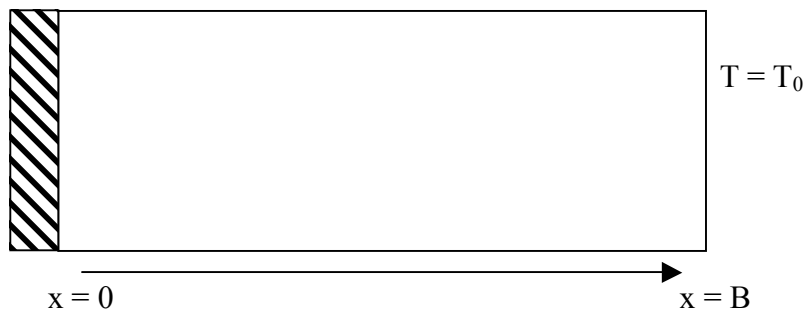


ChEg 356
Spring 2005 Test #1
2/17/05

1. Steady state heat conduction in a slab with heat generation. (40 points)

Consider a slab that has a thickness B in the x direction. The y and z directions are infinite and completely uniform. There is a heat generation rate of S $\text{W}/\text{m}^3\text{-s}$. The wall at $x=0$ is insulated. The wall at $x = B$ is fixed at a temperature $T(x=B) = T_0$. The thermal conductivity is constant with a value of k , $\text{W}/(\text{m K s})$, the density of the slab is ρ (kg/m^3) and the heat capacity is C_p ($\text{W}/\text{K}/\text{kg}$)

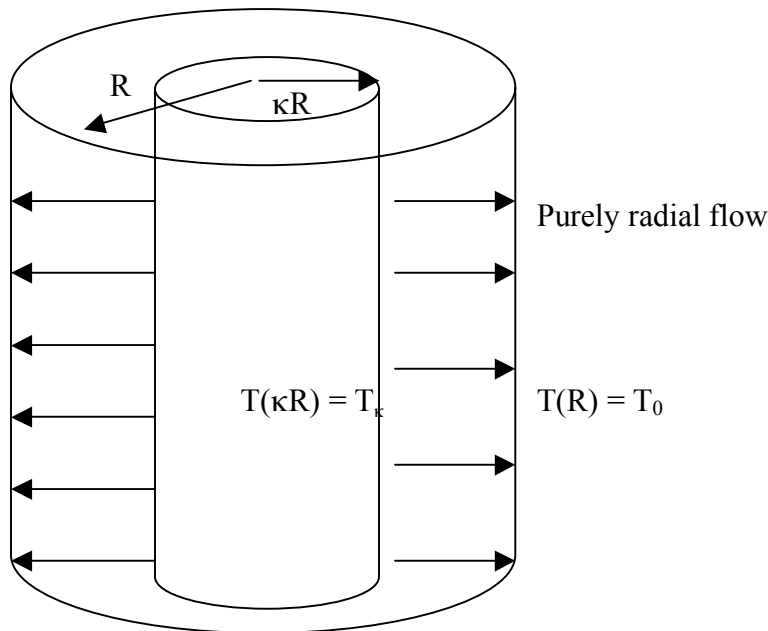


- a. Use a shell balance to derive the differential equation that governs the heat flux.
- b. If Fourier's law is valid for this situation, what is the differential equation for the temperature profile?
- c. Find the temperature profile.
- d. What is the flux at $x = B$?
- e. Now suppose that $S = 0$, what is the temperature profile in the slab?
- f. If the temperature T_0 is changed, (approximately) how long will it be before new steady state is established?

Extra credit (5 points)

- g. Find the differential equation for the temperature profile for the case of finite S (that is again with heat conduction), but this time the thermal conductivity is a function of x , $k = k_0 + k_1 x$.

2. Transpirational cooling in a cylindrical geometry (35 points)

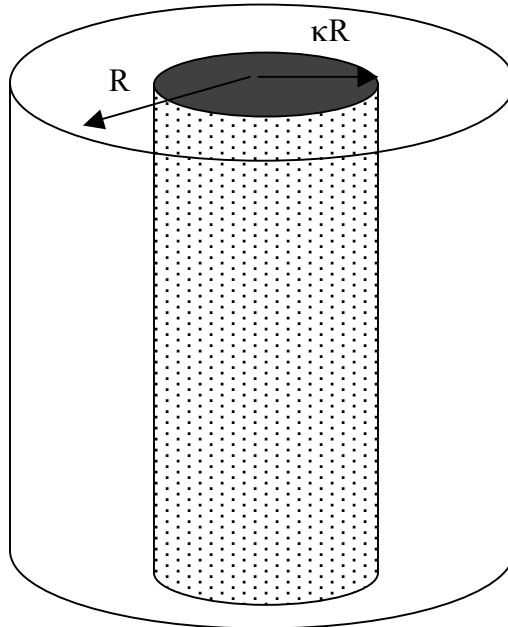


A concentric cylinder device as shown above has a fixed outer temperature $T(R) = T_0$. There is a radial flow outward of fluid, emanating from the inner cylinder, with a strength w with units $\text{kg}/(\text{m}\cdot\text{s})$ (mass/time per unit height of the cylinder). The total flow is constant and goes only in the r direction. The fluid density is ρ , the heat capacity is C_p and the thermal conductivity is k . All of these are constant.

- Find an expression for the velocity field, $v(r)$.
- Use the complete form of the differential energy equation, in terms of temperature, to find an equation that governs this situation. Explain the physical meaning of your terms.
- Solve the ODE for the temperature profile. You may wish to recall that $b \ln(x) = \ln(x^b)$ and that ODE's like this one can be readily solved using a simple substitution variable.
- Find an expression for the heat flux at $r = \kappa R$.

3. An insulated wire. (25 points)

Consider the temperature of a wire that has an insulating material around it.



The wire has a radius κR , the outside of the wire has a radius R . There is heat generation inside the wire by electrical dissipation, S . The thermal conductivity of the wire is k_1 , the thermal conductivity of the insulation is k_2 . They have densities ρ_1 and ρ_2 and heat capacities C_{p1} and C_{p2} . At $r=R$ the heat loss to the surroundings is given by a heat transfer coefficient and Newton's law of cooling $q(R) = h (T(R) - T_0)$.

- Write down the governing equations in terms of temperature for this problem.
- Write down all of the necessary boundary conditions
- Without fitting the boundary conditions, find the general solutions to the differential equations in the wire and the insulation.
- If you solved this problem, the inner region (wire) would not have a logarithmic term, but the outer region would. Explain why.
- It is surprising, but true, that under certain circumstances, adding insulation will increase the heat flux from a wire. Explain how this could be happen.

Table 9.8-1 Summary of Notation for Energy Fluxes

Symbol	Meaning	Reference
$(\frac{1}{2}\rho v^2 + \rho \hat{U})\mathbf{v}$	convective energy flux vector	Eq. 9.7-2
\mathbf{q}	molecular heat flux vector	Eq. 9.1-6
$[\boldsymbol{\pi} \cdot \mathbf{v}]$	molecular work flux vector	Eq. 9.8-4
$\mathbf{e} = \mathbf{q} + [\boldsymbol{\pi} \cdot \mathbf{v}] + (\frac{1}{2}\rho v^2 + \rho \hat{U})\mathbf{v}$ $= \mathbf{q} + [\boldsymbol{\tau} \cdot \mathbf{v}] + (\frac{1}{2}\rho v^2 + \rho \hat{H})\mathbf{v}$	combined energy flux vector	Eq. 9.8-5, 6

In Table 9.8-1 we summarize the notation for the various energy flux vectors introduced in this section. All of them have the same sign convention.

To evaluate the enthalpy in Eq. 9.8-6, we make use of the standard equilibrium thermodynamics formula

$$d\hat{H} = \left(\frac{\partial \hat{H}}{\partial T}\right)_p dT + \left(\frac{\partial \hat{H}}{\partial p}\right)_T dp = \hat{C}_p dT + \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] dp \quad (9.8-7)$$

When this is integrated from some reference state p°, T° to the state p, T , we then get¹

$$\hat{H} - \hat{H}^\circ = \int_{T^\circ}^T \hat{C}_p dT + \int_{p^\circ}^p \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] dp \quad (9.8-8)$$

in which \hat{H}° is the enthalpy per unit mass at the reference state. The integral over p is zero for an ideal gas and $(1/\rho)(p - p^\circ)$ for fluids of constant density. The integral over T becomes $\hat{C}_p(T - T^\circ)$ if the heat capacity can be regarded as constant over the relevant temperature range. It is assumed that Eq. 9.8-7 is valid in nonequilibrium systems where p and T are the *local* values of the pressure and temperature.

QUESTIONS FOR DISCUSSION

1. Define and give the dimensions of thermal conductivity k , thermal diffusivity α , heat capacity \hat{C}_p , heat flux \mathbf{q} , and combined energy flux \mathbf{e} . For the dimensions use m = mass, l = length, T = temperature, and t = time.
2. Compare the orders of magnitude of the thermal conductivities of gases, liquids, and solids.
3. In what way are Newton's law of viscosity and Fourier's law of heat conduction similar? Dissimilar?
4. Are gas viscosities and thermal conductivities related? If so, how?
5. Compare the temperature dependence of the thermal conductivities of gases, liquids, and solids.
6. Compare the orders of magnitudes of Prandtl numbers for gases and liquids.
7. Are the thermal conductivities of gaseous Ne^{20} and Ne^{22} the same?
8. Is the relation $\hat{C}_p - \hat{C}_v = R$ true only for ideal gases, or is it also true for liquids? If it is true for liquids, what formula should be used?
9. What is the kinetic energy flux in the axial direction for the laminar Poiseuille flow of a Newtonian liquid in a circular tube?
10. What is $[\boldsymbol{\pi} \cdot \mathbf{v}] = p\mathbf{v} + [\boldsymbol{\tau} \cdot \mathbf{v}]$ for Poiseuille flow?

¹ See, for example, R. J. Silbey and R. A. Alberty, *Physical Chemistry*, Wiley, 3rd edition (2001), §2.11.

§B.1 NEWTON'S LAW OF VISCOSITY (continued)

Cylindrical coordinates (r, θ, z):

$$\tau_{rr} = -\mu \left[2 \frac{\partial v_r}{\partial r} \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-8})^a$$

$$\tau_{\theta\theta} = -\mu \left[2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-9})^a$$

$$\tau_{zz} = -\mu \left[2 \frac{\partial v_z}{\partial z} \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-10})^a$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (\text{B.1-11})$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu \left[\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right] \quad (\text{B.1-12})$$

$$\tau_{zr} = \tau_{rz} = -\mu \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right] \quad (\text{B.1-13})$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad (\text{B.1-14})$$

^a When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity κ is zero.

Spherical coordinates (r, θ, φ):

$$\tau_{rr} = -\mu \left[2 \frac{\partial v_r}{\partial r} \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-15})^a$$

$$\tau_{\theta\theta} = -\mu \left[2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-16})^a$$

$$\tau_{\phi\phi} = -\mu \left[2 \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot \theta}{r} \right) \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-17})^a$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (\text{B.1-18})$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] \quad (\text{B.1-19})$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu \left[\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right] \quad (\text{B.1-20})$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \quad (\text{B.1-21})$$

^a When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity κ is zero.

§B.2 FOURIER'S LAW OF HEAT CONDUCTION^a

$$[\mathbf{q} = -k\nabla T]$$

Cartesian coordinates (x, y, z):

$$q_x = -k \frac{\partial T}{\partial x} \quad (\text{B.2-1})$$

$$q_y = -k \frac{\partial T}{\partial y} \quad (\text{B.2-2})$$

$$q_z = -k \frac{\partial T}{\partial z} \quad (\text{B.2-3})$$

Cylindrical coordinates (r, θ, z):

$$q_r = -k \frac{\partial T}{\partial r} \quad (\text{B.2-4})$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \quad (\text{B.2-5})$$

$$q_z = -k \frac{\partial T}{\partial z} \quad (\text{B.2-6})$$

Spherical coordinates (r, θ, φ):

$$q_r = -k \frac{\partial T}{\partial r} \quad (\text{B.2-7})$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \quad (\text{B.2-8})$$

$$q_\phi = -k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \quad (\text{B.2-9})$$

^a For mixtures, the term $\sum_a (\bar{H}_a / M_a) \mathbf{j}_a$ must be added to \mathbf{q} (see Eq. 19.3-3).

§B.3 FICK'S (FIRST) LAW OF BINARY DIFFUSION^a

$$[\mathbf{j}_A = -\rho \mathcal{D}_{AB} \nabla \omega_A]$$

Cartesian coordinates (x, y, z):

$$j_{Ax} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial x} \quad (\text{B.3-1})$$

$$j_{Ay} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial y} \quad (\text{B.3-2})$$

$$j_{Az} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial z} \quad (\text{B.3-3})$$

Cylindrical coordinates (r, θ, z):

$$j_{Ar} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial r} \quad (\text{B.3-4})$$

$$j_{A\theta} = -\rho \mathcal{D}_{AB} \frac{1}{r} \frac{\partial \omega_A}{\partial \theta} \quad (\text{B.3-5})$$

$$j_{Az} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial z} \quad (\text{B.3-6})$$

Spherical coordinates (r, θ, ϕ):

$$j_{Ar} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial r} \quad (\text{B.3-7})$$

$$j_{A\theta} = -\rho \mathcal{D}_{AB} \frac{1}{r} \frac{\partial \omega_A}{\partial \theta} \quad (\text{B.3-8})$$

$$j_{A\phi} = -\rho \mathcal{D}_{AB} \frac{1}{r \sin \theta} \frac{\partial \omega_A}{\partial \phi} \quad (\text{B.3-9})$$

^a To get the molar fluxes with respect to the molar average velocity, replace \mathbf{j}_A , ρ , and ω_A by \mathbf{J}_A^* , c , and x_A .

§B.4 THE EQUATION OF CONTINUITY^a

$$[\partial \rho / \partial t + (\nabla \cdot \rho \mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) = 0 \quad (\text{B.4-1})$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \quad (\text{B.4-2})$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho v_\phi) = 0 \quad (\text{B.4-3})$$

^a When the fluid is assumed to have constant mass density ρ , the equation simplifies to $(\nabla \cdot \mathbf{v}) = 0$.

Cylin

$$\rho \left(\frac{\partial v_r}{\partial t} \right)$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} \right)$$

$$\rho \left(\frac{\partial v_z}{\partial t} \right)$$

^b These
assum

Spher

$$\rho \left(\frac{\partial v_r}{\partial t} \right)$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} \right)$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} \right)$$

^c These
assum

§B.5 THE EQUATION OF MOTION IN TERMS OF τ

$$[\rho D\mathbf{v}/Dt = -\nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z) :^a

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} - \left[\frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right] + \rho g_x \quad (\text{B.5-1})$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} - \left[\frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right] + \rho g_y \quad (\text{B.5-2})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left[\frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \tau_{zz} \right] + \rho g_z \quad (\text{B.5-3})$$

^a These equations have been written without making the assumption that τ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, τ_{xy} and τ_{yx} may be interchanged.

Cylindrical coordinates (r, θ, z) :^b

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta r} + \frac{\partial}{\partial z} \tau_{zr} - \frac{\tau_{\theta\theta}}{r} \right] + \rho g_r \quad (\text{B.5-4})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta\theta} + \frac{\partial}{\partial z} \tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} \right] + \rho g_\theta \quad (\text{B.5-5})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta z} + \frac{\partial}{\partial z} \tau_{zz} \right] + \rho g_z \quad (\text{B.5-6})$$

^b These equations have been written without making the assumption that τ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{r\theta} - \tau_{\theta r} = 0$.

Spherical coordinates (r, θ, ϕ) :^c

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi r} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] + \rho g_r \quad (\text{B.5-7})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left[\frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi\theta} + \frac{(\tau_{\theta r} - \tau_{r\theta}) - \tau_{\phi\phi} \cot \theta}{r} \right] + \rho g_\theta \quad (\text{B.5-8})$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} - \left[\frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\phi} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi\phi} + \frac{(\tau_{\phi r} - \tau_{r\phi}) + \tau_{\phi\theta} \cot \theta}{r} \right] + \rho g_\phi \quad (\text{B.5-9})$$

^c These equations have been written without making the assumption that τ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{r\theta} - \tau_{\theta r} = 0$.

§B.6 EQUATION OF MOTION FOR A NEWTONIAN FLUID WITH CONSTANT ρ AND μ

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x \quad (\text{B.6-1})$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y \quad (\text{B.6-2})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \quad (\text{B.6-3})$$

Cylindrical coordinates (r, θ, z):

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] + \rho g_r \quad (\text{B.6-4})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta \quad (\text{B.6-5})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \quad (\text{B.6-6})$$

Spherical coordinates (r, θ, ϕ):

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r \quad (\text{B.6-7})^a$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2} \frac{\partial v_\phi}{\partial \phi} \right] + \rho g_\theta \quad (\text{B.6-8})$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{2 \cot \theta}{r^2} \frac{\partial v_\theta}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] + \rho g_\phi \quad (\text{B.6-9})$$

^a The quantity in the brackets in Eq. B.6-7 is not what one would expect from Eq. (M) for $[\nabla \cdot \nabla \mathbf{v}]$ in Table A.7-3, because we have added to Eq. (M) the expression for $(2/r)(\nabla \cdot \mathbf{v})$, which is zero for fluids with constant ρ . This gives a much simpler equation.

§B.7

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Φ_v =

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§B.8

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^a The system

§B.7 THE DISSIPATION FUNCTION Φ_v FOR NEWTONIAN FLUIDS (SEE EQ. 3.3-3)

Cartesian coordinates (x, y, z) :

$$\Phi_v = 2 \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right]^2 - \frac{2}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]^2 \quad (\text{B.7-1})$$

Cylindrical coordinates (r, θ, z) :

$$\Phi_v = 2 \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \left[\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right]^2 - \frac{2}{3} \left[\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \right]^2 \quad (\text{B.7-2})$$

Spherical coordinates (r, θ, ϕ) :

$$\Phi_v = 2 \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot \theta}{r} \right)^2 \right] + \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right]^2 + \left[\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right]^2 - \frac{2}{3} \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right]^2 \quad (\text{B.7-3})$$

§B.8 THE EQUATION OF ENERGY IN TERMS OF q

$$[\rho \hat{C}_p DT/Dt = -(\nabla \cdot \mathbf{q}) - (\partial \ln \rho / \partial \ln T)_p Dp/Dt - (\boldsymbol{\tau} : \nabla \mathbf{v})]$$

Cartesian coordinates (x, y, z) :

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (\text{B.8-1})^a$$

Cylindrical coordinates (r, θ, z) :

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{1}{r} \frac{\partial}{\partial r} (rq_r) + \frac{1}{r} \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (\text{B.8-2})^a$$

Spherical coordinates (r, θ, ϕ) :

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial q_\phi}{\partial \phi} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (\text{B.8-3})^a$$

^a The viscous dissipation term, $-(\boldsymbol{\tau} : \nabla \mathbf{v})$, is given in Appendix A, Tables A.7-1, 2, 3. This term may usually be neglected, except for systems with very large velocity gradients. The term containing $(\partial \ln \rho / \partial \ln T)_p$ is zero for fluid with constant ρ .

§B.9 THE EQUATION OF ENERGY FOR PURE NEWTONIAN FLUIDS WITH CONSTANT^a ρ AND k

$$[\rho \hat{C}_p DT/Dt = k \nabla^2 T + \mu \Phi_v]$$

Cartesian coordinates (x, y, z):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \Phi_v \quad (\text{B.9-1})^b$$

Cylindrical coordinates (r, θ, z):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \Phi_v \quad (\text{B.9-2})^b$$

Spherical coordinates (r, θ, ϕ):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right] + \mu \Phi_v \quad (\text{B.9-3})^b$$

^a This form of the energy equation is also valid under the less stringent assumptions $k = \text{constant}$ and $(\partial \ln \rho / \partial \ln T)_p D\rho/Dt = 0$. The assumption $\rho = \text{constant}$ is given in the table heading because it is the assumption more often made.

^b The function Φ_v is given in §B.7. The term $\mu \Phi_v$ is usually negligible, except in systems with large velocity gradients.

§B.10 THE EQUATION OF CONTINUITY FOR SPECIES α IN TERMS^a OF j_α

$$[\rho D\omega_\alpha/Dt = -(\nabla \cdot \mathbf{j}_\alpha) + r_\alpha]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_x \frac{\partial \omega_\alpha}{\partial x} + v_y \frac{\partial \omega_\alpha}{\partial y} + v_z \frac{\partial \omega_\alpha}{\partial z} \right) = - \left[\frac{\partial j_{\alpha x}}{\partial x} + \frac{\partial j_{\alpha y}}{\partial y} + \frac{\partial j_{\alpha z}}{\partial z} \right] + r_\alpha \quad (\text{B.10-1})$$

Cylindrical coordinates (r, θ, z):

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_r \frac{\partial \omega_\alpha}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_\alpha}{\partial \theta} + v_z \frac{\partial \omega_\alpha}{\partial z} \right) = - \left[\frac{1}{r} \frac{\partial}{\partial r} (r j_{\alpha r}) + \frac{1}{r} \frac{\partial j_{\alpha \theta}}{\partial \theta} + \frac{\partial j_{\alpha z}}{\partial z} \right] + r_\alpha \quad (\text{B.10-2})$$

Spherical coordinates (r, θ, ϕ):

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_r \frac{\partial \omega_\alpha}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_\alpha}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial \omega_\alpha}{\partial \phi} \right) = \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 j_{\alpha r}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (j_{\alpha \theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial j_{\alpha \phi}}{\partial \phi} \right] + r_\alpha \quad (\text{B.10-3})$$

^a To obtain the corresponding equations in terms of \mathbf{J}_α^* make the following replacements:

Replace	ρ	ω_α	j_α	\mathbf{v}	r_α
by	c	x_α	\mathbf{J}_α^*	\mathbf{v}^*	$R_\alpha - x_\alpha \sum_{\beta=1}^N R_\beta$

**§B.11 THE EQUATION OF CONTINUITY FOR SPECIES A
IN TERMS OF ω_A FOR CONSTANT^a ρ_{AB}**

$$[\rho D\omega_A/Dt = \rho_{AB} \nabla^2 \omega_A + r_A]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial \omega_A}{\partial t} + v_x \frac{\partial \omega_A}{\partial x} + v_y \frac{\partial \omega_A}{\partial y} + v_z \frac{\partial \omega_A}{\partial z} \right) = \rho_{AB} \left[\frac{\partial^2 \omega_A}{\partial x^2} + \frac{\partial^2 \omega_A}{\partial y^2} + \frac{\partial^2 \omega_A}{\partial z^2} \right] + r_A \quad (\text{B.11-1})$$

Cylindrical coordinates (r, θ, z):

$$\rho \left(\frac{\partial \omega_A}{\partial t} + v_r \frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_A}{\partial \theta} + v_z \frac{\partial \omega_A}{\partial z} \right) = \rho_{AB} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega_A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \omega_A}{\partial \theta^2} + \frac{\partial^2 \omega_A}{\partial z^2} \right] + r_A \quad (\text{B.11-2})$$

Spherical coordinates (r, θ, ϕ):

$$\rho \left(\frac{\partial \omega_A}{\partial t} + v_r \frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_A}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial \omega_A}{\partial \phi} \right) = \rho_{AB} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \omega_A}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \omega_A}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \omega_A}{\partial \phi^2} \right] + r_A \quad (\text{B.11-3})$$

^a To obtain the corresponding equations in terms of x_A , make the following replacements:

Replace	ρ	ω_a	\mathbf{v}	r_a
by	c	x_a	\mathbf{v}^*	$R_a - x_a \sum_{\beta=1}^N R_\beta$