11A.3 Transpiration cooling.

a. In the absence of transpiration, Eq. 11.4-1 is indeterminate, but its limiting form is obtainable by expressing the exponential functions as first-order Taylor expansions in w_r (and thus in R_0):

$$\frac{T - T_1}{T_\kappa - T_1} = \frac{1/r - 1/R}{1/\kappa R - 1/R}$$

This profile, dcsignated as Θ_0, is tabulated here for the present geometry:

r, microns	100	200	300	400	500
Θ_0	1.000	0.375	0.1666...	0.0625	0

In the presence of transpiration with the given rate $w_r = 1 \times 10^{-5}$ g/s, the constant R_0 in Eq. 11.4-27 is

$$R_0 = \frac{(1 \times 10^{-5}\ \text{g/s})(0.25\ \text{cal/g·C})}{(4\pi)(6.13 \times 10^{-5}\ \text{cal/cm·s·C})}$$
$$= 0.003245\ \text{cm} = 32.45\ \text{microns}$$

Equation 11.14-27 then gives, with r in microns,

$$\frac{T - T_1}{T_\kappa - T_1} = \frac{(\exp(-32.45/r) - \exp(-32.45/500))}{((\exp(-32.45/200) - \exp(-32.45/500))}$$

A table of this function, here called Θ_w, follows:

r, microns	100	200	300	400	500
Θ_w	1.000	0.406	0.185	0.070	0.000

c. The ratio of the heat conduction to the inner surface $r = \kappa R$ with the latter transpiration rate to that with $w_r = 0$ is, from Eq. 11.4-32,

$$\frac{Q}{Q_0} = \frac{\phi}{\exp\phi - 1}$$
$$= \frac{(R_0(1-\kappa)/\kappa R)}{\exp(R_0(1-\kappa)/\kappa R) - 1}$$
$$= \frac{(32.45)(0.8)/100)}{\exp(32.45)(0.8)/100) - 1}$$
$$= \frac{0.2596}{\exp(0.2596) - 1} = 0.876$$

Thus, this small rate of transpiration reduces the rate of heat conduction to the inner surface by 12.4 percent.

11A.6 Adiabatic frictionless compression of an ideal gas.

The states encountered in such a compression satisfy Eq. 11.4-57,

$$p\rho^{-\gamma} = C_1$$

as well as

$$\frac{p}{\rho T} = R/M$$

Combining these relations, we get

$$\rho T\rho^{-\gamma} = T\rho^{1-\gamma} = C_3$$

Hence, the initial and final states in such a compression satisfy

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

and the final temperature in the case considered here is

$$T_2 = (460 + 100)(10)^{1.4-1} = 1407°\text{R} = 947°\text{F}$$

11B.5 Axial heat conduction in a wire

a. This problem involves purely axial flow of heat (by conduction and convection) so that the energy equation is

$$\rho\hat{C}_p v_z \frac{dT}{dz} = k\frac{d^2T}{dz^2} \quad \text{or} \quad -\frac{\rho\hat{C}_p v}{k}\frac{dT}{dz} = \frac{d^2T}{dz^2} \quad \text{or} \quad -A\frac{dT}{dz} = \frac{d^2T}{dz^2}$$

in which $v_z = -v$, and $A \equiv \rho\hat{C}_p v/k$.

Integration of the differential equation gives

$$-AT = \frac{dT}{dz} + C_1$$

At $z = \infty$, we know that $T = T_\infty$ and $dT/dz = 0$; hence $C_1 = -AT_\infty$. Hence the first-order differential equation becomes

$$\frac{d\Theta}{dz} = -A\Theta \qquad \text{where} \qquad \Theta \equiv (T - T_\infty)/(T_0 - T_\infty)$$

in which (since $\Theta(0) = 1$)

$$\ln\Theta = -Az + \ln C_2 \qquad \text{or} \qquad \Theta \equiv \frac{T - T_\infty}{T_0 - T_\infty} = e^{-Az}$$

This is just Eq. 11B.5-1.

b. For temperature-dependent physical properties we have the following energy equation:

$$-\rho\hat{C}_{p\infty}L(\Theta)v\frac{d\Theta}{dz} = \frac{d}{dz}\left(k_\infty K(\Theta)\frac{d\Theta}{dz}\right) \quad \text{or} \quad -A_\infty L(\Theta)\frac{d\Theta}{dz} = \frac{d}{dz}\left(K(\Theta)\frac{d\Theta}{dz}\right)$$

in which $A_\infty = \rho\hat{C}_{p\infty}v/k_\infty$. The first integration gives

$$+A_\infty\int_z^\infty L(\Theta)\frac{d\Theta}{dz}dz = K(\Theta)\frac{d\Theta}{dz} + C_1$$

We now use the boundary conditions that at $z=\infty$, $\Theta=0$ and $d\Theta/dz=0$, to find that $C_1=0$. The above result may then be written as

$$+A_\infty\int_\Theta^0 L(\overline{\Theta})d\overline{\Theta}=K(\Theta)\frac{d\Theta}{dz} \quad \text{or} \quad -A_\infty\int_0^\Theta L(\overline{\Theta})d\overline{\Theta}=K(\Theta)\frac{d\Theta}{dz}$$

This equation may be integrated to give

$$-A_\infty\int_0^z dz=\int_1^\Theta \frac{K(\overline{\Theta})d\overline{\Theta}}{\int_0^{\overline{\Theta}} L(\overline{\overline{\Theta}})d\overline{\overline{\Theta}}}$$

This result simplifies to that in (*a*) when K and L are both equal to unity. Furthermore, it satisfies the boundary conditions at $z=0$ and $z=\infty$.

c. To show that the last equation in (*b*) satisfies the differential equation (the first equation in (*b*)), we differentiate both sides with respect to z (on the right side, we differentiate with repsect to Θ, using the Leibniz formula, and then multiply by $d\Theta/dz$):

$$-A_\infty=\frac{K(\Theta)}{\int_0^\Theta L(\overline{\overline{\Theta}})d\overline{\overline{\Theta}}}\cdot\frac{d\Theta}{dz} \quad \text{or} \quad -A_\infty\int_0^\Theta L(\overline{\overline{\Theta}})d\overline{\overline{\Theta}}=K(\Theta)\frac{d\Theta}{dz}$$

A second differentiation with respect to z (once again using the Leibniz formula) gives

$$-A_\infty L(\Theta)\frac{d\Theta}{dz}=\frac{d}{dz}\left(K(\Theta)\frac{d\Theta}{dz}\right)$$

and this is the differential equation with which we started.

11B.10 Freezing of a spherical drop

a. The heat conduction equation for the solid phase is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0 \qquad (R_f \le r \le R)$$

Two integrations lead to $T=-(C_1/r)+C_2$. The constants of integration are determined from the boundary conditions

$$\text{B. C. 1: At } r=R_f,\ T=T_0; \qquad \text{B. C. 2: At } r=R,\ -k\frac{dT}{dr}=h(T-T_\infty)$$

This leads to the following expressions:

$$C_1=\frac{T_0-T_\infty}{(1/R)-(1/R_f)-(k/hR^2)};\ C_2=T_0+\frac{T_0-T_\infty}{\left[(1/R)-(1/R_f)-(k/hR^2)\right]R_f}$$

The total heat flow across the spherical surface at $r=R$ is then

$$Q=4\pi R^2\left(-k\frac{dT}{dr}\right)\bigg|_{r=R}=4\pi R^2\left(-\frac{k}{R^2}\right)\frac{T_0-T_\infty}{\left[(1/R)-(1/R_f)-(k/hR^2)\right]}$$

This can be rearranged to give the solution in the text.

b. We now have to equate the heat liberated on freezing at $r=R_f$ to the heat flowing out across the surface at $r=R$:

$$-(\rho\Delta\hat{H}_f)(4\pi R_f^2)\frac{dR_f}{dt}=\frac{h\cdot 4\pi R^2\cdot(T_0-T_\infty)}{1-(hR/k)+(hR^2/kR_f)}$$

Integration then yields

$$-(\rho\Delta\hat{H}_f)\int_R^0\left[1-(hR/k)+(hR^2/kR_f)\right]R_f^2dR_f=hR^2(T_0-T_\infty)\int_0^{t_f}dt$$

where t_f is the time for the freezing of the entire droplet. Evaluation of the integrals then leads to the expression in the text.

11B.11 Temperature rise in a catalyst pellet

a. We make an energy balance over a spherical shell of thickness Δr:

$$4\pi r^2 q_r\big|_r - 4\pi(r+\Delta r)^2 q_r\big|_{r+\Delta r} + 4\pi r^2 \Delta r S_c = 0$$

Then division by $4\pi\Delta r$ gives

$$\frac{(r^2 q_r)\big|_{r+\Delta r} - (r^2 q_r)\big|_r}{\Delta r} - r^2 S_c = 0$$

When the limit is taken that $\Delta r \to 0$ and use is made of the definition of the first derivative, we get

$$\frac{d}{dr}\left(r^2 q_r\right) - r^2 S_c = 0$$

Insertion of Fourier's law then gives

$$\frac{d}{dr}\left(r^2 k \frac{dT}{dr}\right) + r^2 S_c = 0 \quad (**) \quad \text{or} \quad k\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + r^2 S_c = 0$$

for the appropriate equation describing the heat conduction with heat generation by chemical reaction and constant k.

b. From Eq. B.9=3, with the time=derivative term set equal to zero, and all velocities set equal to zero, and all derivatives other than r derivatives set equal to zero gives the heat conduction equation in spherical coordinates for a system with no chemical reaction. Therefore, we have to add a term describing the heat production per unit volume:

$$k\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + S_c = 0$$

which is the same as the result obtained in (*a*).

c. The above differential equation may be integrated in a sequence of steps as follows:

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=-\frac{S_c r^2}{k}; \qquad \left(r^2\frac{dT}{dr}\right)=-\frac{S_c r^3}{3k}+C_1; \qquad \frac{dT}{dr}=-\frac{S_c r}{3k}+\frac{C_1}{r^2}$$

$$T=-\frac{S_c r^2}{6k}-\frac{C_1}{r}+C_2 \qquad (***)$$

The constant C_1 must be zero, because neither the temperature nor its gradient are expected to be infinite. The heat loss to the surroundings provides the second boundary condition needed for getting C_2:

Definition of heat transfer coefficient: $q_r\big|_{r=R}=h\left(T_R-T_g\right)$

From Fourier's law: $q_r\big|_{r=R}=-k\frac{dT}{dr}\bigg|_{r=R}=+\frac{S_c R}{3}$

Equating these expressions: $h\left(T_R-T_g\right)=\frac{S_c R}{3}$

Inserting T_R from (***): $h\left(-\frac{S_c R^2}{6k}+C_2-T_g\right)=\frac{S_c R}{3}$

Solving for C_2: $C_2=\frac{S_c R^2}{6k}+\frac{S_c R}{3h}+T_g$

Thus we finally get the temperature profile within the catalyst pellet:

$$T-T_g=\frac{S_c R^2}{6k}\left[1-\left(\frac{r}{R}\right)^2\right]+\frac{S_c R}{3h}$$

d..When the heat transfer coefficient goes to infinity, the last term in the temperature distribution drops out.

e. The maximum temperature in the system is

$$T_{\max}-T_g=\frac{S_c R^2}{6k}+\frac{S_c R}{3h}=\frac{S_c R^2}{6k}\left(1+\frac{2k}{Rh}\right)$$

f. In Eq. (**) one would have to leave k inside the differential operator, and insert the specific r dependence of both k and S_c .