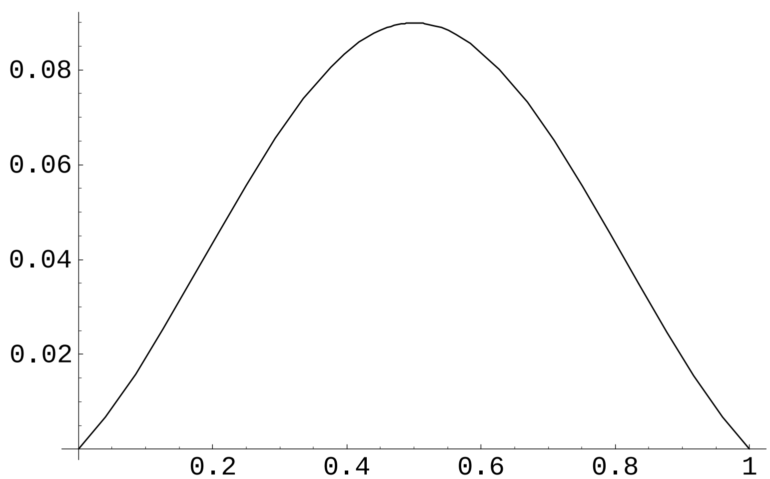
```
pl63 = Plot[ans31x /. k -> 2, {t, 0, 1}]
```



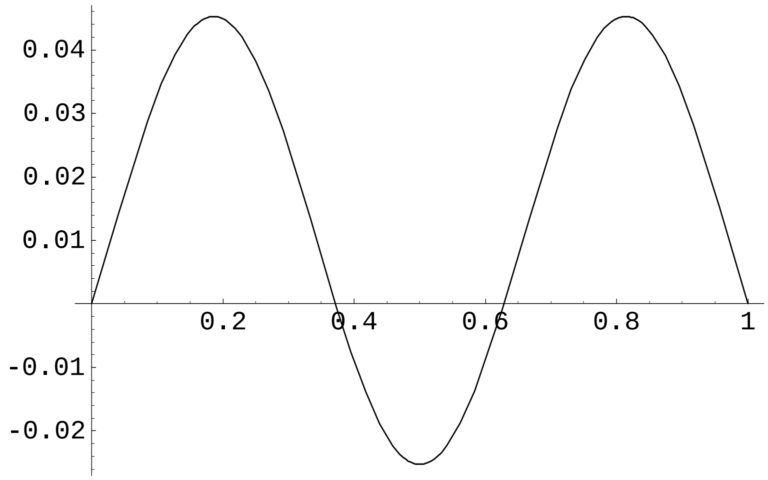
- Graphics -

```
pl64 = Plot[ans31x /. k -> 5, {t, 0, 1}]
```



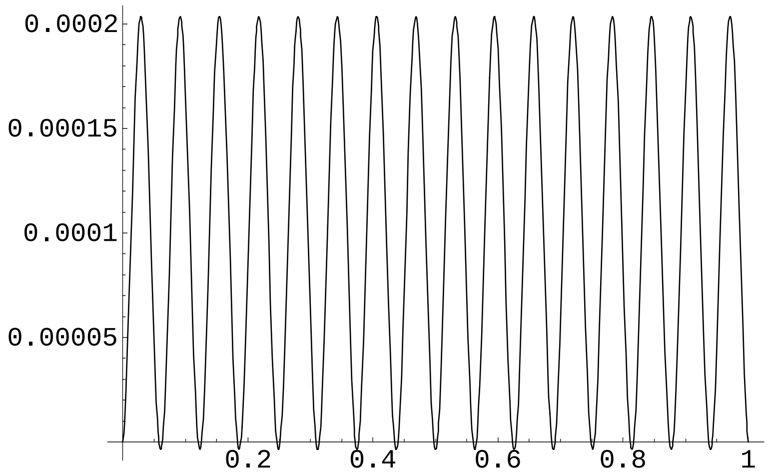
- Graphics -

```
pl65 = Plot[ans31x /. k -> 10, {t, 0, 1}]
```



- Graphics -

```
pl66 = Plot[ans31x /. k -> 100, {t, 0, 1}]
```



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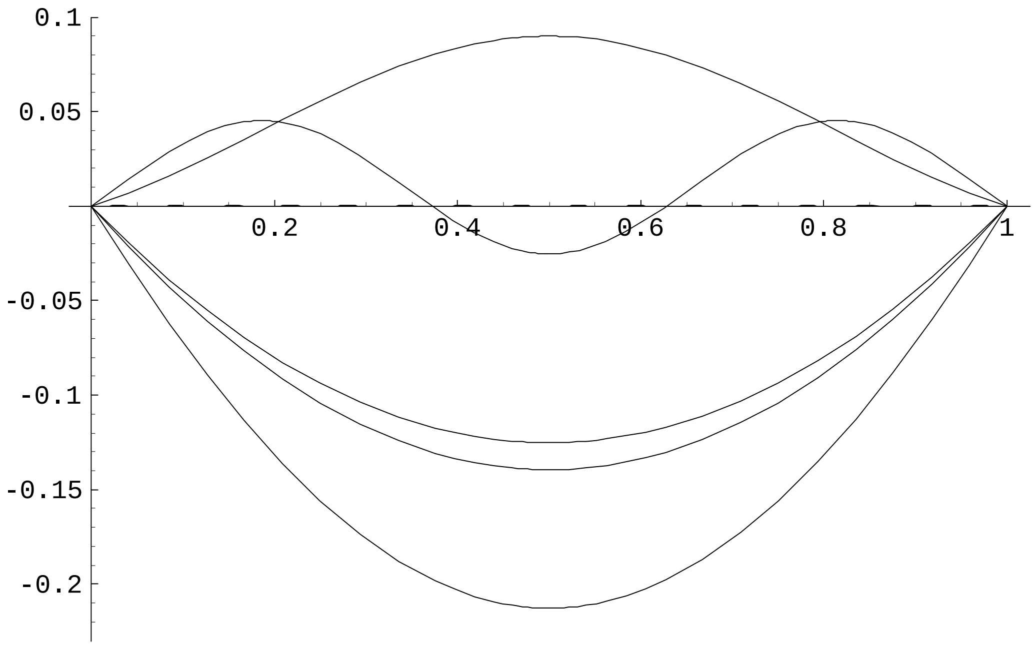
```
Show[pl61, pl62, pl63, pl64, pl65, pl66, PlotRange -> {-.23, .1}]
```



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As the response frequency goes up, the amplitude goes down (does this make sense?)

Now try a higher power of x

Now choose f[x] = x^2.

```
ans4 = Integrate[x^2 greenfunc2, {x, 0, 1}]
```

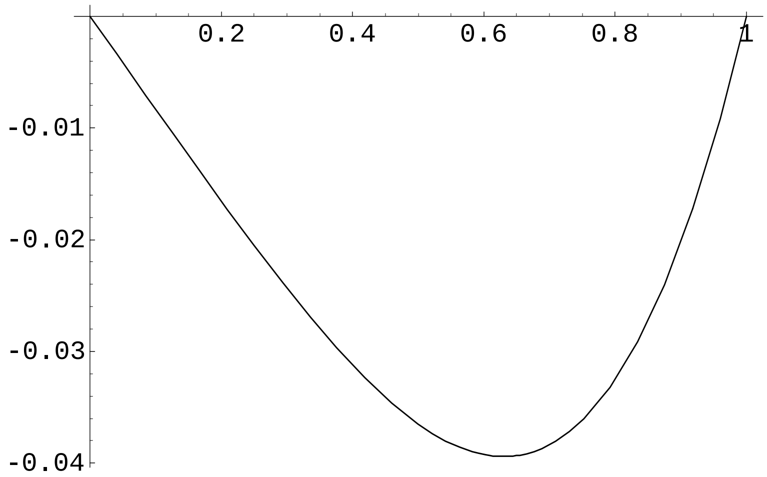
$$-\frac{(k^2+2\cos(k)-2)\csc(k)\sin(k\,t)}{k^4}+\frac{(-k^2\,t^2+(k^2-2)\cos(k-k\,t)-2\,k\sin(k-k\,t)+2)\,\text{UnitStep}(t-1)}{k^4}+\frac{(k^2\,t^2+2\cos(k\,t)-2)\,\text{UnitStep}(t)}{k^4}$$

```
ans4x = FullSimplify[ans4]
```

— *General::spell1 : Possible spelling error: new symbol name "ans4x" is similar to existing symbol "ans4".*

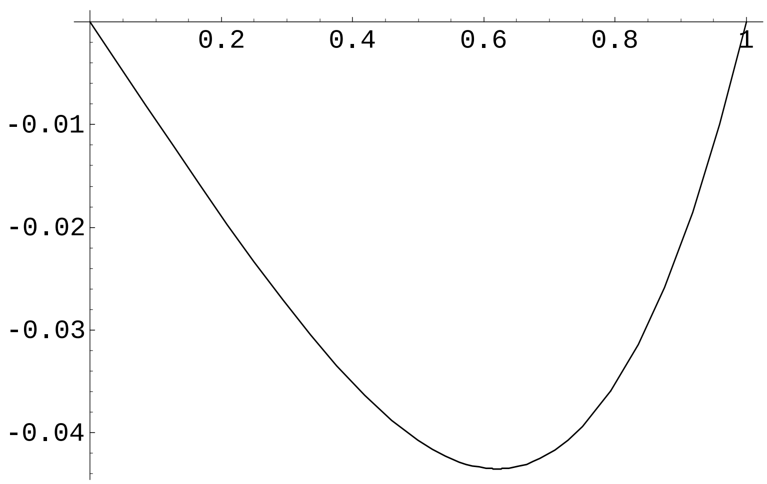
$$\frac{1}{k^4}(-(k^2+2\cos(k)-2)\csc(k)\sin(k\,t)+(-k^2\,t^2+(k^2-2)\cos(k-k\,t)-2\,k\sin(k-k\,t)+2)\,\text{UnitStep}(t-1)+(k^2\,t^2+2\cos(k\,t)-2)\,\text{UnitStep}(t))$$

```
pl71 = Plot[ans4x /. k -> .1, {t, 0, 1}]
```



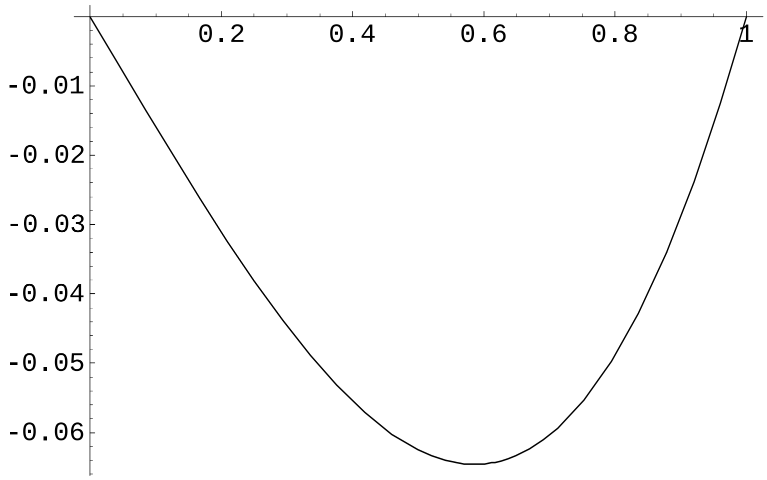
- Graphics -

```
pl72 = Plot[ans4x /. k -> 1, {t, 0, 1}]
```



- Graphics -

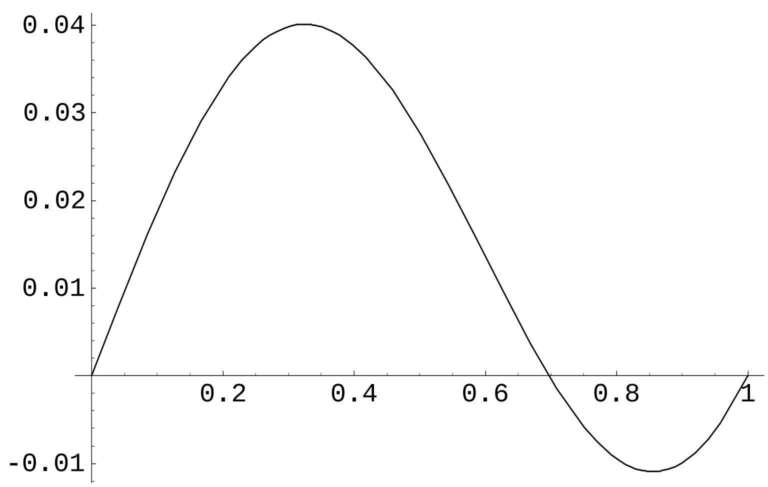
```
pl73 = Plot[ans4x /. k -> 2, {t, 0, 1}]
```



- Graphics -

green2

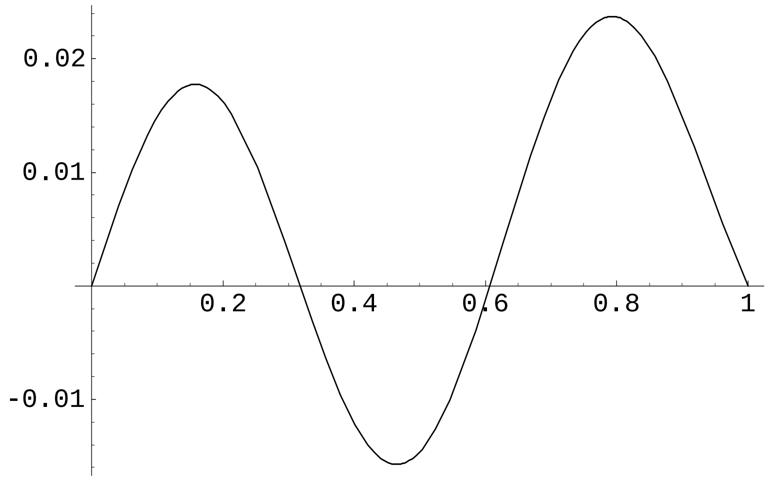
```
pl74 = Plot[ans4x /. k -> 5, {t, 0, 1}]
```



- Graphics -

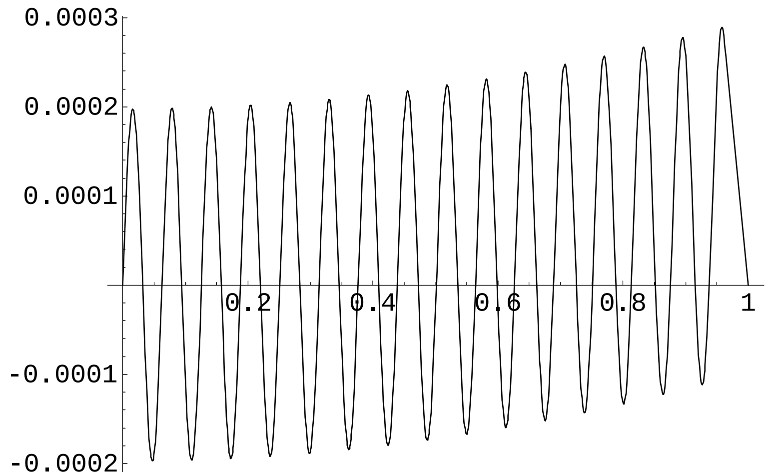
Go down to the DSolve answer

```
pl75 = Plot[ans4x /. k -> 10, {t, 0, 1}]
```



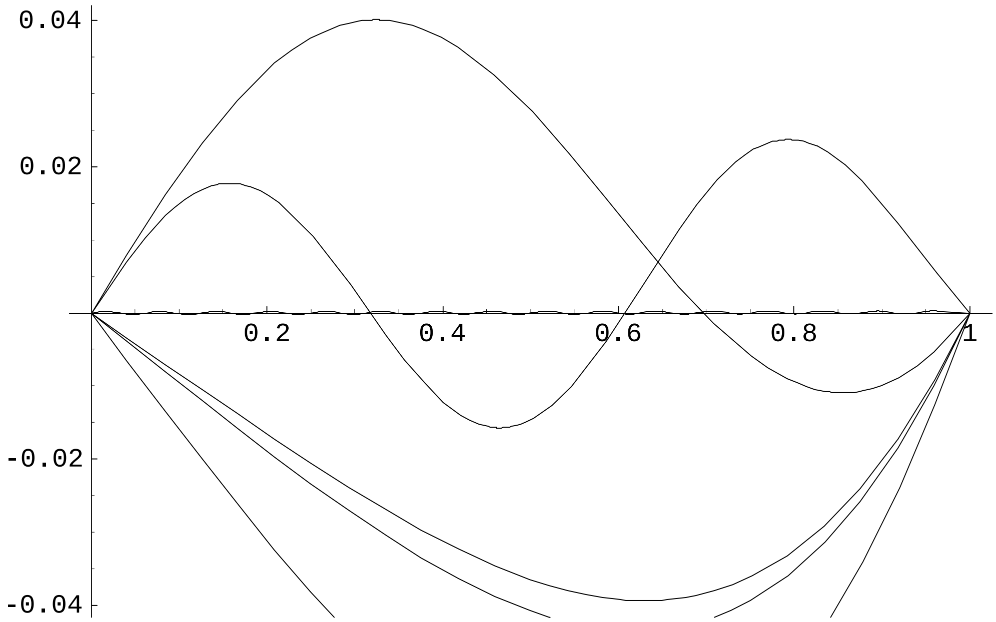
- Graphics -

```
pl76 = Plot[ans4x /. k -> 100, {t, 0, 1}]
```



- Graphics -

```
Show[pl71, pl72, pl73, pl74, pl75, pl76]
```



- Graphics -

We see some asymmetry that is due to the asymmetric forcing. Again the amplitude decreases with frequency.

Just use DSolve(!)

You can check for yourself if we got the correct answers.

```
ans5 =
 DSolve[{D[u[x], {x, 2}] - x^n == 0, u[0] == 0, u[1] == 0}, u[x], x]
```

$$\left\{\left\{u(x) \to \frac{x^{n+2}}{(n+1)(n+2)} + \frac{(-1+0^{n+2})\,x}{(n+1)(n+2)} - \frac{0^{n+2}}{(n+1)(n+2)}\right\}\right\}$$

```
ans5x = FullSimplify[ans5]
```

— *General::spell1 : Possible spelling error: new symbol name "ans5x" is similar to existing symbol "ans5".*

$$\left\{\left\{u(x) \to \frac{x^{n+2} + (-1+0^{n+2})\,x - 0^{n+2}}{(n+1)(n+2)}\right\}\right\}$$

It seems we have to live with the "0's".

```
ewu[x] = u[x] /. ans5x[[1]]
```

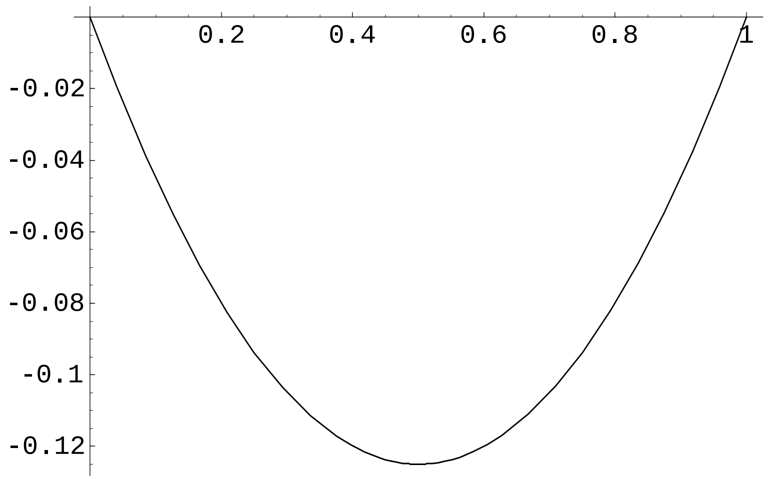
$$\frac{x^{n+2} + (-1+0^{n+2})\,x - 0^{n+2}}{(n+1)(n+2)}$$

```
(ewu[x] /. n -> 0)
```

$$\frac{1}{2}(x^2 - x)$$

dsolveans

```
Plot[Evaluate[ewu[x] /. n -> 0], {x, 0, 1}]
```



- Graphics -

back to Green' *s* function answer

This looks OK.

How about the other L?

We can get a general solution fro x^n, but it involves functions that have possibly ambiguous choices for the path of integration. Thus we just choose x^2.

```
ans6 = DSolve[
  {D[u[x], {x, 2}] + k^2 u[x] - x^2 == 0, u[0] == 0, u[1] == 0}, u[x], x]
```

— *General::spell1 : Possible spelling error: new symbol name "ans6" is similar to existing symbol "ans16".*

$$\left\{\left\{u(x) \to \frac{k^2 x^2 + 2\cos(k x) - (k^2 + 2\cos(k) - 2)\csc(k)\sin(k x) - 2}{k^4}\right\}\right\}$$

```
ans6x = FullSimplify[ans6]
```

— *General::spell : Possible spelling error: new symbol name "ans6x" is similar to existing symbols {ans16x, ans6}.*

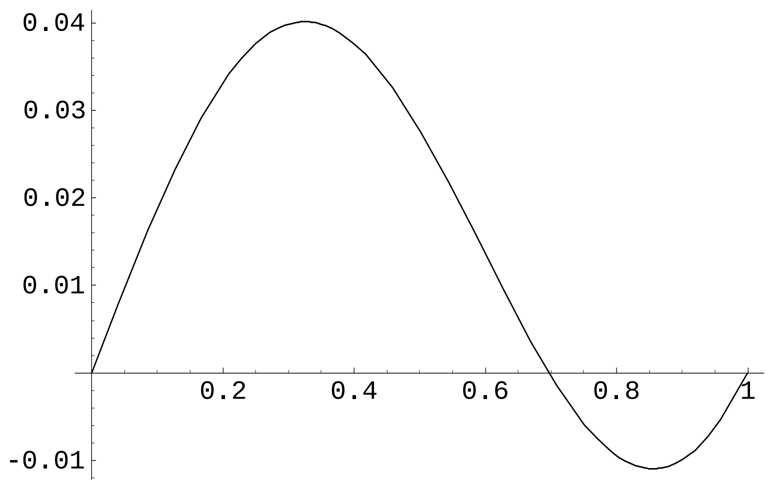
$$\left\{\left\{u(x) \to \frac{k^2 x^2 + 2\cos(k x) - (k^2 + 2\cos(k) - 2)\csc(k)\sin(k x) - 2}{k^4}\right\}\right\}$$

```
ewu2[x] = (u[x] /. ans6x[[1]] /. n -> 2)
```

$$\frac{k^2\,x^2 + 2\cos(k\,x) - (k^2 + 2\cos(k) - 2)\csc(k)\sin(k\,x) - 2}{k^4}$$

dsolve2

```
Plot[Evaluate[ewu2[x] /. k -> 5], {x, 0, 1}]
```

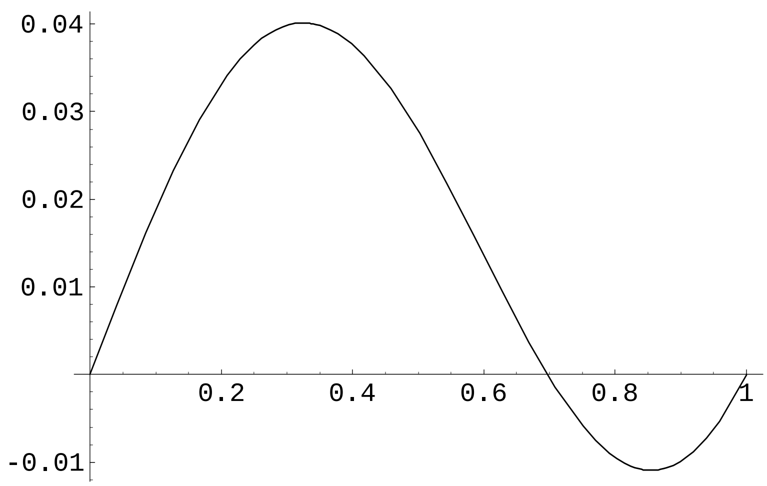


- Graphics -

Go to the Green' s function answer

Or, just look here. Here is the answer from above that used the Green's Function. It matches.

```
Show[pl74]
```



- Graphics -