

# Delay-optimal Power Control Policies

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## Abstract

The delay till success (DTS) is the mean number of transmissions needed, averaged over the fading, until a single packet is successfully received (decoded) over a wireless link. This paper shows that under a mean and a peak power constraint, random power control can significantly reduce the DTS. We derive the optimal power control policies that minimize DTS at one link of given length. For most commonly used fading distributions, these optimal power control policies are random on-off policies, whose parameters depend on the fading statistics and the link distance. We present two applications of this result in the context of noise-limited wireless networks: minimizing the local delay (mean delay for successful nearest-neighbor communication) and minimizing the local anycast delay (mean delay for a transmission to any node).

## I. INTRODUCTION

### A. Motivation and Main Contribution

Consider a fading wireless link where the transmitter keeps sending the same packet. The *delay till success* (DTS) is the mean number of transmissions needed, averaged over the fading, until this packet is successfully received (decoded) at the receiver. Assuming independent and identically distributed (iid) block fading and that the transmitter is allowed to vary only the transmit power, the DTS is a function of the fading statistics and the power control policy. The DTS can be interpreted as the service time of the head-of-line packet, if the transmit buffer and the link are viewed as a queueing system.

This paper shows that under a mean and a peak transmit power constraint, randomly varying the transmit power can significantly reduce the DTS. In particular, we derive the optimal (DTS-minimizing) power control policies for different fading statistics. It turns out that for almost all popular fading distributions (Rayleigh, Nakagami- $m$ , Rician, lognormal) the optimal power control policy is a random on-off policy.

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## B. Related Work

Recently, [1]–[4] introduced the notion of the *local delay*, which is a fundamental source of delay in wireless networks. It is the mean time until a node successfully transmits to its nearest neighbor in a wireless network whose node locations are governed by a point process, averaged over fading, channel access, and the point process. The local delay can be characterized in networks both with [1], [3], [4] and without [1], [2], [4] interference. In [2], the author shows that power control can significantly reduce the local delay in noise-limited networks, but the optimum power control policy is not derived. The DTS can be viewed as the *conditional local delay*, *i.e.*, the local delay conditioned on the link distance (see Section II-B for details).

Besides its use in reducing the local delay, power control can benefit both point-to-point wireless communication and wireless networks in many different ways (see [5]–[8] and the references therein).

In the context of point-to-point communication, typical power control policies include water-filling, dynamic programming, channel inversion. Water-filling is typically used to maximize the throughput under energy constraint [9]–[13]. Power control policies based on dynamic programming is useful in reducing the queueing delay under power constraints, or, conversely, reducing energy consumption under queueing delay constraints [8], [14]–[17]. All the above power control policies require instantaneous channel state information (CSI) at the transmitter, while no such assumption is made in this paper.

In wireless networks, power control is often considered as a tool of interference management, see, *e.g.*, [7] and the references therein. While the conclusion has been drawn that random transmit power control may improve the network performance [18], [19] in the presence of interference, here we consider the noise-limited case, with an explicit focus on delay-optimality.

## C. Applications to Wireless Networks

The optimal power control schemes devised in this paper have two direct applications in the context of noise-limited wireless networks: the minimization of the *local delay* [1]–[4]; and the minimization of the *local anycast delay*. The local delay is the DTS averaged over the random distances in an ensemble of links. The local anycast delay is the mean time until a packet is successfully received (decoded) in *any* of a set of the desired receivers. Since the DTS-minimizing power control policies (where the link distance is fixed) are also delay-optimal

for random link distances if they are known at the transmitter, this paper is the first to provide and prove the optimal power control schemes in terms of reducing the local delay.

#### D. Organization

The rest of the paper is organized as follows: Section II introduces the system model and defines the DTS (or conditional local delay), local delay and local anycast delay. In Section III, we provide and prove the optimal power control policy for Rayleigh fading, while Section IV extends the results to general fading distributions. Concluding remarks are provided in Section V.

## II. PROBLEM FORMULATION

### A. Reception Model

The basic model we use in this paper is the one provided in [2]. The received power is

$$P_r = PHr^{-\alpha},$$

where  $P$  is the transmit power,  $H$  is the iid (power) fading coefficient,  $r$  is the link distance, and  $\alpha$  is the path loss exponent. We use an SNR condition to define whether a transmission is successful. A transmission is regarded successful if  $P_r > \theta$ , where  $\theta$  incorporates both the SNR threshold and the noise power. Then, we can write the success probability of a single transmission as a (deterministic) function of  $r$  as

$$p_s(r) = \mathbb{P}(PHr^{-\alpha} > \theta),$$

where  $\mathbb{P}(\cdot)$  denotes the probability measure and  $P$  can be a stochastic function of  $r$  as  $r$  is considered as a constant that can be learned by the transmitter.  $H$  is assumed to be iid over time and unknown to the transmitter.

### B. Delay Definitions

1) *DTS and Local Delay*: The *delay till success* (DTS) is defined as the mean number of time slots that the receiver needs to successfully receive (decode) the message over a link distance of distance  $r$ . With iid fading and iid transmit power  $P$  (or constant transmit power), the event of successful transmission is iid over time. Thus, the time to the first successful transmission is geometrically distributed with mean

$$D_r = \frac{1}{p_s(r)}, \quad (1)$$

which is, by definition, the DTS. If the link distance is a random variable  $R$ , which is constant over time and known at the transmitter, the *local delay* [1]–[4] is the ensemble average of the DTS, *i.e.*,

$$D = \mathbb{E}_R[D_R] = \mathbb{E}_R \left[ \frac{1}{p_s(R)} \right]. \quad (2)$$

Such a situation arises when considering a noise-limited static random wireless network, which can be modeled as a collection of links with spatially random but temporally fixed distances (Fig. 1). Hence the DTS can also be interpreted as the local delay conditioned on the link distance, and we may use the two terms DTS and conditional local delay interchangeably.

2) *The Local Anycast Delay:* Consider the case where a transmitter wants to transmit the message to any one of the  $n$  desired receivers (Fig. 2). Let  $r_i$  be the distance from the transmitter to each receiver and  $H_i$  be the fading coefficient from the transmitter to each receiver, where  $i \in [n]^1$ , and the  $H_i$  are iid both over time and space. Then the local anycast delay, defined as the mean time until the message is successfully decoded at *any of the desired receivers*, is

$$\begin{aligned} D_a &= \frac{1}{1 - \mathbb{P}(PH_1r_1^{-\alpha} \leq \theta, PH_2r_2^{-\alpha} \leq \theta, \dots, PH_nr_n^{-\alpha} \leq \theta)} \\ &= \frac{1}{\mathbb{P}(P \max\{H_i r_i^{-\alpha}, i \in [n]\} > \theta)}. \end{aligned} \quad (3)$$

Comparing (3) with (1), it is obvious that  $D_a$  is equivalent to the conditional local delay  $D_1$ , where the link distance  $r = 1$  and the fading subject to the distribution of  $\max\{H_i r_i^{-\alpha}, i \in [n]\} > \theta$ . Since  $H_i$  is iid over space, this fading distribution can be completely characterized by  $\mathbb{P}(\max\{H_i r_i^{-\alpha}, i \in [n]\} \leq x) = \prod_{i=1}^n \mathbb{P}(H_i r_i^{-\alpha} \leq x)$ .

### C. The Optimal Stationary Power Control Policy

We concentrate on stationary power control policies, *i.e.*, the statistics of the transmit power  $P$  in different time slots are the same, and define the optimal stationary power control policy to be the stationary power control policy that minimizes the conditional local delay (or, delay till success). Without loss of generality, we consider a unit mean power constraint and a peak power constraint  $P_{\max}$ , with  $P_{\max} > 1$  (otherwise, the mean power constraint will always be loose), and call a policy to be *valid* if and only if it satisfies both the constraints. In other words, a valid policy has  $\mathbb{E}P = 1$  and  $P \leq P_{\max}$ .

<sup>1</sup>We use  $[n]$  to denote the set  $\{1, 2, 3, \dots, n\}$ .

Let  $\mathcal{P}$  be the class of probability density functions (pdfs) with support at most  $[0, P_{\max}]$  and mean 1. The problem is to find the pdf  $f_{P|r}^*$  of the transmit power  $P(r)$ , where

$$f_{P|r}^* \triangleq \arg \min_{f_{P|r} \in \mathcal{P}} D_r = \arg \max_{f_{P|r} \in \mathcal{P}} \mathbb{P}(P(r)Hr^{-\alpha} > \theta). \quad (4)$$

Initial efforts to reduce the local (unicast) delay using power control are made in [2]. (2) shows that the power control policy minimizing  $D_r$  for all  $r$  is the power control policy that minimizes the local delay. Thus, an important application of the results on conditional local delay is the discovery of a local delay-minimizing power control policy.

For the local anycast delay, we observe from (3) that with the appropriate adjustment in the fading distribution, the optimal power control policy can be applied to minimize the local anycast delay also.

### III. RAYLEIGH FADING

#### A. Random On-off Is the Optimal Policy

In the iid fading case, the conditional local delay (or, delay till success) is simply the inverse of the success probability  $\mathbb{P}(HPPr^{-\alpha} > \theta)$ . For Rayleigh fading,

$$\mathbb{P}(HPPr^{-\alpha} > \theta) = \int_0^\infty \bar{F}_P \left( \frac{\theta r^\alpha}{h} \right) e^{-h} dh = \theta r^\alpha \int_0^\infty \bar{F}_P(x^{-1}) e^{-\theta r^\alpha x} dx,$$

where  $\bar{F}_P(x)$  is the complementary cumulative distribution function (ccdf) of the randomly controlled power  $P$ . Thus it must be monotonically decreasing,  $\bar{F}_P(x) = 0 \forall x > P_{\max}$ , and, by the mean power constraint  $\int_0^\infty \bar{F}_P(x) dx \leq 1$ .

To simplify the notation, we define the following function

$$F'(x) \triangleq \bar{F}_P(x^{-1}), \quad \forall x > 0, \quad (5)$$

which is the cumulative distribution function (cdf) of  $P^{-1}$ . The constraints on  $\bar{F}_P$  are mapped to the constraints that  $F'(x)$  is monotonically increasing,  $F'(x) = 0 \forall x < P_{\max}^{-1}$ ,  $\lim_{x \rightarrow \infty} F'(x) \leq 1$  and  $\mathbb{E}P = \int_0^\infty x^{-2} F'(x) dx \leq 1$ .

Therefore, the problem now becomes to find the  $F^*(x)$ , defined as the optimal  $F'(x)$  satisfying all the requirements above and maximizing  $\int_0^\infty F'(x) e^{-\theta r^\alpha x} dx$ . In order to maintain full generality of the transmit power distribution, we do not require  $\lim_{x \rightarrow \infty} F'(x) = 1$ . This is because

$$\lim_{x \rightarrow \infty} F'(x) = \lim_{x \rightarrow \infty} \mathbb{P}(P^{-1} \leq x) = \lim_{x \rightarrow \infty} \mathbb{P}(P \geq \frac{1}{x}) = \mathbb{P}(P > 0),$$

which is less than 1 when  $\mathbb{P}(P = 0) > 0$ . In fact, as will be shown later,  $\mathbb{P}(P = 0) > 0$  is often the case for the optimal power control policies.

**Lemma 1.** *The desired function  $F^*(x)$  satisfies  $F^*(x) = F^*(x_M)$ ,  $\forall x > x_M$ , where  $x_M \triangleq \max\{P_{\max}^{-1}, \frac{1}{\theta r^\alpha}\}$ .*

*Proof:* See Appendix A. ■

Analogously, we have the following lemma.

**Lemma 2.** *If  $1 \leq \theta r^\alpha \leq P_{\max}$ , we have  $F^*(x) = 0$ ,  $\forall x < \frac{1}{\theta r^\alpha}$ .*

*Proof:* Similar to the proof of Lemma 1, we start with the case where  $F^*(x)$  is simple and then generalize to the case of all valid cdfs.

Consider the case where  $F^*(x)$  is a simple function and write it as (11). Assuming  $\frac{1}{\theta r^\alpha} \in [b_l, b_{l+1})$ , we can construct

$$\tilde{F}(x) = F^*(x) - \sum_{n=1}^l a_n \mathbf{1}_{[b_n, \infty)}(x) + \sum_{n=1}^l \frac{a_n}{b_n \theta r^\alpha} \mathbf{1}_{[\frac{1}{\theta r^\alpha}, \infty)}(x),$$

where  $\mathbf{1}_A(\cdot)$  is the indicator function. Suppose that  $F^*(x_0) > 0$  for some  $x_0 < \frac{1}{\theta r^\alpha}$ , we know  $\tilde{F}(x) \neq F^*(x)$ , since  $\tilde{F}(x) = 0$ ,  $\forall x < \frac{1}{\theta r^\alpha}$ . Meanwhile, it can be verified that  $\int_0^\infty x^{-2} \tilde{F}(x) dx = \int_0^\infty x^{-2} F^*(x) dx$ . By Lemma 1,  $F^*(x) = F^*(\frac{1}{\theta r^\alpha})$ ,  $\forall x > \frac{1}{\theta r^\alpha}$  and thus  $\tilde{F}(x) \leq 1$  (because  $\int_1^\infty x^{-2} dx = 1$ ). All other constraints over  $\tilde{F}(x)$  to be a valid candidate for  $F'(x)$  are automatically satisfied. Also,

$$\begin{aligned} \int_0^\infty e^{-\theta r^\alpha x} \tilde{F}(x) dx - \int_0^\infty e^{-\theta r^\alpha x} F^*(x) dx &= \sum_{n=1}^l \int_0^\infty \left( \frac{a_n}{b_n \theta r^\alpha} \mathbf{1}_{[\frac{1}{\theta r^\alpha}, \infty)} - a_n \mathbf{1}_{[b_n, \infty)} \right) e^{-\theta r^\alpha x} dx \\ &= \sum_{n=1}^l \frac{a_n}{b_n \theta r^\alpha} \left( \frac{1}{\theta r^\alpha} e^{-\theta r^\alpha \frac{1}{\theta r^\alpha}} - b_n e^{-\theta r^\alpha b_n} \right), \end{aligned}$$

which is strictly larger than zero due to the fact that  $b_n \leq \frac{1}{\theta r^\alpha}$ ,  $\forall n \leq l$  by assumption and the monotonicity of  $x e^{-\theta r^\alpha x}$  in  $[0, \frac{1}{\theta r^\alpha}]$ . Therefore, we found  $\tilde{F}(x)$  as a strictly better candidate than  $F^*(x)$ , contradicting the assumption that it is the desired function. The generalization from simple function to general functions are the same as in the proof of Lemma 1. ■

Similarly, it can be shown that:

**Lemma 3.** *If  $\theta r^\alpha < 1$ , we have  $F^*(x) = 0$ ,  $\forall x < 1$ .*

Although special care must be taken to make sure that  $\tilde{F}(x) \leq 1$ , the proof of Lemma 3 directly follows from that of Lemma 2 and is therefore omitted.

Combining Lemmas 1, 2, 3 and the requirements we have for a valid  $F'(x)$ , we conclude that  $F^*(x)$  is of the form:

$$F^*(x) = \begin{cases} 1_{[1, \infty)}(x), & \theta r^\alpha \leq 1 \\ \frac{1}{\theta r^\alpha} 1_{[\frac{1}{\theta r^\alpha}, \infty)}(x), & 1 < \theta r^\alpha \leq P_{\max} \\ P_{\max}^{-1} 1_{[P_{\max}^{-1}, \infty)}(x), & \theta r^\alpha > P_{\max} \end{cases} .$$

As stated earlier, there is a one to one mapping between  $F'(x)$  and  $\bar{F}_P(x)$  (and thus  $F_P(x)$ ). Hence, the result above directly leads to the following theorem.

**Theorem 1.** *For Rayleigh fading, given a link distance  $r$ , the optimal distribution of the transmit power  $P$  that minimizes DTS is*

$$F_P(x) = \begin{cases} 1_{[1, \infty)}(x), & \theta r^\alpha \leq 1 \\ (1 - \frac{1}{\theta r^\alpha}) 1_{[0, \theta r^\alpha)}(x) + 1_{[\theta r^\alpha, \infty)}(x), & 1 < \theta r^\alpha \leq P_{\max} \\ (1 - P_{\max}^{-1}) 1_{[0, P_{\max})}(x) + 1_{[P_{\max}, \infty)}(x), & \theta r^\alpha > P_{\max} \end{cases}$$

More concisely, we can define  $\xi \triangleq \max\{1, \min\{P_{\max}, \theta r^\alpha\}\}$ . Then Theorem 1 says: the optimal random power control strategy is a random on-off policy with transmit probability  $\xi^{-1}$  and transmit power  $\xi$ .

**Definition 1.** *A link is said to be in the peak-power-limited regime if the optimal power control policy is to transmit at power  $P_{\max}$  with probability  $P_{\max}^{-1}$ .*

**Definition 2.** *A link is said to be in the mean-power-limited regime if constant power transmission ( $P \equiv 1$ ) is the optimal power control policy.*

The optimal strategy *maximizes* the variance of the transmit power in the peak-power-limited regime, while *minimizing* this variance in the mean-power-limited regime.

Theorem 1 also indicates that in order to apply the optimal power control policy, the transmitter needs to know either  $r$  and  $\alpha$  or  $r^\alpha$ . Since  $\mathbb{E}H = 1$ ,  $r^\alpha$  can be easily obtained by simply taking the average of the received power.

**Corollary 1.** *Without peak power constraint, but with the mean power limited to  $\mathbb{E}P = 1$ , the optimal (DTS-minimizing) random power control policy is*

$$F_P(x) = \begin{cases} 1_{[1, \infty)}(x), & \theta r^\alpha \leq 1 \\ (1 - \frac{1}{\theta r^\alpha}) 1_{[0, \theta r^\alpha)}(x) + 1_{[\theta r^\alpha, \infty)}(x), & \theta r^\alpha > 1. \end{cases}$$

The exact value of the local delay depends on the distribution of the link distance  $R$ . An important case is the Rayleigh distribution, since it is the distribution of the nearest-neighbor distance in a 2-dimensional network, whose nodes are distributed as a Poisson point process (PPP) [21]. It is shown in [2] that with such a distribution of  $R$ , the local delay is unbounded if Rayleigh fading is considered and no power control is applied (except for the case of  $\alpha = 2$ ). The natural question is whether random power control can make the local delay finite in the same scenario, which is answered by the following proposition.

**Proposition 1.** *Without a peak power constraint, (random) power control can reduce the local delay to be finite while keeping the mean transmit power at each node unit even if the link distance is Rayleigh distributed. However, with a peak power constraint, power control cannot achieve a finite local delay if the link distance is Rayleigh distributed and  $\alpha > 2$ .*

*Proof:* The first part (no peak power constraint) can be shown by directly applying the result in Corollary 1. When link distances are Rayleigh distributed,

$$\begin{aligned} D &= \mathbb{E} \left[ \frac{1}{P_{s|R}} \right] = 2\pi\lambda \int_0^{\theta^{-\frac{1}{\alpha}}} r e^{\theta r^\alpha - \lambda\pi r^2} dr + 2\pi\lambda\theta e \int_{\theta^{-\frac{1}{\alpha}}}^{\infty} r^{\alpha+1} e^{-\lambda\pi r^2} dr \\ &\leq e(1 - e^{-\lambda\pi\theta^{-\frac{2}{\alpha}}}) + \theta e(\lambda\pi)^{-\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2} + 1, \lambda\pi\theta^{-\frac{2}{\alpha}}) < \infty, \end{aligned}$$

where  $\Gamma(\cdot, \cdot)$  is the upper incomplete gamma function.

To show the second part, we realize that, with only a peak power constraint, the minimum local delay is achieved when the transmit power is  $P_{\max}$  at each link in each time slot. Then the proposition trivially follows from the fact that any constant power in is not sufficient to keep the local delay finite when the link distance is Rayleigh distributed and  $\alpha > 2$  [2]. ■

### B. Comparison of Random Power Control Schemes

In this subsection, we compare the DTS performance (in the presence of Rayleigh fading) of several power control policies, defined as follows:

**Definition 3.** *The optimal power control (OPC) policy is the power control policy defined in Theorem 1.*

**Definition 4.** *The peak power control (PPC) policy transmits at power  $P_{\max}$  with probability  $P_{\max}^{-1}$  and does not transmit with probability  $1 - P_{\max}^{-1}$ , regardless of the value of  $r$ .*

**Definition 5.** *The uniform power control (UPC) policy transmits at power  $P$  each time with  $P$  uniformly distributed in  $[1 - \Delta, 1 + \Delta]$ . Here,  $\Delta \triangleq \min\{1, P_{\max} - 1\}$ .*

**Definition 6.** *The hybrid uniform power control (HUPC) policy transmits with probability  $\frac{2}{P_{\max}+1}$ . If transmitting, the transmit power is uniformly distributed between 1 and  $P_{\max}$ .*

**Definition 7.** *The 1-bit power control (1BPC) policy transmits at constant power ( $P = 1$ ) when  $\theta r^\alpha \leq \frac{\log P_{\max}}{1 - P_{\max}^{-1}}$ . When  $\theta r^\alpha > \frac{\log P_{\max}}{1 - P_{\max}^{-1}}$ , the policy transmits at power  $P_{\max}$  with probability  $P_{\max}^{-1}$  and does not transmit with probability  $1 - P_{\max}^{-1}$ .*

While the peak power control (PPC) policy, the uniform power control (UPC) policy, and the hybrid uniform power control (HUPC) policy are all suboptimal, their complexity is lower than OPC's in the sense that they do not require the link distance information  $R$ . Meanwhile, their constructions are inspired by Theorem 1 in different ways. For example, in the peak-power-limited regime PPC is the same as OPC. The intuition behind HUPC is that Theorem 1 implies that for all realizations of  $R$  it is always suboptimal to transmit with power in  $(0, 1)$ .

The 1-bit power control (1BPC) policy is proposed as a trade-off between OPC and other kinds of power control policies that do not utilize the link distance information. In practice, although the link distance can always be measured, its precise value might be difficult to acquire, *e.g.*, it may take too long to accurately measure. In such occasions, the performance of OPC becomes difficult to realize, and 1BPC turns out to be more suitable, since it only requires 1 bit of information about the link distance.

The intuition of 1BPC lies in the observation that OPC is constant power transmission in the mean-power-limited regime and PPC in the peak-power-limited regime. Therefore, 1BPC switches between these two types of power control policies, utilizing the 1 bit information and achieving minimum DTS in the two regimes. Outside these two regimes, the switching point  $\frac{\log P_{\max}}{1 - P_{\max}^{-1}}$  is chosen in a way that 1BPC always achieves the smaller DTS achievable by either

constant power transmission or PPC.

It is not difficult to find that if the link distance  $r$  is known and OPC is applied, the conditional local delay is

$$D_r = \begin{cases} P_{\max} e^{-\frac{\theta r^\alpha}{P_{\max}}}, & \theta r^\alpha \geq P_{\max} \\ \theta r^\alpha e, & 1 < \theta r^\alpha < P_{\max} \\ e^{\theta r^\alpha}, & \theta r^\alpha \leq 1. \end{cases}$$

In comparison, we can see that with constant power transmission, the conditional local delay is always equal to  $\exp(\theta r^\alpha)$ . When  $P_{\max} \geq 2$ , the transmit power of UPC is uniformly distributed in  $[0, 2]$ . Its conditional local delay can be calculated as  $\left(\exp(-\frac{1}{2}\theta r^\alpha) - \frac{1}{2}\theta r^\alpha \int_{\frac{1}{2}\theta r^\alpha}^{\infty} \frac{\exp(-x)}{x} dx\right)^{-1}$ . Straightforward (but tedious) manipulation reveals the conditional local delay for HUPC to be

$$\frac{P_{\max}^2 - 1}{2} \left( P_{\max} e^{-\frac{\theta r^\alpha}{P_{\max}}} - e^{-\theta r^\alpha} - \theta r^\alpha \int_{\frac{\theta r^\alpha}{P_{\max}}}^{\theta r^\alpha} \frac{e^{-x}}{x} dx \right)^{-1}.$$

The calculation of the conditional local delay for 1BPC is similar to that of OPC.

Figs. 3 compares all the power control policies defined above along with constant power transmission ( $P \equiv 1$ ). The parameters in the figure are chosen in a way that the mean-power-limited regime is  $r < 1$  and the peak-power-limited regime is  $r > 2$  for the sake of easy illustration. The figure shows that in the peak-power-limited regime (large  $r$ ) the conditional local delay grows exponentially with  $r$  for all power control policies. This is mainly due to the peak power constraint. However, for different power control schemes the exponent is quite different, which can result in many orders of difference in conditional local delay. As expected, in this regime, OPC, PPC and 1BPC perform the best among all schemes, and constant power transmission is the worst. Both UPC and HUPC appear to be trade-offs between the best and the worst.

In the mean-power-limited regime (small  $r$ ), the difference in the conditional local delay between the different schemes can be at most a factor of 4 (Fig. 3). Still, UPC and HUPC perform between the two extremes. Fig. 3 also shows that 1BPC is not considerably inferior to OPC even in its suboptimal regime ( $1 < \theta r^\alpha < P_{\max}$ ), and thus is a good substitute for OPC in many cases.

#### IV. GENERAL FADING DISTRIBUTIONS

##### A. The Optimality of Random On-off

Results in Section III raise a more general question: Is the random on-off policy still optimal in reducing the (conditional) local delay if the fading is not Rayleigh? To answer this question, this subsection derives more general sufficient conditions for the optimality of random on-off policies. We use  $g(x)$  to denote the pdf of the fading random variable  $H$ .

**Lemma 4.** *For a given  $r$ , if there exists a constant  $\vartheta < \infty$  such that  $x \int_x^\infty g(\theta r^\alpha y) dy$  is strictly monotonically decreasing for all  $x > \vartheta$ , we have*

$$F^*(x) = F^*(x_M), \quad \forall x > x_M,$$

where  $F^*(x)$  is the desired function as defined before, and  $x_M \triangleq \max\{P_{\max}^{-1}, \vartheta\}$ .

*Proof:* Using the definition of  $\bar{F}_p(x)$  and  $F'(x)$  as before, we have

$$\mathbb{P}(HP_r^{-\alpha} > \theta) = \int_0^\infty \bar{F}_p\left(\frac{\theta r^\alpha}{h}\right) g(h) dh = \theta r^\alpha \int_0^\infty F'(x) g(\theta r^\alpha x) dx.$$

As in the proof of Lemma 1, we prove Lemma 4 by contradiction. Starting with simple functions, we write  $F^*(x)$  as in (11) and construct  $\tilde{F}(x)$  as (12). Straightforward manipulation shows

$$\begin{aligned} & \int_0^\infty \tilde{F}(x) g(\theta r^\alpha x) dx - \int_0^\infty F^*(x) g(\theta r^\alpha x) dx \\ &= \sum_{n=l+1}^j \frac{a_n - a_{n-1}}{b_n} \left( x_M \int_{x_M}^\infty g(\theta r^\alpha x) dx - b_n \int_{b_n}^\infty g(\theta r^\alpha x) dx \right), \end{aligned} \quad (6)$$

which is strictly larger than zero by the monotonicity of  $x \int_x^\infty g(\theta r^\alpha y) dy$ . Therefore, the lemma is proved for simple functions. The generalization to non-simple functions is just as in Lemma 1. ■

A simple sanity check would be to consider the Rayleigh fading case, where  $g(x) = \exp(-x)$ . Then,  $x \int_x^\infty g(\theta r^\alpha y) dy = \frac{x}{\theta r^\alpha} \exp(-\theta r^\alpha x)$ , which is strictly monotonically decreasing for  $x \geq \frac{1}{\theta r^\alpha}$ . This retrieves Lemma 1. Similarly, Lemmas 2 and 3 can be generalized as follows:

**Lemma 5.** *For a given  $r$ , let  $\varsigma$  be any constant such that  $x \int_x^\infty g(\theta r^\alpha y) dy$  is strictly monotonically increasing for all  $0 \leq x < \varsigma$ . Then the desired function  $F^*(x)$  must have*

$$F^*(x) = 0, \quad \forall x < x_m,$$

where  $x_m \triangleq \min\{1, \varsigma\}$ .

The proof is analogous to that of Lemma 4. For Rayleigh fading,  $x \int_x^\infty g(\theta r^\alpha y) dy$  is strictly increasing for all  $0 \leq x \leq \frac{1}{\theta r^\alpha}$ .

**Theorem 2.** *Let  $\bar{G}(x)$  denote the cdf of  $H$ . If there exists some  $x_0 > 0$ , such that  $x\bar{G}(\theta r^\alpha x)$  is strictly increasing on  $[0, x_0)$  and strictly decreasing on  $(x_0, \infty)$ , the optimal power control policy is a random on-off policy with transmit power  $\xi$  and transmit probability  $\xi^{-1}$ , where  $\xi \triangleq \max\{1, \min\{P_{\max}, x_0^{-1}\}\}$ .*

Theorem 2 is simply a combination of Lemmas 4 and 5. In Appendix B, we show that Nakagami- $m$  fading satisfies the condition in Theorem 2, which leads to the following corollary.

**Corollary 2.** *The optimal power control policy for Nakagami- $m$  fading is a random on-off policy.*

### B. Peak-power-limited and Mean-power-limited Regimes

For a more general class of fading distributions, the conditions in Theorem 2 may not be satisfied. The simplest example may be the (discrete) fading distribution with pdf

$$g(x) = q_1 \delta(x - h_1) + q_2 \delta(x - h_2), \quad (7)$$

where  $0 \leq h_1 < h_2 < \infty$ ,  $q_1 h_1 + q_2 h_2 = 1$ , and  $q_1 + q_2 = 1$ . Then,  $x\bar{G}(x) = x1_{[0, h_1/\theta r^\alpha)}(x) + q_2 x 1_{[h_1/\theta r^\alpha, h_2/\theta r^\alpha)}$ , which does not satisfy the conditions in Theorem 2 for two reasons: 1) there is no *strict* monotonicity for  $x > \frac{h_2}{\theta r^\alpha}$ ; 2) even if we relax the strictness requirement, there is still no such  $x_0$  that  $x\bar{G}(x)$  is monotonically increasing on  $[0, x_0)$  and decreasing on  $(x_0, \infty)$ , as long as  $q_2 > 0$ . Thus, results so far are not applicable in this case. However, some of the results can still be obtained in particular regimes of  $r$  even when the conditions in Theorem 2 are not met.

**Theorem 3.** *For a general fading distribution with cdf  $\bar{G}(x)$ , fixed threshold  $\theta$ , and link distance  $r_0$ , if  $x\bar{G}(\theta r_0^\alpha x)$  is monotonically decreasing for all  $x > P_{\max}^{-1}$ , the random on-off peak power control policy with on power  $P_{\max}$  achieves the minimum conditional local delay. Moreover, for all  $r > r_0$ , the same policy is still delay-optimal.*

*Proof:* When the monotonicity of  $x\bar{G}(\theta r_0^\alpha x)$  is strict, the proof of the first part of Theorem 3 trivially follows from Lemma 4, since  $F'(x) = 0 \forall x < P_{\max}^{-1}$ . In the non-strict case, Lemma 4 needs to be slightly generalized, *i.e.*, (6) is no longer strictly larger than zero. Yet, (6) is still no less than zero, which ensures that the constructed  $\tilde{F}(x)$  produces a conditional local delay no larger than the minimum conditional local delay. Thus, the first part of the theorem is proved.

For the second part, let  $P_{\max}^{-1} < x_1 < x_2$  and  $r > r_0$ . Noting that  $x \int_x^\infty g(\theta r_0^\alpha t) dt = \frac{x}{\theta r^\alpha} \int_{\theta r^\alpha x}^\infty g(t) dt$  and using the monotonicity of  $x \int_x^\infty g(\theta r_0^\alpha t) dt$ , we have

$$\begin{aligned} \frac{x_1}{\theta r^\alpha} \int_{\theta r^\alpha x_1}^\infty g(t) dt &= \frac{\theta r_0^\alpha}{\theta r^\alpha} \left(\frac{r_0}{r}\right)^\alpha \frac{\left(\frac{r}{r_0}\right)^\alpha x_1}{\theta r_0^\alpha} \int_{\theta r_0^\alpha \left(\frac{r}{r_0}\right)^\alpha x_1}^\infty g(t) dt \\ &> \frac{\theta r_0^\alpha}{\theta r^\alpha} \left(\frac{r_0}{r}\right)^\alpha \frac{\left(\frac{r}{r_0}\right)^\alpha x_2}{\theta r_0^\alpha} \int_{\theta r_0^\alpha \left(\frac{r}{r_0}\right)^\alpha x_2}^\infty g(t) dt = \frac{x_2}{\theta r^\alpha} \int_{\theta r^\alpha x_2}^\infty g(t) dt. \end{aligned}$$

Thus, the monotonicity of  $x \int_x^\infty g(\theta r^\alpha t) dt$  is proved for all  $r > r_0$ . ■

Note that Theorem 3 does *not* imply that, for general fading, there must exist a peak-power-limited regime where the random on-off peak power control (PPC) is delay-optimal. To show this, one can consider a fading distribution with an oscillating tail in the pdf, where  $x\bar{G}(\theta r^\alpha x)$  does not have a monotonic tail for all  $0 < r < \infty$ .

Likewise, we can deduce the following theorem:

**Theorem 4.** *Constant power transmission minimizes the conditional local delay, if  $x\bar{G}(\theta r_0^\alpha x)$  is monotonically increasing for all  $x < 1$ . Moreover, the optimality still holds for all  $r < r_0$ .*

For the particular example we raised at the beginning of this subsection, where the fading coefficient has a pdf as in (7), Theorems 3 and 4 indicate: 1) when  $P_{\max} h_2 < \theta r^\alpha$ , the random on-off power control policy achieves minimum conditional local delay; 2) when  $h_1 > \theta r^\alpha$ , constant power transmission minimizes local delay. These two facts are intuitive in this example. Because, when  $P_{\max} h_2 < \theta r^\alpha$ , even full power transmission ( $P = P_{\max}$ ) cannot achieve a successful transmission, and thus the conditional local delay is always  $\infty$ . When  $h_1 > \theta r^\alpha$ , constant-power transmission ( $P = 1$ ) always succeeds. So, the minimum conditional local delay  $D_r = 1$  is achieved by such policy.

In addition to the toy example above, Theorems 3 and 4 are useful when the fading distribution has a very complicated shape, making  $x\bar{G}(\theta r^\alpha x)$  non-unimodal.

### C. Numerical Approach

**Theorem 5.** *If the transmit power can  $P$  only be chosen from a finite set of power levels  $W = \{w_0, w_1, \dots, w_N\}$ , where  $0 = w_0 < w_1 < \dots < w_N = P_{\max}$ , and  $\bar{G}(x)$  is the ccdf of the fading coefficient  $H$ , then the optimal power control policy is of the form*

$$F_P^*(x) = \sum_{k=0}^N p_k w_k, \quad (8)$$

where  $(p_0, p_1, \dots, p_N) \in [0, 1]^{N+1}$  is the solution of the following linear programming problem:

$$\begin{aligned} & \underset{\{p_k, 0 \leq k \leq N\}}{\text{maximize}} && \sum_{k=0}^N p_k \bar{G}\left(\frac{\theta r^\alpha}{w_k}\right) \\ & \text{subject to} && p_k \geq 0, \quad k = 0, \dots, N \\ & && \sum_{k=0}^N p_k = 1, \quad \sum_{k=0}^N p_k w_k = 1. \end{aligned}$$

### D. Examples

1) *Nakagami- $m$  Fading:* The optimality of random on-off in the presence of Nakagami- $m$  fading is shown in Corollary 2. Then, the implementation of the optimal policy hinges on finding the corresponding  $x_0$ , which is the solution of  $\Gamma(m, m\theta r^\alpha x) = (m\theta r^\alpha x)^m e^{-m\theta r^\alpha x}$ . Numerically solving this equation yields the optimal policy as well as the minimum conditional local delay. Fig. 4 compares the minimum conditional local delay for different  $m$  ( $m = 1$  is the Rayleigh fading case). As expected, when  $r$  is small, a larger  $m$  yields a lower conditional local delay, since there is less chance for the channel to be in a bad condition. On the other hand, for large  $r$ , Nakagami fading with a larger  $m$  has a larger conditional local delay, since the chance of a particularly large channel gain is considerably smaller than in the Rayleigh fading case.

In particular, for any two curves in Fig. 4, there is a crossover point slightly larger than  $r = 2$ . Before this point, channel (fading) randomness increases the conditional local delay, *i.e.*, a larger  $m$  results in a smaller delay. After this point, channel (fading) randomness helps reduce the conditional local delay, *i.e.*, a larger  $m$  results in a larger delay.

2) *Rician Fading and Lognormal Shadowing:* The conditions in Theorem 3 are not restrictive. In fact, almost all practical continuous fading distributions satisfy them although it can be tedious to prove. In particular, apart from the Rayleigh fading and Nakagami- $m$  fading, two of the most common types of fading, Rician fading and lognormal shadowing, satisfy these conditions. In the

following, we use the numerical approach introduced above to verify the optimality of random on-off policies.

The ccdf of the Rician fading is  $\bar{G}(x) = Q\left(\frac{s}{\sigma}, \frac{\sqrt{x}}{\sigma}\right)$ , where  $s^2$  is the line of sight (LOS) power component,  $2\sigma^2$  is the non-LOS power component, and  $Q(\cdot, \cdot)$  is the Marcum Q function. The mean power is the sum of these two power components. Let  $K = s^2/2\sigma^2$ , and fix the mean of  $H$  to be one. The ccdf can be written as

$$\bar{G}(x) = Q(\sqrt{2K}, \sqrt{2(K+1)x}). \quad (9)$$

If  $H$  represents the effect of lognormal shadowing and  $\mathbb{E}H = 1$ , the ccdf of  $H$  is

$$\bar{G}(x) = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{\ln x + \sigma^2/2}{\sigma\sqrt{2}}\right), \quad (10)$$

where  $\sigma^2$  is proportional to the variance of the received power in dB, and  $\operatorname{erf}(\cdot)$  is the error function.

Fig. 5 and Fig. 6 show the cdf of the optimal power control policy for different link distances. For small link distances, *e.g.*,  $r = 0.5$ , constant power transmission is optimal. For large distances, *e.g.*,  $r = 2$ , peak power control (PPC) is optimal. Between these two regimes, *e.g.*,  $r = 1.5$ , the optimal policy is a random on-off power control policy with certain a transmit probability in  $[P_{\max}^{-1}, 1]$ . In any case, the random on-off policy is optimal.

3) *Local Anycast Delay*: As mentioned in Section II-B, the optimal policy in this paper can be directly applied as the optimal policy that minimizes the local anycast delay. In particular, we provide the following corollary which follows from Theorem 2 and is proven in Appendix C.

**Corollary 3.** *When the desired receivers are located at the same distance to the transmitter and Rayleigh fading is considered, the optimal policy that minimizes the local anycast delay is a random on-off policy.*

Similar to the Nakagami- $m$  fading case, in general, there is no closed form expression for the (optimal) transmit probability. However, this optimal configuration is implied by the solution of  $\frac{d}{dx}L(x) = 0$ , where  $L(x)$  is defined in Appendix C.

Table I compares the transmit power  $\xi$  of the optimal random on-off policy for different number of receivers and different common link distances  $r$ . It shows that with a larger number of desired receivers, the optimal policy tends to reduce the transmit power of each transmission

attempt while increasing the transmit probability  $\xi^{-1}$ . Fig. 7 shows how the minimum local anycast delay decreases as more desired receivers are available. In both Table I and Fig. 7, the case  $n = 1$  corresponds to the single-link Rayleigh fading case.

## V. CONCLUSIONS

This paper provides a set of power control policies that minimize the conditional local delay (or delay till success) for channels with different fading statistics. We give sufficient conditions under which the random on-off policy is optimal and show that almost all common fading models satisfy these conditions, including Rayleigh fading, Nakagami- $m$  fading, Rician fading and lognormal shadowing. These results naturally leads to a solution for minimizing the local delay in random but fixed wireless networks, and also provide the solution for minimizing the local anycast delay.

Although we focused on minimizing DTS, given a single-bit Automatic Repeat reQuest (ARQ), the optimal policies derived also maximize the network throughput and minimize the queueing delay. To see this, recall that (4) shows that the optimal power control policy is the policy that maximizes the transmission success probability of a link of length  $r$ . For a single rate of transmission, the success probability equals the long term throughput. Thus, given a single ARQ, the delay-optimal power control policies also maximize the throughput in noise-limited networks. At the same time, since the transmission success probability can be interpreted as the service rate of the transmission queue, with the ARQ, the DTS-minimizing power control policy minimizes the queueing delay as well.

While the literature studying power allocation over time, frequency or space is extensive, this paper shows that power control can improve the performance of wireless communication through assigning transmit power in another dimension, the probability space.

As in many cases the optimal policy derived in this paper is a ‘peaky’ scheme (random on-off), our results bear interesting relations to some of the results also suggesting ‘peaky’ transmissions, *e.g.*, [22]–[24].

In a wireless network where interference is not negligible, our optimal scheme can be used as a lower-layer power control policy. When a specific link is activated by an upper layer interference-managing MAC scheme (*e.g.*, CSMA, LMAC [25], etc.), individual links can apply our policy in this paper to minimize the conditional local delay, and thus the local delay.

## APPENDIX A

## PROOF OF LEMMA 1

*Proof:* First, consider the case that  $F^*(x)$  is a simple function. Since,  $F^*(x)$  is monotonically increasing, we can write it as

$$F^*(x) = \sum_{i=0}^N a_i \mathbf{1}_{[b_i, b_{i+1})}(x), \quad (11)$$

where  $0 = a_0 < a_1 < a_2 < \dots < a_N \leq 1$  and  $0 = b_0 < b_1 < b_2 < \dots < b_{N+1} = \infty$ . Suppose there exists a  $x_0 > x_M$ , such that  $F^*(x_0) \neq F^*(x_M)$ , i.e.,  $F^*(x_0) > F^*(x_M)$ , and assume  $x_0 \in [b_j, b_{j+1})$ ,  $x_M \in [b_l, b_{l+1})$ , for some  $l, j \in \mathbb{N}$  such that  $0 < l < j$ . Then, let

$$\tilde{F}(x) \triangleq F^*(x) - \sum_{n=l+1}^j (a_n - a_{n-1}) \mathbf{1}_{[b_n, \infty)}(x) + x_M \sum_{n=l+1}^j \frac{a_n - a_{n-1}}{b_n} \mathbf{1}_{[x_M, \infty)}(x). \quad (12)$$

It can be easily verified that  $\int_0^\infty x^{-2} \tilde{F}(x) dx = \int_0^\infty x^{-2} F^*(x) dx$  and  $\tilde{F}(x)$  satisfies all the requirements for a valid  $F'(x)$  over  $[0, \infty)$ . Moreover,

$$\begin{aligned} & \int_0^\infty e^{-\theta r^\alpha x} \tilde{F}(x) dx - \int_0^\infty e^{-\theta r^\alpha x} F^*(x) dx \\ &= \int_0^\infty e^{-\theta r^\alpha x} x_M \sum_{n=l+1}^j \frac{a_n - a_{n-1}}{b_n} \mathbf{1}_{[x_M, \infty)}(x) dx - \int_0^\infty e^{-\theta r^\alpha x} \sum_{n=l+1}^j (a_n - a_{n-1}) \mathbf{1}_{[b_n, \infty)}(x) dx \\ &= \sum_{n=l+1}^j \frac{a_n - a_{n-1}}{b_n \theta r^\alpha} (x_M e^{-\theta r^\alpha x_M} - b_n e^{-\theta r^\alpha b_n}), \end{aligned}$$

which is strictly larger than zero because of the monotonicity of  $x e^{-\theta r^\alpha x}$  at  $[\frac{1}{\theta r^\alpha}, \infty)$  and the fact that  $b_n > x_M \geq \frac{1}{\theta r^\alpha} \forall n \geq l+1$ . This contradicts the assumption that  $F^*(x)$  is the function which maximizes  $\int_0^\infty F'(x) e^{-\theta r^\alpha x} dx$  and satisfies all the constraints.

For general  $F^*(x)$ , consider a sequence of simple functions  $(F_k^*)_1^\infty$  such that  $F_i^* < F_j^* < F^*$ ,  $\forall i < j$  and  $\lim_{k \rightarrow \infty} F_k^* = F^*$ . By the monotone convergence theorem,  $\lim_{k \rightarrow \infty} \int_0^\infty x^{-2} F_k^*(x) dx = \int_0^\infty x^{-2} F^*(x) dx$  and  $\lim_{k \rightarrow \infty} \int_0^\infty e^{-\theta r^\alpha x} F_k^*(x) dx = \int_0^\infty e^{-\theta r^\alpha x} F^*(x) dx$ . Using the construction in the proof for the simple functions, we are able to produce another sequence of simple functions  $(\tilde{F}_k)_1^\infty$ , such that  $\int_0^\infty e^{-\theta r^\alpha x} \tilde{F}_k(x) dx > \int_0^\infty e^{-\theta r^\alpha x} F_k^*(x) dx, \forall k$ . Meanwhile,  $\lim_{k \rightarrow \infty} \tilde{F}_k \neq F^*$ , since  $\tilde{F}_k(x_0) = \tilde{F}_k(\frac{1}{\theta r^\alpha})$ . Thus, the limiting function of  $\tilde{F}_k(x)$  is a strictly better candidate for  $F'(x)$  than  $F^*(x)$ . ■

## APPENDIX B

THE OPTIMAL POWER CONTROL POLICY UNDER NAKAGAMI- $m$  FADING

Let  $K(x) = x\Gamma(m, m\theta r^\alpha x)$ . With Theorem 2, the following proposition is sufficient to prove Corollary 2.

**Proposition 2.** *There exists a unique  $x_0 \in (0, \frac{m+1}{m\theta r^\alpha})$ , such that  $\frac{d}{dx}K(x) |_{x=x_0} = 0$ .  $K(x)$  is strictly increasing on  $(0, x_0)$  and strictly decreasing on  $(x_0, \infty)$ .*

*Proof:* Since  $K(x)$  is twice-differentiable, the monotonicity in the proposition can be shown by evaluating the derivatives of  $K(x)$ . To prove the first part of the proposition, we first notice that there must exist at least one  $x_0 \in (0, \frac{m+1}{m\theta r^\alpha})$ , such that  $\frac{d}{dx}K(x) |_{x=x_0} = 0$ . This is due to the continuity of  $K(x)$  as well as the fact that  $K(0) = \lim_{x \rightarrow \infty} K(x) = 0$  and  $\frac{d}{dx}K(x) |_{x=0} = \Gamma(m) > 0$ .

In the following, we prove the uniqueness of  $x_0$  by contradiction. Assume there is another point  $x_1 \neq x_0$  and  $\frac{d}{dx}K(x) |_{x=x_1} = 0$ . Without loss of generality, consider  $x_1 > x_0$  (otherwise, we can exchange the subscript). Because  $\lim_{t \rightarrow \infty} \frac{d}{dx}K(x) |_{x=t} = 0$  and

$$\frac{d^2}{dx^2}K(x) = (m\theta r^\alpha x)^m e^{-m\theta r^\alpha x} \left( m\theta r^\alpha - \frac{m+1}{x} \right), \quad (13)$$

which is strictly positive when  $x > \frac{m+1}{m\theta r^\alpha}$ , we must have  $x_0 < x_1 < \frac{m+1}{m\theta r^\alpha}$ . However, (13) also indicates  $\frac{d}{dx}K(x)$  is strictly decreasing on  $(0, \frac{m+1}{m\theta r^\alpha})$ . Then,  $\frac{d}{dx}K(x) |_{x=x_0} = \frac{d}{dx}K(x) |_{x=x_1} = 0$  implies  $x_0 = x_1$ , which contradicts the assumption that  $x_1 \neq x_0$ .

Since  $K(x)$  is continuous and  $K(0) = \lim_{t \rightarrow \infty} K(t) = 0$ , the uniqueness of  $x_0$  implies that there are at most two monotonic regions of  $K(x)$  over  $[0, \infty)$ . Combined with the fact that  $\frac{d}{dx}K(x) |_{x=0} = \Gamma(m) > 0$ , we conclude that  $K(x)$  is strictly increasing on  $[0, x_0)$  and strictly decreasing on  $[x_0, \infty)$ . ■

## APPENDIX C

## LOCAL ANYCAST DELAY

In Rayleigh fading case, the distribution of fading coefficients  $H_i$  is exponential with unit mean. As we are considering the case where the link distances to each of the  $n$  desired receivers are the same, the cdf of  $\max\{H_i r_i^{-\alpha}\}$  is then  $G(x) = (1 - e^{-r^\alpha x})^n$ , where  $r$  is the link distance. Let  $L(x) \triangleq x(1 - G(\theta x))$ . With Theorem 2, the following proposition suffices to show Corollary 3.

**Proposition 3.** *There exists a unique  $x_0$ , such that  $L(x)$  is monotonically increasing on  $[0, x_0]$  and monotonically decreasing on  $[x_0, \infty)$ .*

*Proof:* Since  $L(x)$  is differentiable on  $[0, \infty)$  and its derivative is continuous, it suffices to show: there exists a unique  $x_0$ , such that  $\frac{d}{dx}L(x) |_{x=x_0} = 0$ , and  $\frac{d}{dx}L(x)$  is positive on  $[0, x_0]$  and negative on  $[x_0, \infty)$ . Observing that  $\lim_{x \rightarrow 0^+} \frac{d}{dx}L(x) > 0$  and that  $\frac{d}{dx}L(x)$  approaches zero from below when  $x \rightarrow \infty$ , we can deduce the latter directly from the former. Thus, the key is to show  $\frac{d}{dx}L(x) = 0$  has a unique solution on  $[0, \infty)$ .

This is proved in three steps: first, we show that there can be at most one solution of  $\frac{d}{dx}L(x) = 0$  on  $[0, \frac{1}{\theta r^\alpha}]$ ; second, we show there can be at most one solution of  $\frac{d}{dx}L(x) = 0$  on  $[\frac{1}{\theta r^\alpha}, \infty)$ ; third, we observe that cannot be two solutions of  $\frac{d}{dx}L(x) = 0$  on  $[0, \infty)$ .

First, the derivative of  $L(x)$  can be expanded as

$$\frac{d}{dx}L(x) = 1 - (1 - e^{-\theta r^\alpha x})^n - n\theta r^\alpha x e^{-\theta r^\alpha x} (1 - e^{-\theta r^\alpha x})^{n-1}, \quad (14)$$

which is strictly decreasing on  $[0, \frac{1}{\theta r^\alpha}]$  due to the monotonicity of  $e^{-\theta r^\alpha x}$ , the monotonicity of  $x e^{-\theta r^\alpha x}$  on  $[0, \frac{1}{\theta r^\alpha}]$ . Thus there cannot be more than one solution of  $\frac{d}{dx}L(x) = 0$  on  $[0, \frac{1}{\theta r^\alpha}]$ .

Second,  $\frac{d}{dx}L(x) = 0$  can be rearranged as  $1 - (1 - n\theta r^\alpha x)e^{-\theta r^\alpha x} = (1 - e^{-\theta r^\alpha x})^{1-n}$ , where the left side is a strictly increasing function of  $x$  for  $x > \frac{n-1}{n\theta r^\alpha}$  and the right side is a decreasing function of  $x$ . Thus, there can be at most one solution of  $\frac{d}{dx}L(x) = 0$  on  $[\frac{1}{\theta r^\alpha}, \infty) \in (\frac{n-1}{n\theta r^\alpha}, \infty)$ .

Third, there can be only an odd number of zero crossings of  $\frac{d}{dx}L(x)$  on  $[0, \infty)$  since  $\frac{d}{dx}L(x)$  is continuous,  $\lim_{x \rightarrow 0^+} \frac{d}{dx}L(x) > 0$ , and  $\frac{d}{dx}L(x)$  approaches zero from below as  $x \rightarrow \infty$ . Combining with results above, we conclude there is a unique zero crossing of  $\frac{d}{dx}L(x)$  on  $[0, \infty)$ . ■

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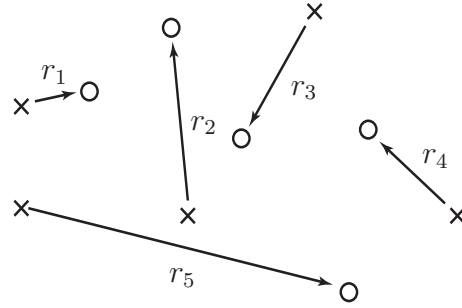


Fig. 1: A collection of links with random distances. Transmitters are denoted by  $\times$  and receivers are denoted by  $\circ$ . The distances  $r_k, k \in [5]$ , are iid drawn from some distribution  $f_R(x)$ .

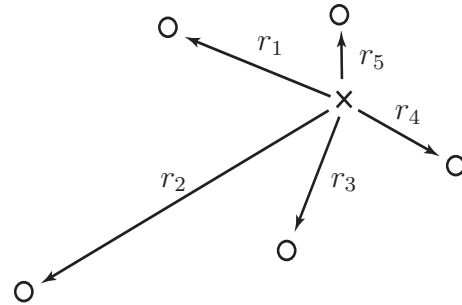


Fig. 2: Broadcast in wireless network. Transmitters are denoted by  $\times$  and receivers are denoted by  $\circ$ . The distances  $r_k, k \in [5]$ , are deterministic and known to the transmitter.

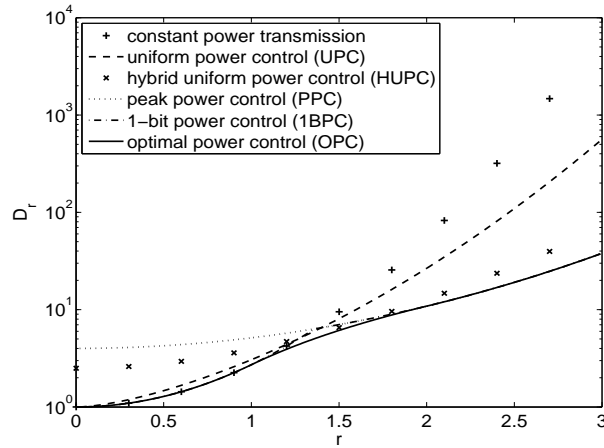


Fig. 3: Comparison of the conditional local delay for different power control schemes. Here,  $P_{\max} = 4$ ,  $\theta = 1$ ,  $\alpha = 2$ .

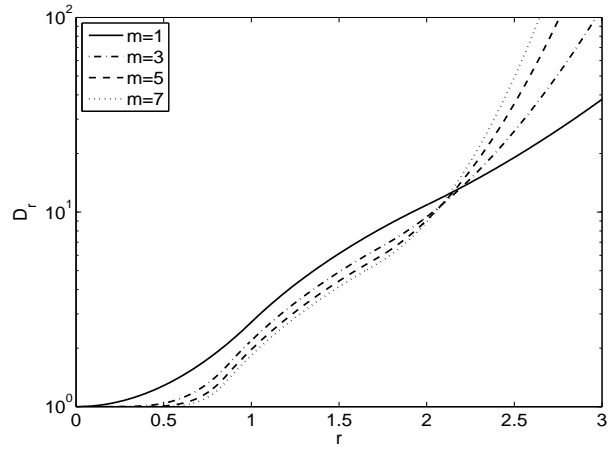


Fig. 4: Minimum conditional local delay for Nakagami- $m$  fading, where  $P_{\max} = 4$ ,  $\theta = 1$ ,  $\alpha = 2$ .

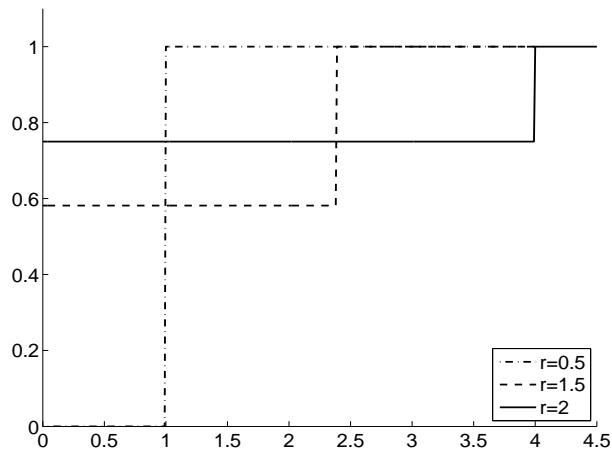


Fig. 5: Numerically obtained  $F_P^*(x)$  for Rician fading.  $P_{\max} = 4$ ,  $\theta = 1$ ,  $\alpha = 2$ ,  $K = 1$ .

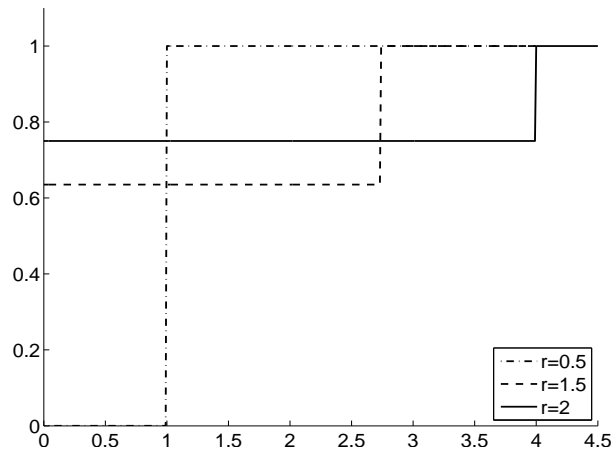


Fig. 6: Numerically obtained  $F_P^*(x)$  for lognormal fading.  $P_{\max} = 4$ ,  $\theta = 1$ ,  $\alpha = 2$ ,  $\sigma = 1$ .

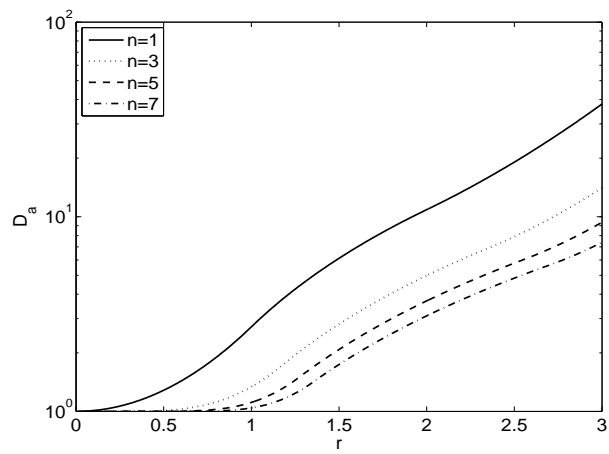


Fig. 7: Minimum local anycast delay in the Rayleigh fading case.  $P_{\max} = 4$ ,  $\theta = 1$ ,  $\alpha = 2$ .

