Spatial Outage Capacity of Poisson Bipolar Networks

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***Abstract*—We introduce a new notion of capacity, termed *spatial outage capacity* (SOC), which is defined as the maximum density of concurrently active links that have a success probability greater than a predefined threshold. For Poisson bipolar networks, we provide exact analytical and approximate expressions for the density of successful transmissions under each link's outage constraint. In the high-reliability regime, we obtain an exact closed-form expression of the SOC, which gives the asymptotic scaling behavior of the SOC.**

***Index Terms*—Bipolar network, interference, Poisson point process, SIR, spatial outage capacity, stochastic geometry**

I. Introduction

A. Motivation

Stochastic geometry provides the mathematical tools to study wireless networks where node locations are modeled by a random point process. By *spatial averaging*, stochastic geometry allows us to evaluate the statistics of the wireless network such as interference distribution and average success probability. In this approach, the performance evaluation is usually done with respect to the *typical user or typical link*. This approach leads to the tractable performance metrics for the given network parameters, which in turn, allows us to choose network parameters that optimize the network performance.

While such a macroscopic view based on spatial averaging is important, it does not give fine-grained information about the network, such as the link-wise performance characterized by the link success probability (or outage probability). Due to random node locations, the success probability of each link is a random variable that depends on path loss, fading, and interferer locations. In fact, as Fig. 1 shows, for the same average success probability, depending on the network parameters, the distribution of link success probabilities in a Poisson bipolar network varies significantly. Thus the link success probability distribution is a much more comprehensive metric than the average success probability that is usually considered.

In this regard, we introduce a new notion of capacity, termed *spatial outage capacity* (SOC). The SOC is defined as the maximum density of concurrently active links that have a success probability greater than a certain threshold. Thus the definition of SOC is based on the distribution of link success probabilities across the network.

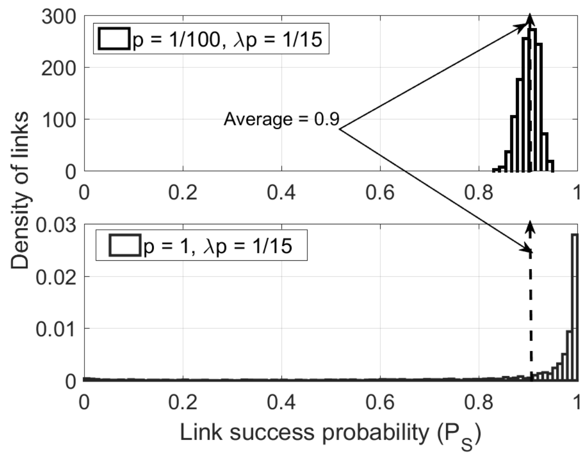

Fig. 1. The distribution of link success probabilities in a Poisson bipolar network obtained via Monte-Carlo simulations for $p = 1/100$ and $p = 1$. Both cases have the same average success probability of $p_{\rm s} = 0.9$, but we see a different distribution of link success probabilities for different values of the pair density λ and transmit probability p. For $p = 1/100$, the link success probabilities lie between 0.83 and 0.95 (concentrated around their average), while for $p = 1$, they are spread more widely. The SIR threshold $\theta = -10$ dB, distance between a transmitter and its receiver $R = 1$, path loss exponent $\alpha = 4$, and $\lambda p = 1/15$.

Definition 1 (Spatial outage capacity)**.** *For a stationary and ergodic point process model, the SOC is*

$$S(\theta, x) \triangleq \sup_{\lambda, p} \ \lambda p \eta(\theta, x), \tag{1}$$

where $\theta \in \mathbb{R}^+$, $x \in (0,1)$, $\lambda > 0$, and $p \in (0,1]$.

In (1), λ is the total intensity of potential transmitters, p is the fraction of nodes that are active at a time, and $\eta(\theta, x)$ is the fraction of links in each realization of the point process that have a signal-to-interference ratio (SIR) greater than θ with probability at least x. Due to the ergodicity of the point process, $\eta(\theta, x)$ is also the probability that the typical link has a success probability of at least x. We can further denote the density of concurrently active links that have a success probability greater than x as

$$\tau(\theta, x) \triangleq \lambda p \eta(\theta, x), \tag{2}$$

which results in

$$S(\theta, x) = \sup_{\lambda, p} \ \tau(\theta, x). \tag{3}$$

The SOC provides a useful practical measure which tells us the maximum number of active users per unit area a wireless network can handle at a time while guaranteeing minimum reliability (i.e., success probability) for *each active link*.

B. Background

The fraction $\eta(\theta, x)$ in (1) is termed *meta distribution* of the SIR in [1]. Given the point process, the meta distribution is the complementary cumulative distribution function (ccdf) of the link success probability $P_{\rm s}$ averaged only over the fading and the medium access scheme (if random) of interferers. Hence $P_{\rm s}$ is a random variable given as

$$P_{\rm s}(\theta) \triangleq \mathbb{P}(\mathsf{SIR} > \theta \mid \Phi), \tag{4}$$

where θ is the SIR threshold, and the meta distribution is given by

$$\eta(x, \theta) \triangleq \mathbb{P}^{!{\rm t}}(P_{\rm s}(\theta) > x), \tag{5}$$

where $\mathbb{P}^{!{\rm t}}$ denotes the reduced Palm probability, given the presence of an active transmitter at the prescribed location, and the SIR is calculated at its dedicated receiver. Given the random locations of nodes, $\eta(x, \theta)$ is the probability that the link under consideration has the success probability at least x. The standard success probability follows as

$$p_{\rm s}(\theta) = \mathbb{P}(\mathsf{SIR} > \theta) = \mathbb{E}^{!{\rm t}}(P_{\rm s}(\theta)) = \int_0^1 \eta(\theta, x) {\rm d}x. \tag{6}$$

C. Contributions

The contributions of the paper are as follows:

- We introduce a new notion of capacity—spatial outage capacity—based on the link success probability distribution.
- For the Poisson bipolar network with ALOHA, we evaluate the density of concurrently active successful transmissions under the reliability constraint.
- We show the trade-off between the density of active transmissions and the fraction of successful transmissions.
- In the high-reliability regime where the target outage probability is close to 0, we give a closed-form expression of the SOC, and prove that the SOC is achieved at $p = 1$.

D. Related Work

For Poisson bipolar networks, the success probability of the typical link $p_{\rm s}$ is studied in [2] and [3]. The notion of *transmission capacity* is introduced in [4], which is defined as the maximum density of successful transmissions provided the outage probability *of the typical user* stays below a predefined threshold ϵ. While the results obtained in [4] are certainly important, the transmission capacity does not represent the actual maximum density of successful transmissions for the target outage probability, as claimed in [4], since the metric implicitly assumes that each link is typical. We illustrate this through the following example.

Example 1. *For Poisson bipolar networks with SIR threshold* $\theta = -10$ dB*, distance between a transmitter and its receiver* $R = 1$*, path loss exponent* $\alpha = 4$*, and target outage probability* $\epsilon = 0.1$*, the transmission capacity is* 0.0608 *(see [5, (4.15)]). At this value of transmission capacity, for* $p = 1$*, actually only* 82% *of active transmissions achieve an outage probability smaller than* 0.1*. On the other hand, by considering the fraction of active nodes that meet the target outage probability of* 0.1 *and optimizing over* λ *and* p*, we obtain the actual maximum density of concurrently active transmissions that have outage probability smaller than* ϵ*, which is the SOC introduced in this paper and has a value of* 0.09227.

A version of the transmission capacity that considers the link success probability distribution is introduced in [6], but it does not consider a medium access control (MAC) scheme, i.e., all nodes always transmit ($p = 1$). Here, we consider the general case with the transmit proabability $p \in (0, 1]$. The choice of p is important as it significantly affects the link success probability distribution as shown in Fig. 1. Also, the transmission capacity defined in [6] requires a certain fraction of users to satisfy the outage constraint. Thus their transmission capacity definition corresponds to the minimum fraction of users that satisfy the outage constraint. Such constraint is not required by our definition of SOC, and the SOC corresponds to the actual density of users that satisfy the outage constraint.

The meta distribution $\eta(\theta, x)$ for Poisson bipolar networks and cellular networks is studied in [1], where a closed-form expression for the moments of $P_{\rm s}$ is obtained, and an exact integral expression and simple bounds for $\eta(\theta, x)$ are provided. A key result in [1] is that, for constant transmitter density λp, as the Poisson bipolar network becomes very dense ($\lambda \to \infty$) with a very small transmit probability ($p \to 0$), all links have the same success probability, which is same as the average success probability $p_{\rm s}$.

II. Network Model

We consider the Poisson bipolar network model in which the locations of transmitters form a homogeneous Poisson point process (PPP) $\Phi \subset \mathbb{R}^2$ with density λ [7, Def. 5.8]. Each transmitter has a dedicated receiver at a distance R in a uniformly random direction. In a time slot, each node in Φ independently transmits at unit power with probability p and stays silent with probability $1-p$. Thus, the active transmitters form a homogeneous PPP with density λp. We consider a standard power law path loss model with path loss exponent α. We assume that a channel is subject to independent Rayleigh fading with channel power gains as i.i.d. exponential random variables with mean 1.

We focus on the interference-limited case, where the received SIR is a key quantity of interest. The average success probability $p_{\rm s}$ of the typical link depends on the SIR. From [3], [7], [8], it is known that

$$p_{\rm s}(\theta) = \exp\left(-\lambda p C \theta^{\delta}\right), \tag{7}$$

where $C \triangleq \pi R^2 \Gamma(1+\delta)\Gamma(1-\delta)$ with $\delta \triangleq 2/\alpha$.

III. Spatial Outage Capacity

A. Exact Formulation

Observe from Def. 1 that, the SOC depends on $\eta(\theta, x) = \mathbb{P}(P_{\rm s}(\theta) > x \mid \Phi)$. Let $M_b(\theta)$ denote the bth moment of $P_{\rm s}(\theta)$, i.e.,

$$M_b(\theta) \triangleq \mathbb{E}\left((P_{\rm s}(\theta))^b\right). \tag{8}$$

Then the average success probability is $p_{\rm s}(\theta) \equiv M_1(\theta)$.

From [1, Thm. 1], we can express $M_b(\theta)$ as

$$M_b(\theta) = \exp\left(-\lambda C \theta^\delta D_b(p,\delta)\right), \quad b \in \mathbb{C}, \tag{9}$$

where

$$D_b(p,\delta) \triangleq \sum_{k=1}^{\infty} \binom{b}{k}\binom{\delta-1}{k-1} p^k, \quad p, \delta \in (0,1], \tag{10}$$

which is termed *diversity polynomial* in [9]. For $b = 1$ (first moment), $D_1(p,\delta) = p$, and we get the expression of $p_{\rm s}(\theta)$ as in (7). We can also express $D_b(p,\delta)$ using the Gaussian hypergeometric function ${}_2F_1$ as

$$D_b(p,\delta) = pb\,{}_2F_1(1-b, 1-\delta; 2; p). \tag{11}$$

Using the Gil-Pelaez theorem [10], the exact expression of $\tau(\theta,x) = \lambda p \eta(\theta,x)$ can be obtained in integral form from that of $\eta(\theta,x)$ given in [1, Cor. 3] as follows.

$$\tau(\theta,x) = \frac{\lambda p}{2} - \frac{\lambda p}{\pi}\int_0^\infty \frac{\sin(u\log x + \lambda C\theta^\delta \Im(D_{ju}))}{u e^{\lambda C \theta^\delta \Re(D_{ju})}} {\rm d}u, \tag{12}$$

where $j \triangleq \sqrt{-1}$, $D_{ju} = D_{ju}(p,\delta)$ is given by (10), while $\Re(z)$ and $\Im(z)$ are the real and imaginary parts of the complex number z, respectively. Note that the SOC is obtained by maximizing (12) over λ and p.

B. Approximation with Beta Distribution

We can accurately approximate (12) in a semi-closed form using beta distribution, which is indeed a good approximation (and a simple one) as shown in [1]. The rationale behind such approximation is that the support of the link success probability $P_{\rm s}$ is $[0,1]$, making the beta distribution a natural choice. With the beta distribution approximation, from [1, Sec. II.F], τ is approximated as

$$\tau \approx \lambda p \left(1 - I_x\left(\frac{\mu\beta}{1-\mu}, \beta\right)\right), \tag{13}$$

where $I_x(y,z) \triangleq \int_0^x t^{y-1}(1-t)^{z-1}{\rm d}t / B(y,z)$ is the regularized incomplete beta function with $B(\cdot,\cdot)$ denoting beta function, $\mu = M_1$, and $\beta = (M_1 - M_2)(1-M_1)/(M_2 - M_1^2)$.

The advantage of the beta approximation is the faster computation of τ compared to the exact expression without losing much accuracy [1, Tab. I, Fig. 4] (also see Fig. 6 of this paper). In general, it is difficult to obtain the SOC analytically due to the form of τ given in (12) and (13). But we can obtain the SOC numerically with ease. We can also gain useful insights considering some specific scenarios, on which we focus in following three subsections of the paper.

C. Constrained SOC

1) Constant λp: For constant λp (or, equivalently, a $p_{\rm s}$), we now study how the density of successful transmissions τ with reliability constraint x behaves in an ultra-dense network. Given θ, R, α, and x, this case is equivalent to asking how τ varies as $\lambda \to \infty$ while letting $p \to 0$ for constant transmitter density λp (constant $p_{\rm s}$).

Theorem 1 ($p \to 0$ for constant λp)**.** *Let $\nu = \lambda p$. Then, for constant ν while letting $p \to 0$, the SOC constrained on the density of concurrent transmissions is*

$$\mathcal{S}_\nu = \begin{cases} \lambda p, & \text{if } x < p_{\rm s} \\ 0, & \text{if } x > p_{\rm s}. \end{cases} \tag{14}$$

Proof: Applying Chebyshev's inequality to (5), for $x < p_{\rm s} = M_1$, we have

$$\eta(\theta,x) > 1 - \frac{\mathrm{var}(P_{\rm s}(\theta))}{(x-M_1)^2}, \tag{15}$$

where $\mathrm{var}(P_{\rm s}(\theta)) = M_2 - M_1^2$. From [1, Cor. 1], we know that $\mathrm{var}(P_{\rm s}(\theta)) \to 0$ as $p \to 0$ for constant ν. Thus the lower bound in (15) approaches 1, leading to $\eta(\theta,x) = 1$, in turn, resulting in the SOC constrained on the density of concurrent transmissions, which is equal to $S_\nu = \lambda p$.

On the other hand, for $x > M_1$,

$$\eta(\theta,x) \le \frac{\mathrm{var}(P_{\rm s}(\theta))}{(x-M_1)^2}. \tag{16}$$

As we let $p \to 0$ for constant ν, the upper bound in (16) approaches 0, leading to $\eta(\theta,x) = 0$, in turn, resulting in the SOC constrained on the density of concurrent transmissions, which is equal to $S_\nu = 0$. ■

In fact, as $\mathrm{var}(P_{\rm s}(\theta)) \to 0$, the cdf (in turn ccdf) of $P_{\rm s}(\theta)$ approaches a step function, and $\eta(\theta,x)$ (i.e., ccdf of $P_{\rm s}(\theta)$) leaps from 1 to 0 at the mean of $P_{\rm s}(\theta)$, i.e., $p_{\rm s}$. This behavior justifies (14).

2) $\lambda p \to 0$: For $\lambda p \to 0$, τ increases linearly, which we prove in the next theorem.

Theorem 2 (τ as $\lambda p \to 0$)**.** *As $\lambda p \to 0$,*

$$\tau \sim \lambda p, \quad \lambda p \to 0. \tag{17}$$

Proof: As $\lambda p \to 0$, the mean $M_1 = p_{\rm s}$ approaches 1. Thus the variance of $P_{\rm s}$ approaches 0 as $\mathrm{var}(P_{\rm s}) = M_1^2(M_1^{p(\delta-1)} - 1)$. Since $x \in (0,1)$, $x < M_1$ as $\lambda p \to 0$. Using Chebyshev's inequality for $x < M_1$ as in (15) and substituting $\mathrm{var}(P_{\rm s}) = 0$ in it, the lower bound in (15) approaches 1, leading to $\eta(\theta,x) = 1$. Thus, $\tau \sim \lambda p$ as $\lambda p \to 0$, having slope 1 with respect to λp. ■

Remark 1. *The case $\lambda p \to 0$ can be interpreted in two ways: 1) $\lambda \to 0$ for constant p and 2) $p \to 0$ for constant λ. Thm. 2 is valid for both cases.*

Thm. 2 can be understood as follows. As $\lambda p \to 0$, the density of active transmitters is very small. Thus each transmission succeeds with high probability, and η is 1 in

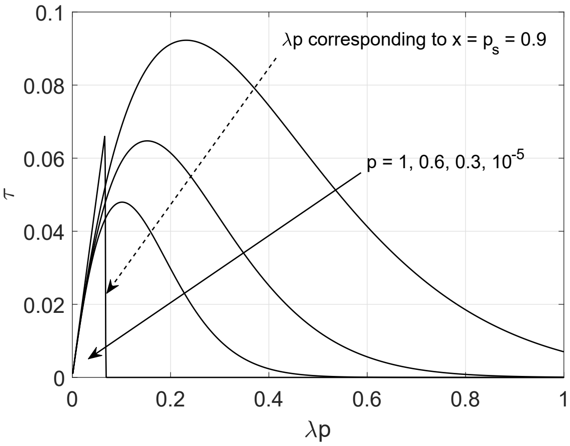

Fig. 2. The density of successful transmissions τ as depicted in (2) for different values of transmit probability p for $\theta = -10$ dB, $R = 1$, $\alpha = 4$, and $x = 0.9$. Observe that the slope of τ is one for small λp.

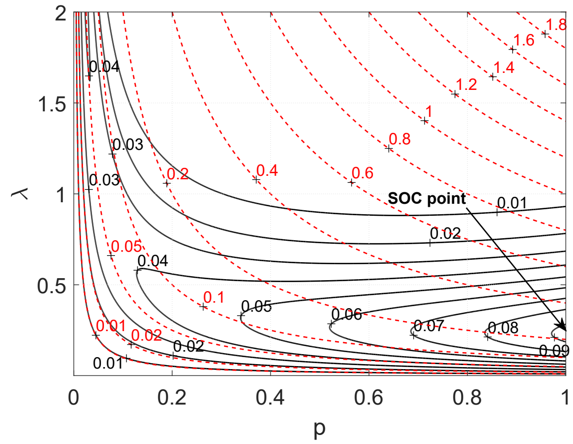

Fig. 3. Counter plots of τ and the product λp for $\theta = -10$ dB, $R = 1, \alpha = 4$, and $x = 0.9$. The solid lines represent the contour curves for τ and the dashed lines represent the contour curves for λp. The numbers in "black" and "red" indicate the contour levels for τ and λp, respectively. The "SOC point" corresponds to the supremum of τ, and gives the SOC equal to $S = 0.09227$. The values of λ and p at the SOC point are 0.23 and 1, respectively, and the corresponding average success probability is $p_s = 0.6984$.

this regime, increasing the density of successful transmissions linearly with λp.

Remark 2. *As p gets smaller, the probability that a node makes a transmission attempt in a slot reduces, increasing the delay. Since the mean delay is larger than $1/p$, it would get large for small values of p. Thus, a delay constraint prohibits p from getting too small. The case of constant p is relevant since it can be interpreted as a delay constraint.*

Fig. 2 illustrates Thm. 2. Also, observe that, as $p \to 0$ ($p = 10^{-5}$ in Fig. 2), τ increases linearly with λp till the product λp reaches to the value that corresponds to $p_s = x = 0.9$, and then leaps to 0. This behavior is in accordance with Thm. 1. In general, as λp increases, τ increases first and then decreases after a tipping point. This is due to the two opposite effects of λp on τ. Because of the term λp in the expression of τ (see (2)), the increase in λp increases τ first, but contributes to more interference at the same time, which reduces the fraction $\eta(\theta, x)$ of links that have reliability at least x, thereby reducing τ.

The contour plot Fig. 3 visualizes the trade-off between λp and $\eta(\theta, x)$. The contour curves for small products λp run nearly parallel to those for τ, indicating that $\eta(\theta, x)$ is close to 1. Contrary, for large values of λp, the decrease in $\eta(\theta, x)$ dominates τ. Specifically the contour curves for $\lambda p = 0.01$ and $\lambda p = 0.02$ match almost exactly with those for $\tau = 0.01$ and $\tau = 0.02$, respectively. This behavior is in accordance with Thm. 2. Also, notice that, for larger values of λ ($\lambda > 0.4$ for Fig. 3), τ, i.e., the density of successful transmissions under the reliability constraint, first increases and then decreases with the increase in p. This behavior is due to the following trade-off in p. For a small p, there are few active transmitters in the network per unit area, but a higher fraction of transmissions are successful. On the other hand, a larger p means more active transmitters per unit area, but also a higher interference which reduces the fraction of

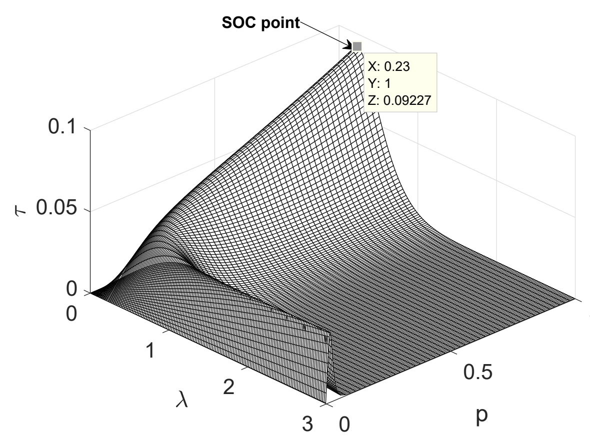

Fig. 4. Three-dimensional plot of τ corresponding to the contour plot Fig. 3.

successful transmissions. For small λ ($\lambda < 0.4$), the increase in the density of active transmitters dominates the increase in interference, and τ increases monotonically with p. An interesting observation in Fig. 3 is that the average success probability p_s at the SOC point is 0.6984 for 90% reliability. The three-dimensional plot corresponding to the contour plot Fig. 3 is shown in Fig. 4.

D. High-reliability Regime

In the high-reliability regime, the reliability threshold is $x \to 1$. In terms of the outage probability, the outage probability $\epsilon = 1 - x$ of a link is close to 0, i.e., $\epsilon \to 0$.

In this section, we investigate the behavior of τ and SOC in the high-reliability regime. In this regard, we first state a simplified version of de Bruijn's Tauberian theorem (see [11, Thm. 4.12.9]) which allows a convenient formulation of

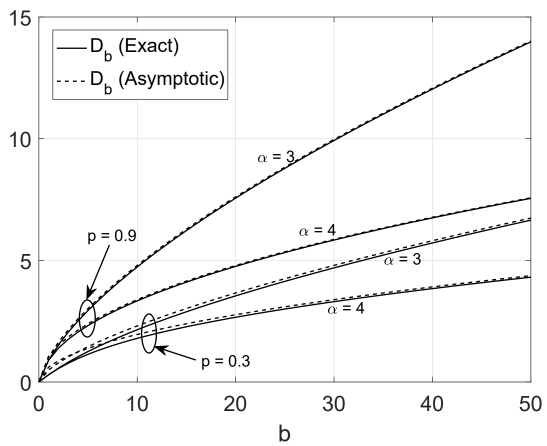

Fig. 5. The solid lines represent the exact $D_b(p,\delta)$ as in (10), while the dashed lines represent asymptotic form of $D_b(p,\delta)$ as in (21). Observe that (21) is a good approximation of (10) and is asymptotically tight.

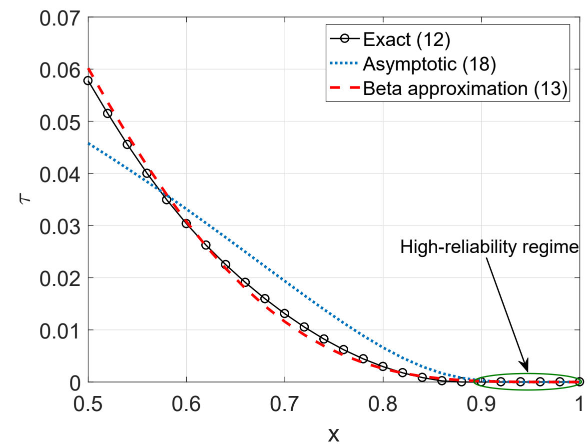

Fig. 6. The solid line with marker 'o' represents the exact expression of τ as in (12), the dotted line represents the asymptotic expression of τ given by (18) as $\epsilon \to 0$, and the dashed line represents the approximation by the beta distribution given by (13). Observe that (18) is asymptotically tight, while the beta approximation is good. $\theta = 0$ dB, $R = 1, \alpha = 4, \lambda = 1/2$, and $p = 1/3$.

$\eta(\theta, 1-\epsilon) = \mathbb{P}(P_{\mathrm{s}}(\theta) > 1-\epsilon)$ as $\epsilon \to 0$, in terms of the Laplace transform. The following simplified version of de Bruijn's Tauberian theorem suffices for our purposes.

Theorem 3 (de Bruijn's Tauberian theorem [12, Thm. 1])**.** *For a non-negative random variable Y, the Laplace transform $\mathbb{E}[\exp(-sY)] \sim \exp(rs^u)$ for $s \to \infty$ is equivalent to $\mathbb{P}(Y \leq \epsilon) \sim \exp(q/\epsilon^v)$ for $\epsilon \to 0$, when $1/u = 1/v + 1$ (for $u \in (0,1)$ and $v > 0$), and the constants r and q are related as $|ur|^{1/u} = |vq|^{1/v}$.*

Theorem 4. *For $\epsilon \to 0$, the density of successful transmissions $\tau(\theta, 1-\epsilon)$ is asymptotically equal to*

$$\tau(\theta, 1-\epsilon) \sim \lambda p \exp\left(-\left(\frac{\theta p}{\epsilon}\right)^{\kappa} \frac{(\delta\lambda C')^{\frac{\kappa}{\delta}}}{\kappa}\right), \quad \epsilon \to 0, \quad (18)$$

where $\kappa = \frac{\delta}{1-\delta} = \frac{2}{\alpha-2}$ and $C' = \pi R^2 \Gamma(1-\delta)$.

Proof: From (2) and (5),

$$\tau(\theta, 1-\epsilon) = \lambda p \mathbb{P}(P_{\mathrm{s}}(\theta) > 1-\epsilon). \quad (19)$$

Using Thm. 3, we first prove that

$$\mathbb{P}(P_{\mathrm{s}}(\theta) > 1-\epsilon) \sim \exp\left(-\left(\frac{\theta p}{\epsilon}\right)^{\kappa} \frac{(\delta\lambda C')^{\frac{\kappa}{\delta}}}{\kappa}\right), \quad \epsilon \to 0, \quad (20)$$

which gives the desired result in (18). Note that for $b \in \mathbb{R}$,

$$D_b(p,\delta) \sim p^{\delta} b^{\delta} / \Gamma(1+\delta), \quad b \to \infty, \quad (21)$$

where $D_b(p,\delta)$ is given by (10). In Fig. 5, we illustrate how quickly D_b approaches the asymptote.

Let $Y = -\log P_{\mathrm{s}}$. The Laplace transform of Y is $\mathbb{E}(\exp(-sY)) = \mathbb{E}(P_{\mathrm{s}}^s) = M_s$. Here, M_s is the sth moment of P_{s} given by (8). Using (9) and (21), we have

$$M_s \sim \exp\left(-\frac{\lambda C(\theta p)^{\delta} s^{\delta}}{\Gamma(1+\delta)}\right), \quad s \to \infty.$$

From Thm. 3, we have $r = -\frac{\lambda C(\theta p)^{\delta}}{\Gamma(1+\delta)}$, $u = \delta$, $v = \delta/(1-\delta) = \kappa$, and thus

$$q = \frac{1}{\kappa} (\delta\lambda C')^{\frac{\kappa}{\delta}} (\theta p)^{\kappa},$$

where $C' = \pi R^2 \Gamma(1-\delta)$.

Using Thm. 3, we can now write

$$\begin{aligned}\mathbb{P}(Y \leq \epsilon) &= \mathbb{P}(P_{\mathrm{s}}(\theta) \geq \exp(-\epsilon)) \\ &\sim \exp\left(-\frac{(\theta p)^{\kappa} (\delta\lambda C')^{\frac{\kappa}{\delta}}}{\kappa\epsilon^{\kappa}}\right), \quad \epsilon \to 0,\end{aligned}$$

which equals (20) since $\exp(-\epsilon) \sim 1-\epsilon$ as $\epsilon \to 0$, and the desired result in (18) follows from (19). ■

For the special case of $p = 1$ (all transmitters are active), $\mathbb{P}(P_{\mathrm{s}}(\theta) \geq 1-\epsilon)$ as in (18) simplifies to

$$\mathbb{P}(P_{\mathrm{s}}(\theta) \geq 1-\epsilon) \sim \exp\left(-\frac{(\delta\lambda C'\theta^{\delta})^{\frac{\kappa}{\delta}}}{\kappa\epsilon^{\kappa}}\right), \quad \epsilon \to 0,$$

which is in agreement with [6, Thm. 2], which was derived in a less direct way than Thm. 4. Fig. 6 shows the tightness of (18) in the high-reliability regime and also the accuracy of the beta approximation given by (13).

We now investigate the scaling of $S(\theta, 1-\epsilon)$ in the high-reliability regime.

Corollary 1 (SOC in high-reliability regime)**.** *For $\epsilon \to 0$, the SOC is asymptotically equal to*

$$S(\theta, 1-\epsilon) \sim \left(\frac{\epsilon}{\delta\theta}\right)^{\delta} \frac{\exp(-(1-\delta))}{\pi R^2 \Gamma(1-\delta)}, \quad (22)$$

and it is achieved at $p = 1$.

Proof: Let us denote

$$\xi(\theta, \epsilon) = \left(\frac{\theta}{\epsilon}\right)^{\kappa} \frac{(\delta C')^{\kappa/\delta}}{\kappa}.$$

From (18), we can then write

$$\tau(\theta, 1-\epsilon) \sim \lambda p \exp\left(-\lambda^{\kappa/\delta} p^\kappa \xi(\theta,\epsilon)\right), \quad \epsilon \to 0.$$

Thus we have

$$S(\theta, 1-\epsilon) \sim \sup_{\lambda, p} \; f(\lambda, p), \quad \epsilon \to 0,$$

where $f(\lambda, p) = \lambda p \exp(-\lambda^{\kappa/\delta} p^\kappa \xi(\theta,\epsilon))$. First, fix $p \in (0, 1]$. As $\epsilon \to 0$, we can then write

$$\frac{\partial f}{\partial \lambda} = \underbrace{p \exp\left(-\lambda^{\kappa/\delta} p^\kappa \xi(\theta,\epsilon)\right)}_{>0} \left[1 - \frac{\kappa \xi(\theta,\epsilon)}{\delta} \lambda^{\kappa/\delta} p^\kappa\right].$$

Setting $\frac{\partial f}{\partial \lambda} = 0$, we obtain the critical point as

$$\lambda_0 = \left(\frac{\delta}{\xi(\theta,\epsilon)\kappa p^\kappa}\right)^{\delta/\kappa}.$$

Note that, for fixed p, f is strictly increasing for $\lambda \in (0, \lambda_0]$ and strictly decreasing for $\lambda > \lambda_0$. Hence we have

$$\begin{aligned} S(\theta, 1-\epsilon) &\sim \sup_{p} \; f(\lambda_0, p), \quad \epsilon \to 0, \\ &= \sup_p \; p^{1-\delta} \left(\frac{\delta}{\kappa \xi(\theta,\epsilon)}\right)^{\delta/\kappa} \exp\left(-\frac{\delta}{\kappa}\right). \end{aligned}$$

Observe that $f(\lambda_0, p)$ monotonically increases with p, and thus attains the maximum at $p = 1$. Thus the SOC is achieved at $p = 1$ and is given by (22) after simplification. ∎

Remark 3. *From Cor. 1, we observe that, as $\epsilon \to 0$, the exponents of θ and ϵ are the same. In the high-reliability regime, the SOC scales in ϵ similar to the transmission capacity defined in [6], while the transmission capacity defined in [4] scales linearly in ϵ (see [5, (4.29)]).*

For $\alpha = 4$, the expression of SOC in (22) simplifies to

$$S(\theta, 1-\epsilon) \; \sim \left(\frac{2\epsilon}{\theta e}\right)^{\frac{1}{2}} \frac{1}{\pi^{\frac{3}{2}} R^2}.$$

For $\alpha = 4$, Fig. 7 plots τ versus λ and p for $x = 0.993$ (high-reliability regime). In this case, the SOC is achieved at $p = 1$ which is in accordance with Cor. 1.

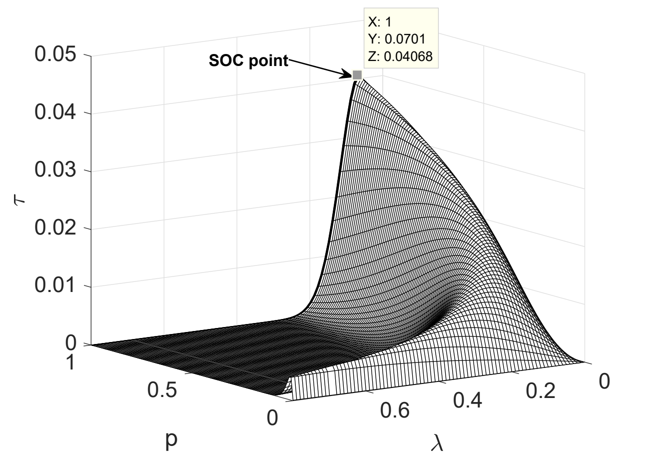

Fig. 7. Three-dimensional plot of τ for $x = 0.993, \theta = -10$ dB, $R = 1$, and $\alpha = 4$. Observe that $p = 1$ achieves the SOC. The average success probability p_s at the SOC point is 0.8964.

IV. Conclusions

The first main contribution is a new notion of capacity, termed spatial outage capacity (SOC), which is the maximum density of concurrently active intermissions while ensuring a certain reliability. The SOC gives fine-grained information about the network compared to the transmission capacity whose framework is based on the average success probability. The SOC has applications in wireless networks with strict reliability constraints.

Secondly, for Poisson bipolar networks with ALOHA, we have obtained an exact analytical expression and a simple approximation for the density τ of concurrently active links that have a success probability greater than a certain threshold. The SOC can be easily calculated numerically as the supremum of τ obtained by optimizing over the density λ and the transmit probability p. When constrained on the density of concurrent transmissions, i.e., for constant λp, while letting $p \to 0$, the supremum of τ is equal to the product λp if the reliability threshold is smaller than the average success probability and zero if the reliability threshold is larger than the average success probability. In the high-reliability regime where the outage probability of a link goes to 0, we give a closed-form expression of the SOC and show that $p = 1$ achieves the SOC.

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