Generalized eigenvalue problems

10/6/98

For a problem where $AB(\lambda)\, y = 0$, we expect that non trivial solutions for y will exist only for certain values of λ. Thus this problem appears to be an eigenvalue problem, but not of the usual form. $(A - I\lambda)\, x = 0$. Can we convert $AB(\lambda)\, y = 0$ to the standard form? Yes, we realize a "generalized" version of $AB(\lambda)\, y = 0$ is $Ax = B\lambda\, x$.

First try a simple arbitrary matrix

Here is the AB matrix

```
AB={{1-λ,2,-3+λ},{2 λ, -3, 4+λ},{11,-3 λ+3, 2-λ}}
```

$$\begin{pmatrix} 1-\lambda & 2 & \lambda-3 \\ 2\lambda & -3 & \lambda+4 \\ 11 & 3-3\lambda & 2-\lambda \end{pmatrix}$$

We can get A by simply removing λ.

```
A=AB/.λ->0
```

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 4 \\ 11 & 3 & 2 \end{pmatrix}$$

Now looking at the definition we can get B by subtracting A from AB and factoring out λ.

```
B=-(AB-A)/λ
```

$$\begin{pmatrix} 1 & 0 & -1 \\ -2 & 0 & -1 \\ 0 & 3 & 1 \end{pmatrix}$$

NOw check to see what we have

```
A - B λ
```

$$\begin{pmatrix} 1-\lambda & 2 & \lambda-3 \\ 2\lambda & -3 & \lambda+4 \\ 11 & 3-3\lambda & 2-\lambda \end{pmatrix}$$

So it works.

Now how do we solve, Ax = B λ x?

Form $B^{-1} A x = B^{-1} B \lambda x$.

This gives:

$(B^{-1} A - I \lambda) x = 0$, which is of the usual form of an eigen value problem.

```
temp1=Inverse[B].(A - B λ)
```

$$\begin{pmatrix} \frac{1-\lambda}{3} - \frac{2\lambda}{3} & \frac{5}{3} & \frac{1}{3}(-\lambda-4) + \frac{\lambda-3}{3} \\ \frac{2(1-\lambda)}{9} + \frac{2\lambda}{9} + \frac{11}{3} & \frac{1}{3}(3-3\lambda) + \frac{1}{9} & \frac{2-\lambda}{3} + \frac{2(\lambda-3)}{9} + \frac{\lambda+4}{9} \\ -\frac{2}{3}(1-\lambda) - \frac{2\lambda}{3} & -\frac{1}{3} & \frac{1}{3}(-\lambda-4) - \frac{2(\lambda-3)}{3} \end{pmatrix}$$

We now have an eigen value problem

```
temp2=Simplify[temp1]
```

$$\begin{pmatrix} \frac{1}{3}-\lambda & \frac{5}{3} & -\frac{7}{3} \\ \frac{35}{9} & \frac{10}{9}-\lambda & \frac{4}{9} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}-\lambda \end{pmatrix}$$

```
temp3=Det[temp2]
```

$$-\lambda^3 + \frac{19\,\lambda^2}{9} + \frac{59\,\lambda}{9} - \frac{29}{9}$$

```
eigs=Solve[%==0,λ]
```

$$\Big\{\Big\{\lambda \to \frac{19}{27} - \frac{(1+i\sqrt{3})\sqrt[3]{10274+81\,i\sqrt{268190}}}{27\,2^{2/3}} - \frac{977\,(1-i\sqrt{3})}{27\sqrt[3]{2\,(10274+81\,i\sqrt{268190})}}\Big\}, \Big\{\lambda \to \frac{19}{27} - \frac{(1-i\sqrt{3})\sqrt[3]{10274+81\,i\sqrt{268190}}}{27\,2^{2/3}} - \frac{977\,(1+i\sqrt{3})}{27\sqrt[3]{2\,(10274+81\,i\sqrt{268190})}}\Big\},$$
$$\Big\{\lambda \to \frac{19}{27} + \frac{977\,2^{2/3}}{27\sqrt[3]{10274+81\,i\sqrt{268190}}} + \frac{1}{27}\sqrt[3]{2\,(10274+81\,i\sqrt{268190})}\Big\}\Big\}$$

```
Chop[N[eigs]]
```

$\{\{\lambda \to 0.441819\}, \{\lambda \to -1.99196\}, \{\lambda \to 3.66125\}\}$

Which is the same as

```
Chop[N[Eigenvalues[Inverse[B].A]]]
```

$\{0.441819, -1.99196, 3.66125\}$

We might also expect that less work is required:

```
Det[AB]
```

$-9\lambda^3 + 19\lambda^2 + 59\lambda - 29$

```
Solve[%==0,λ]
```

$$\left\{\left\{\lambda \to \frac{19}{27} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{10274 + 81\, i\sqrt{268190}}}{27\; 2^{2/3}} - \frac{977\left(1 - i\sqrt{3}\right)}{27\sqrt[3]{2\left(10274 + 81\, i\sqrt{268190}\right)}}\right\}, \left\{\lambda \to \frac{19}{27} - \frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{10274 + 81\, i\sqrt{268190}}}{27\; 2^{2/3}} - \frac{977\left(1 + i\sqrt{3}\right)}{27\sqrt[3]{2\left(10274 + 81\, i\sqrt{268190}\right)}}\right\}, \left\{\lambda \to \frac{19}{27} + \frac{977\; 2^{2/3}}{27\sqrt[3]{10274 + 81\, i\sqrt{268190}}} + \frac{1}{27}\sqrt[3]{2\left(10274 + 81\, i\sqrt{268190}\right)}\right\}\right\}$$

```
Chop[N[%]]
```

$\{\{\lambda \to 0.441819\}, \{\lambda \to -1.99196\}, \{\lambda \to 3.66125\}\}$

■ Now try a matrix from the numerical solution to channel flow stability

The complete motivation for this example can be found in the mathematica notebook, Linear Stability of Pressure Driven Channel Flow.

The matrix that arises when solving the Orr-Sommerfeld equation using Chebyshev polynomials in a spectral method is shown below as AB. The physical problem is the stability of laminar channel flow to turbulence (although there is a problem with this) The term rr is the Reynolds number and α is the wave number. In a spectral method, the solution is expressed in terms of a set of orthogonal functions with unknown coefficients. Chebyshev polynomials are a convenient set for numerical problems on regular finite domains. They are solutions to **Chebyshev's** differential equation and the first few are

```
chebs=Table[ChebyshevT[i,y],{i,0,5}]
```

$\{1, y, 2y^2 - 1, 4y^3 - 3y, 8y^4 - 8y^2 + 1, 16y^5 - 20y^3 + 5y\}$

Here is the type of orthogonity that these display. You can think of this integration as an appropriate inner product for these functions on the domain, -1,1.

```
Table[Integrate[chebs*ChebyshevT[i,y]*
(1-y^2)^(-1/2),{y,-1,1}],{i,0,5}]
```

$$\begin{pmatrix} \pi & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\pi}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\pi}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\pi}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\pi}{2} \end{pmatrix}$$

The matrix is

$$\begin{pmatrix} \alpha^4 + \frac{1}{2} i\, \text{rr}\, \alpha^3 - i\, \text{rr}\, \lambda\, \alpha^3 - \frac{1152\, \alpha^2}{7} + \frac{10\, i\, \text{rr}\, \alpha}{7} + \frac{576}{7} i\, \text{rr}\, \lambda\, \alpha + \frac{271872}{7} & -\frac{1}{4} i\, \text{rr}\, \alpha^3 - \frac{912\, \alpha^2}{7} + \frac{12\, i\, \text{rr}\, \alpha}{7} + \frac{456}{7} i\, \text{rr}\, \lambda\, \alpha + \frac{238080}{7} & -\frac{416\, \alpha^2}{7} + \frac{4\, i\, \text{rr}\, \alpha}{7} + \frac{208}{7} i\, \text{rr}\, \lambda\, \alpha + \frac{138048}{7} \\ -\frac{1}{2} i\, \text{rr}\, \alpha^3 - \frac{2496\, \alpha^2}{7} + \frac{48\, i\, \text{rr}\, \alpha}{7} + \frac{1248}{7} i\, \text{rr}\, \lambda\, \alpha + \frac{460800}{7} & \alpha^4 + \frac{1}{2} i\, \text{rr}\, \alpha^3 - i\, \text{rr}\, \lambda\, \alpha^3 - \frac{1920\, \alpha^2}{7} + \frac{52\, i\, \text{rr}\, \alpha}{7} + \frac{960}{7} i\, \text{rr}\, \lambda\, \alpha + \frac{405120}{7} & -\frac{1}{4} i\, \text{rr}\, \alpha^3 - \frac{864\, \alpha^2}{7} + \frac{8\, i\, \text{rr}\, \alpha}{7} + \frac{432}{7} i\, \text{rr}\, \lambda\, \alpha + 23 \\ \frac{4}{7} i\, \text{rr}\, \alpha^3 - \frac{3072\, \alpha^2}{7} + \frac{48\, i\, \text{rr}\, \alpha}{7} + \frac{1536}{7} i\, \text{rr}\, \lambda\, \alpha + 34560 & \frac{2}{7} i\, \text{rr}\, \alpha^3 - \frac{2544\, \alpha^2}{7} + \frac{52\, i\, \text{rr}\, \alpha}{7} + \frac{1272}{7} i\, \text{rr}\, \lambda\, \alpha + 30720 & \alpha^4 + \frac{13}{14} i\, \text{rr}\, \alpha^3 - i\, \text{rr}\, \lambda\, \alpha^3 - \frac{960\, \alpha^2}{7} + \frac{134\, i\, \text{rr}\, \alpha}{7} + \frac{480}{7} i\, \text{rr} \end{pmatrix}$$

We can see this better with some numbers substituted

```
AB/.{α->1,rr->100}
```

$$\begin{pmatrix} \frac{56900\, i\, \lambda}{7} + \left(\frac{270727}{7} + \frac{1350\, i}{7}\right) & \frac{45600\, i\, \lambda}{7} + \left(\frac{237168}{7} + \frac{1025\, i}{7}\right) & \frac{20800\, i\, \lambda}{7} + \left(\frac{137632}{7} + \frac{400\, i}{7}\right) \\ \frac{124800\, i\, \lambda}{7} + \left(65472 + \frac{4450\, i}{7}\right) & \frac{95300\, i\, \lambda}{7} + \left(57601 + \frac{5550\, i}{7}\right) & \frac{43200\, i\, \lambda}{7} + \left(33888 + \frac{625\, i}{7}\right) \\ \frac{153600\, i\, \lambda}{7} + \left(\frac{238848}{7} + \frac{5200\, i}{7}\right) & \frac{127200\, i\, \lambda}{7} + \left(\frac{212496}{7} + \frac{5400\, i}{7}\right) & \frac{47300\, i\, \lambda}{7} + \left(\frac{133447}{7} + \frac{14050\, i}{7}\right) \end{pmatrix}$$

λ is apparently an eigen value because we have AB.y = 0, but the form is not
Ay = λ y . The term λ is not just on the diagonals. It is really AAy = λ BB y. This is the form of a generalized eigenvalue problem. Let's see how to construct the problem in this form. First get the AA matrix,

```
AA=AB/.λ->0
```

$$\begin{pmatrix} \alpha^4+\frac{1}{2}\,i\,\mathrm{rr}\,\alpha^3-\frac{1152\,\alpha^2}{7}+\frac{10\,i\,\mathrm{rr}\,\alpha}{7}+\frac{271872}{7} & -\frac{1}{4}\,i\,\mathrm{rr}\,\alpha^3-\frac{912\,\alpha^2}{7}+\frac{12\,i\,\mathrm{rr}\,\alpha}{7}+\frac{238080}{7} & -\frac{416\,\alpha^2}{7}+\frac{4\,i\,\mathrm{rr}\,\alpha}{7}+\frac{138048}{7} \\ -\frac{1}{2}\,i\,\mathrm{rr}\,\alpha^3-\frac{2496\,\alpha^2}{7}+\frac{48\,i\,\mathrm{rr}\,\alpha}{7}+\frac{460800}{7} & \alpha^4+\frac{1}{2}\,i\,\mathrm{rr}\,\alpha^3-\frac{1920\,\alpha^2}{7}+\frac{52\,i\,\mathrm{rr}\,\alpha}{7}+\frac{405120}{7} & -\frac{1}{4}\,i\,\mathrm{rr}\,\alpha^3-\frac{864\,\alpha^2}{7}+\frac{8\,i\,\mathrm{rr}\,\alpha}{7}+\frac{238080}{7} \\ \frac{4}{7}\,i\,\mathrm{rr}\,\alpha^3-\frac{3072\,\alpha^2}{7}+\frac{48\,i\,\mathrm{rr}\,\alpha}{7}+34560 & \frac{2}{7}\,i\,\mathrm{rr}\,\alpha^3-\frac{2544\,\alpha^2}{7}+\frac{52\,i\,\mathrm{rr}\,\alpha}{7}+30720 & \alpha^4+\frac{13}{14}\,i\,\mathrm{rr}\,\alpha^3-\frac{960\,\alpha^2}{7}+\frac{134\,i\,\mathrm{rr}\,\alpha}{7}+19200 \end{pmatrix}$$

```
BB=Expand[(AB-AA)/λ]
```

$$\begin{pmatrix} \frac{576\,i\,\mathrm{rr}\,\alpha}{7}-i\,\mathrm{rr}\,\alpha^3 & \frac{456\,i\,\mathrm{rr}\,\alpha}{7} & \frac{208\,i\,\mathrm{rr}\,\alpha}{7} \\ \frac{1248\,i\,\mathrm{rr}\,\alpha}{7} & \frac{960\,i\,\mathrm{rr}\,\alpha}{7}-i\,\mathrm{rr}\,\alpha^3 & \frac{432\,i\,\mathrm{rr}\,\alpha}{7} \\ \frac{1536\,i\,\mathrm{rr}\,\alpha}{7} & \frac{1272\,i\,\mathrm{rr}\,\alpha}{7} & \frac{480\,i\,\mathrm{rr}\,\alpha}{7}-i\,\mathrm{rr}\,\alpha^3 \end{pmatrix}$$

Thus the problem is
(AA + λ BB).{y1,y2} = 0

We can check

```
Expand[AB-(AA  + λ BB ) ]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now solve this

```
temp=(AA  + λ BB ) /.{α->1,rr->100}
```

$$\begin{pmatrix} \frac{56900\,i\,\lambda}{7}+\left(\frac{270727}{7}+\frac{1350\,i}{7}\right) & \frac{45600\,i\,\lambda}{7}+\left(\frac{237168}{7}+\frac{1025\,i}{7}\right) & \frac{20800\,i\,\lambda}{7}+\left(\frac{137632}{7}+\frac{400\,i}{7}\right) \\ \frac{124800\,i\,\lambda}{7}+\left(65472+\frac{4450\,i}{7}\right) & \frac{95300\,i\,\lambda}{7}+\left(57601+\frac{5550\,i}{7}\right) & \frac{43200\,i\,\lambda}{7}+\left(33888+\frac{625\,i}{7}\right) \\ \frac{153600\,i\,\lambda}{7}+\left(\frac{238848}{7}+\frac{5200\,i}{7}\right) & \frac{127200\,i\,\lambda}{7}+\left(\frac{212496}{7}+\frac{5400\,i}{7}\right) & \frac{47300\,i\,\lambda}{7}+\left(\frac{133447}{7}+\frac{14050\,i}{7}\right) \end{pmatrix}$$

```
temp2=Expand[Inverse[-BB/.{α->1,rr->100}] . temp]
```

$$\begin{pmatrix} -\lambda-\left(\frac{222437}{839902}+\frac{659846447\,i}{41995100}\right) & -\frac{934127}{1679804}-\frac{141664062\,i}{10498775} & \frac{37498}{419951}-\frac{77017372\,i}{10498775} \\ \frac{152977}{839902}+\frac{317976\,i}{2099755} & \left(\frac{371367}{839902}-\frac{349871\,i}{41995100}\right)-\lambda & -\frac{1093615}{1679804}-\frac{872436\,i}{10498775} \\ \frac{109304}{419951}+\frac{23377152\,i}{419951} & \frac{211074}{419951}+\frac{507434958\,i}{10498775} & \left(\frac{977461}{839902}+\frac{1128276721\,i}{41995100}\right)-\lambda \end{pmatrix}$$

```
N[temp2]
```

$$\begin{pmatrix} -1.\,\lambda-(0.264837+15.7125\,i) & -0.556093-13.4934\,i & 0.0892914-7.33584\,i \\ 0.182137+0.151435\,i & (0.442155-0.00833123\,i)-1.\,\lambda & -0.651037-0.0830988\,i \\ 0.260278+55.6664\,i & 0.502616+48.3328\,i & (1.16378+26.8669\,i)-1.\,\lambda \end{pmatrix}$$

Which now looks like an eigenvalue problem.

```
Det[temp2]
```

$$-\lambda^3+\left(\frac{160913}{119986}+\frac{66868629\,i}{5999300}\right)\lambda^2-\left(\frac{4986894039}{599930000}+\frac{63065813\,i}{5999300}\right)\lambda+\left(\frac{4700270937}{1199860000}+\frac{93185918087\,i}{59993000000}\right)$$

```
Solve[%==0,λ]
```

$$\{\{\lambda \to \left(\frac{160913}{359958} + \frac{22289543\, i}{5999300}\right) + \frac{1}{179979000000}\Big(\Big(\frac{1}{2}\big((14061412569455523149600000000000000 - 77041533825259768430582400000000000\, i) + \sqrt{(3366680112867341769963970493203084194931082016000000000000000000000000 - 17311451207751413351397581093908658401058208000000000000000000000000000\, i)}\big)\Big)^\wedge(1/3)\Big) - \left(\frac{1060843452532464400}{179979} + \frac{3936601731000000\, i}{59993}\right)\Big/\Big(2^{2/3}\big((14061412569455523149600000000000000 - 77041533825259768430582400000000000\, i) + \sqrt{(3366680112867341769963970493203084194931082016000000000000000000000000 - 17311451207751413351397581093908658401058208000000000000000000000000000\, i)}\big)^\wedge(1/3)\Big)\},$$

$$\{\lambda \to \left(\frac{160913}{359958} + \frac{22289543\, i}{5999300}\right) - \frac{1}{359958000000}\Big(\big(1 + i\sqrt{3}\big)\Big(\frac{1}{2}\big((14061412569455523149600000000000000 - 77041533825259768430582400000000000\, i) + \sqrt{(3366680112867341769963970493203084194931082016000000000000000000000000 - 17311451207751413351397581093908658401058208000000000000000000000000000\, i)}\big)\Big)^\wedge(1/3)\Big) + \Big(\Big(\frac{530421726266232200}{179979} + \frac{1968300865500000\, i}{59993}\Big)\big(1 - i\sqrt{3}\big)\Big)\Big/\Big(2^{2/3}\big((14061412569455523149600000000000000 - 77041533825259768430582400000000000\, i) + \sqrt{(3366680112867341769963970493203084194931082016000000000000000000000000 - 17311451207751413351397581093908658401058208000000000000000000000000000\, i)}\big)^\wedge(1/3)\Big)\},$$

$$\{\lambda \to \left(\frac{160913}{359958} + \frac{22289543\, i}{5999300}\right) - \frac{1}{359958000000}\Big(\big(1 - i\sqrt{3}\big)\Big(\frac{1}{2}\big((14061412569455523149600000000000000 - 77041533825259768430582400000000000\, i) + \sqrt{(3366680112867341769963970493203084194931082016000000000000000000000000 - 17311451207751413351397581093908658401058208000000000000000000000000000\, i)}\big)\Big)^\wedge(1/3)\Big) + \Big(\Big(\frac{530421726266232200}{179979} + \frac{1968300865500000\, i}{59993}\Big)\big(1 + i\sqrt{3}\big)\Big)\Big/\Big(2^{2/3}\big((14061412569455523149600000000000000 - 77041533825259768430582400000000000\, i) + \sqrt{(3366680112867341769963970493203084194931082016000000000000000000000000 - 17311451207751413351397581093908658401058208000000000000000000000000000\, i)}\big)^\wedge(1/3)\Big)\}\}$$

```
N[%]
```

$\{\{\lambda \to 0.516142 - 0.149591\, i\}, \{\lambda \to 0.42435 - 0.510491\, i\}, \{\lambda \to 0.400606 + 11.8062\, i\}\}$

Which is the same as

```
N[Eigenvalues[temp2/.λ->0]]
```

$\{0.516142 - 0.149591\, i, 0.42435 - 0.510491\, i, 0.400606 + 11.8062\, i\}$

We can also check:

```
Det[AB]
```

$$\alpha^{12} + \frac{27}{14} i\,\text{rr}\,\alpha^{11} - 3 i\,\text{rr}\,\lambda\,\alpha^{11} - \frac{9\,\text{rr}^2\,\alpha^{10}}{8} - 3\,\text{rr}^2\,\lambda^2\,\alpha^{10} + \frac{27}{7}\text{rr}^2\,\lambda\,\alpha^{10} - 576\,\alpha^{10} - \frac{3}{16} i\,\text{rr}^3\,\alpha^9 + i\,\text{rr}^3\,\lambda^3\,\alpha^9 - \frac{27}{14} i\,\text{rr}^3\,\lambda^2\,\alpha^9 - 912\, i\,\text{rr}\,\alpha^9 + \frac{9}{8} i\,\text{rr}^3\,\lambda\,\alpha^9 +$$
$$1440\, i\,\text{rr}\,\lambda\,\alpha^9 + \frac{2337\,\text{rr}^2\,\alpha^8}{7} + 1152\,\text{rr}^2\,\lambda^2\,\alpha^8 - 1354\,\text{rr}^2\,\lambda\,\alpha^8 + \frac{726912\,\alpha^8}{7} - \frac{267}{28} i\,\text{rr}^3\,\alpha^7 - 288\, i\,\text{rr}^3\,\lambda^3\,\alpha^7 + 442\, i\,\text{rr}^3\,\lambda^2\,\alpha^7 + \frac{1125504}{7} i\,\text{rr}\,\alpha^7 -$$
$$\frac{1044}{7} i\,\text{rr}^3\,\lambda\,\alpha^7 - \frac{1453824}{7} i\,\text{rr}\,\lambda\,\alpha^7 - \frac{404316\,\text{rr}^2\,\alpha^6}{7} - \frac{705792}{7}\text{rr}^2\,\lambda^2\,\alpha^6 + \frac{1098432}{7}\text{rr}^2\,\lambda\,\alpha^6 + \frac{39591936\,\alpha^6}{7} - \frac{852}{7} i\,\text{rr}^3\,\alpha^5 - \frac{21120}{7} i\,\text{rr}^3\,\lambda^3\,\alpha^5 +$$
$$\frac{42816}{7} i\,\text{rr}^3\,\lambda^2\,\alpha^5 + \frac{45035520}{7} i\,\text{rr}\,\alpha^5 - \frac{21720}{7} i\,\text{rr}^3\,\lambda\,\alpha^5 - \frac{59387904}{7} i\,\text{rr}\,\lambda\,\alpha^5 - \frac{9353088\,\text{rr}^2\,\alpha^4}{7} - \frac{19722240}{7}\text{rr}^2\,\lambda^2\,\alpha^4 + \frac{28061952}{7}\text{rr}^2\,\lambda\,\alpha^4 +$$
$$165888000\,\alpha^4 + \frac{144}{7} i\,\text{rr}^3\,\alpha^3 - \frac{36864}{7} i\,\text{rr}^3\,\lambda^3\,\alpha^3 + \frac{34560}{7} i\,\text{rr}^3\,\lambda^2\,\alpha^3 + \frac{949248000}{7} i\,\text{rr}\,\alpha^3 - \frac{6912}{7} i\,\text{rr}^3\,\lambda\,\alpha^3 - 165888000\, i\,\text{rr}\,\lambda\,\alpha^3 - \frac{3350016\,\text{rr}^2\,\alpha^2}{7} -$$
$$6635520\,\text{rr}^2\,\lambda^2\,\alpha^2 + \frac{33914880}{7}\text{rr}^2\,\lambda\,\alpha^2 + 990904320\,\alpha^2 + 194641920\, i\,\text{rr}\,\alpha - 495452160\, i\,\text{rr}\,\lambda\,\alpha + 4246732800$$

```
%39/.{rr->100,α->1}
```

$$-\frac{59993000000\, i\,\lambda^3}{7} - \left(\frac{668686290000}{7} - \frac{80456500000\, i}{7}\right)\lambda^2 + \left(\frac{630658130000}{7} - \frac{498689403900\, i}{7}\right)\lambda - \left(\frac{93185918087}{7} - \frac{235013546850\, i}{7}\right)$$

```
N[%]
```

$-8.57043\times 10^{9}\, i\, \lambda^3 - (9.55266\times 10^{10} - 1.14938\times 10^{10}\, i)\, \lambda^2 + (9.0094\times 10^{10} - 7.12413\times 10^{10}\, i)\, \lambda - (1.33123\times 10^{10} - 3.35734\times 10^{10}\, i)$

```
Solve[%==0,λ]
```

$\{\{\lambda \to 0.400606 + 11.8062\, i\}, \{\lambda \to 0.42435 - 0.510491\, i\}, \{\lambda \to 0.516142 - 0.149591\, i\}\}$

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%38
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$\{0.516142 - 0.149591\, i, 0.42435 - 0.510491\, i, 0.400606 + 11.8062\, i\}$