10A.1 Heat loss from an insulated pipe.

We use the notation of Fig. 10.6-2. When the temperatures at the inner and outer surfaces are known, Eq. 10.6-29 can be reduced to

$$\frac{Q_0}{L} = \frac{2\pi L(T_0 - T_3)}{\left[\frac{\ln(r_1/r_0)}{k_{01}} + \frac{\ln(r_2/r_1)}{k_{12}} + \frac{\ln(r_3/r_2)}{k_{23}}\right]}$$

The r_i for this problem are:

$$\begin{aligned} r_0 &= 2.067/2 = 1.0335 \text{ in} \\ r_1 &= 1.0335 + 0.154 = 1.19 \text{ in} \\ r_2 &= 1.19 + 2 = 3.19 \\ r_3 &= 3.19 + 2 = 5.19 \end{aligned}$$

Insertion of numerical values into the above formula gives:

$$\begin{aligned} \frac{Q_0}{L} &= \frac{2\pi(250 - 90 \text{ F})}{\left[\frac{\ln(1.19/1.0335)}{26.1} + \frac{\ln(3.19/1.19)}{0.04} + \frac{\ln(5.19/3.19)}{0.03} \text{ F}\cdot\text{hr}\cdot\text{ft/Btu}\right]} \\ &= \frac{320\pi}{0.0054 + 24.7 + 16.2} = 24 \text{ Btu/hr per foot of pipe} \end{aligned}$$

10A.7 Viscous heating in a ball-point pen.

The paralel-plate approximation in §10.4 is used here to estimate the viscous heating of the fluid; more accurate results will be presented in Chapter 11. Multiplication of Eq. 10.4-9 by $(T_b - T_0)$, and setting $T_b = T_0$, gives the temperature profile in the ink,

$$T - T_0 = \frac{1}{2}\frac{\mu v_b^2}{k}(x/b)[1 - (x/b)]$$

valid when both adjoining surfaces are at temperature T_0.

At $x = b/2$ the temperature rise attains its maximum value

$$(T - T_0)_{\max} = \frac{1}{8}\frac{\mu v_b^2}{k}$$

Insertion of the data for this problem gives

$$\begin{aligned}(T - T_0)_{\max} &= \frac{1}{8}\frac{(10^4 \times 0.01\mathrm{g/cm{\cdot}s})(100 \times 2.54/60\ \mathrm{cm/s})^2}{(5 \times 10^{-4} \times 4.1840 \times 10^7\ \mathrm{g{\cdot}cm/s^3{\cdot}K})} \\ &= 0.011\ \mathrm{K}\end{aligned}$$

as the maximum dissipative temperature rise in the ink. Thus, the warming of the ink by viscous dissipation will be negligible compared with the warming of the pen by contact with the hand of the user.

10B.1 Heat conduction from a sphere to a stagnant fluid

a. The heat is being conducted in the r direction only. Therefore, we select a shell of thickness Δr over which we make the energy balance:

$$\Delta r q_r 4\pi r^2 q_r\big|_r - 4\pi(r+\Delta r)^2 q_r\big|_{r+\Delta r} = 0 \text{ or } 4\pi\left(r^2 q_r\right)\big|_r - 4\pi\left(r^2 q_r\right)\big|_{r+\Delta r} = 0$$

We now divide by Δr and then take the limit as Δr goes to zero

$$\lim_{\Delta r \to 0} \frac{\left(r^2 q_r\right)\big|_{r+\Delta r} - \left(r^2 q_r\right)\big|_r}{\Delta r} = 0$$

We then use the definition of the first derivative to get

$$\frac{d}{dr}\left(r^2 q_r\right) = 0 \qquad \text{and} \qquad \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$$

In the second equation we have inserted Fourier's law of heat conduction with constant thermal conductivity.

b. Integration of this equation twice with respect to r gives

$$r^2 \frac{dT}{dr} = C_1 \text{ and } T = -\frac{C_1}{r} + C_2$$

The boundary conditions then gives $C_1 = -R(T_R - T_\infty)$, $C_2 = T_\infty$, and

$$\frac{T - T_\infty}{T_R - T_\infty} = \frac{R}{r}$$

c. The heat flux at the surface is

$$q_r\big|_{r=R} = -k\frac{dT}{dr}\bigg|_{r=R} = +kR(T_R - T_\infty)\frac{1}{r^2}\bigg|_{r=R} = \frac{k(T_R - T_\infty)}{R} \equiv h(T_R - T_\infty)$$

so that $h = k/R = 2k/D$ and Nu $= 2$.

d. Bi contains k of the solid; Nu contains k of the fluid.

10B.6 Insulation thickness for a furnace wall

Let the regions be labeled as follows:

Refractory brick	"01"
Insulating brick	"12"
Steel	"23"

and we may used the formulas given in Eqs. 10.6-8, 9, and 10.

The minimum wall thickness will occur when $T_1 = 2000°\text{F}$. If for the sake of being on the safe side, let $T_0 = 2500°\text{F}$. Then for the region "01" the thickness must be

$$x_1 - x_0 = \frac{k_{01}(T_0 - T_1)}{q_0} = \frac{\frac{1}{2}(4.1+3.6)(2500-2000)}{5000} = 0.39\text{ft}$$

Here we have taken the thermal conductivity of the refractory brick to be the arithmetic average of the values the thermal conductivity at 2000°F and 2500°F (the latter estimated by linear extrapolation from the given data).

For the remaining two regions, we may add Eqs. 10.6-9 and 10 t get

$$T_1 - T_3 = q_0\left(\frac{x_2 - x_1}{k_{12}} + \frac{x_3 - x_2}{k_{23}}\right)$$

or, taking the steel temperature to be 100,

$$2000 - 100 = 5000\left(\frac{x_2 - x_1}{\frac{1}{2}(0.9+1.8)} + \frac{(0.25)\frac{1}{12}}{26.1}\right)$$

This gives $x_2 - x_1 = 0.51\text{ft}$.

10B.7 Forced-convection heat transfer in flow beween parallel plates

a. Since the temperature depends on both x and z, we make an energy balance over a region of volume $W\Delta x\Delta z$, in which W is the dimension of the slit in the y direction. The various contributions to the energy balance are:

Total energy in at x: $\quad e_x|_x W\Delta z$

Total energy out at $x+\Delta x$: $\quad e_x|_{x+\Delta x} W\Delta z$

Total energy in at z: $\quad e_z|_z W\Delta x$

Total energy out at $z+\Delta z$: $\quad e_z|_{z+\Delta z} W\Delta x$

Work done on fluid by gravity: $\quad \rho v_z g_z W\Delta x\Delta z$

When these terms are added together and divided by $W\Delta x\Delta z$, we get

$$-\frac{\partial e_x}{\partial x}-\frac{\partial e_z}{\partial z}-\rho v_z g=0$$

since gravity is acting in the -z direction.

Now we use Eqs. 9.8-6 and 9.8-8 to write out the x and z components of the combined energy flux:

$$e_x=\tau_{xz}v_z+q_x=-\left(\mu\frac{\partial v_z}{\partial x}\right)v_z-k\frac{\partial T}{\partial x}$$

$$\begin{aligned}e_z&=\left(\tfrac{1}{2}\rho v_z^2\right)v_z+\rho\hat{H}v_z+\tau_{zz}v_z+q_z\\&=\left(\tfrac{1}{2}\rho v_z^2\right)v_z+\left(p-p^{\circ}\right)v_z+\rho\hat{C}_p\left(T-T^{\circ}\right)v_z-\left(2\mu\frac{\partial v_z}{\partial z}\right)v_z-k\frac{\partial T}{\partial z}\end{aligned}$$

Substituting these expressions into the energy balance, and making use of the fact that v_z depends only on x gives

$$\rho\hat{C}_p v_z \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\left(\frac{\partial v_z}{\partial x}\right)^2 + \left(-\frac{\partial p}{\partial z} + \mu\frac{\partial^2 v_z}{\partial x^2} - \rho g\right)$$

The term in the last parentheses is zero by the equation of motion, the term just before that is the viscous heating (which we neglect), and in the first parentheses we neglect the heat conduction in the z-direction.

b. Hence we get

$$\rho\hat{C}_p v_{z,\max}\left[1-\left(\frac{x}{B}\right)^2\right]\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial x^2} \qquad \text{or} \qquad \left(1-\sigma^2\right)\frac{\partial \Theta}{\partial \zeta} = \frac{\partial^2 \Theta}{\partial \sigma^2}$$

with the boundary and initial conditions: at $\sigma = \pm 1$, $\pm(\partial\Theta/\partial\sigma) = 1$, and at $\zeta = 0$, $\Theta = 0$.

c. For large *z* we propose the solution $\Theta(\sigma,\zeta) = C_0\zeta + \Psi(\sigma)$. Then $\Psi(\sigma)$ has to satisfy the ordinary differential equation

$$\frac{\partial^2 \Psi}{\partial \sigma^2} = C_0\left(1-\sigma^2\right)$$

which is easily integrated. The expression for $\Theta(\sigma,\zeta)$ is then

$$\Theta(\sigma,\zeta) = C_0\zeta + C_0\left(\tfrac{1}{2}\sigma^2 - \tfrac{1}{12}\sigma^4\right) + C_1\sigma + C_2$$

Application of the boundary conditions at $\sigma = \pm 1$ gives $C_1 = 0$ and $C_0 = \frac{3}{2}$. The remaining constant has to be obtained from an integral condition:

$$\zeta = \int_0^1 \Theta(\sigma,\zeta)\left(1-\sigma^2\right)d\sigma$$

This gives $C_2 = -\frac{39}{280}$. Combining these results we get

$$\Theta(\sigma,\zeta) = \tfrac{3}{2}\zeta + \tfrac{3}{2}\left(\tfrac{1}{2}\sigma^2 - \tfrac{1}{12}\sigma^4\right) - \tfrac{39}{280}$$

which is in accordance with Eq. 10B.7-4.

10B.15 Radial temperature gradients in an annular chemical reactor

a. Consider a cylindrical shell of thickness Δr and length L. We make an energy balance over this shell, by paralleling the derivation in Eqs. 10.2-2 to 6, replacing the electrical heat source by the chemical heat source S_c. Hence we have (cf. Eq. 10.2-6):

$$\frac{d}{dr}(rq_r) = S_c r$$

Into this we substitute Fourier's law for heat conduction in the r-direction to get

$$\frac{d}{dr}\left(r\left(-k_{\text{eff}}\frac{dT}{dr}\right)\right) = S_c r \qquad \text{or} \qquad k_{\text{eff}}\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -S_c$$

provided that the effective thermal conductivity does not vary with position. The boundary conditions are:

B. C. 1: at $r = r_0$, $T = T_0$

B. C. 2: at $r = r_0$, $\dfrac{dT}{dr} = 0$

b. A natural choice for the dimensionless radial coordinate involves division of *r* by either the inner or outer radius; we choose the inner radius and write $\xi = r/r_0$. Then the differential equation becomes:

$$\frac{k_{\text{eff}}}{r_0^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dT}{d\xi}\right) = -S_c \qquad \text{or} \qquad \frac{k_{\text{eff}}}{r_0^2}\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d(T-T_0)}{d\xi}\right) = -S_c$$

From this, it is evident that $k_{\text{eff}}(T-T_0)/S_c r_0^2$ is dimensionless. By inserting a factor of 4, we get $\Theta = 4k_{\text{eff}}(T-T_0)/S_c r_0^2$. The insertion of the factor of 4 is arbitrary, but it makes the final dimensionless answer somewhat simpler. In terms of these dimensionless quantities the partial differential equation becomes:

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\Theta}{d\xi}\right)=-4$$

with boundary conditions

B. C. 1: at $\xi = 1,\quad \Theta = 0$
B. C. 2: at $\xi = 1,\quad d\Theta/d\xi = 0$

c. Integration twice leads to

$$\Theta = -\xi^2 + C_1 \ln\xi + C_2$$

Application of the boundary conditions gives $C_1 = 2$ and $C_2 = 1$, so that

$$\Theta = 1 - \xi^2 + 2\ln\xi$$

d. The dimensionless temperature at the outer wall is then

$$\Theta(a) = 1 - a^2 + 2\ln a \qquad (a = r_1/r_0)$$

The volume averaged reduced temperature is:

$$\langle\Theta\rangle = \frac{\int_0^{2\pi}\int_1^a\left(1-\xi^2+2\ln\xi\right)\xi d\xi d\theta}{\int_0^{2\pi}\int_1^a \xi d\xi d\theta} = \frac{\left[\frac{1}{2}\xi^2 - \frac{1}{4}\xi^4 + \xi^2\ln\xi - \frac{1}{2}\xi^2\right]_1^a}{\left[\frac{1}{2}\xi^2\right]_1^a}$$

$$= = -\tfrac{1}{2}\left(a^2+1\right) + 2\frac{a^2\ln a}{a^2-1}$$

e. The temperature at the outer wall is

$$T = T_0 + \frac{S_c r_0^2}{4k_{\text{eff}}}\left(1 - a^2 + 2\ln a\right)$$

$$= 900 + \frac{\left(4800\ \frac{\text{cal}}{\text{hr}\cdot\text{cm}^3}\right)\left(3.97\times10^{-3}\ \frac{\text{Btu}}{\text{cal}}\right)\left(2.54\times12\ \frac{\text{cm}}{\text{ft}}\right)\left(\frac{0.45}{12}\right)^2}{4(0.3)}$$

$$\times\left(1-(1.11)^2+2\ln(1.11)\right)$$
$$=900+(0.681)(1-1.23+0.21)$$
$$=900+(63.3)(1-1.23+0.21)=899°\text{F}$$

f. If the inner and outer radii were doubled, the temperature difference between the walls would be four times as great.