EE 87021: Advanced Topics in Random Wireless Networks

Summary of Lecture 14 (Oct. 13, 2008)

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1 Executive Summary

Starting with the definitions of the reduced Palm distribution and the reduced Campbell measure, we discussed some of their properties. We proved Slivnyak's theorem, which states that for a Poisson point process, the distribution of the original process is equal to its reduced Palm distribution. We also revisited second moment measures for stationary point processes and introduced the J- and K-functions.

2 Palm Distributions and Relative Frequencies

Example: If $Y = \{\varphi \in \mathbb{N} : \Phi(\{o\}) > 0\}$, what is the Palm probability $P_o(Y)$?

$$P_o(Y) = \frac{\lambda(Y)}{\lambda} = \frac{\mathbb{E}\left(\Phi_Y\left([0,1]^d\right)\right)}{\lambda} = \frac{\mathbb{E}\#\{n : x_n \in [0,1]^d, \mathbf{1}_Y(\Phi_{-x_n})\}}{\lambda} = 1.$$

Note that if Φ is ergodic, the Palm probability can be alternatively calculated $\forall Y \in \mathcal{N}$ as

$$P_o(Y) = \lim_{m\to\infty} \frac{\Phi_Y([-m,m]^d)}{\Phi([-m,m]^d)}.$$

3 Reduced Palm Distributions

It is useful to consider the *reduced Campbell measure* $\mathcal{C}^!$ which is defined as follows.

$$\begin{aligned}\int_{\mathbb{N}} \sum_{x\in\varphi} f(x, \varphi \setminus \{x\}) P(\mathrm{d}\varphi) &= \int_{\mathbb{N}} \sum_{x\in\varphi} f(x, \varphi - \delta_x) P(\mathrm{d}\varphi) \\ &= \int_{\mathbb{R}^d\times\mathbb{N}} f(x,\varphi)\mathcal{C}^!(\mathrm{d}(x,\varphi)). \end{aligned} \quad (1)$$

As the Campbell measure $\mathcal{C}$ is related to the Palm distribution, $\mathcal{C}^!$ is related to the *reduced Palm distribution*.

Definition: The reduced Palm distribution $P_o^!(Y)$ is defined as $P_o^!(Y) = \mathbb{P}(\Phi \setminus \{o\} \in Y || o)$. In the stationary case,

$$P_o^!(Y) = \int_{\mathbb{N}} \sum_{x\in\varphi\cap B} \frac{\mathbf{1}_Y\left((\varphi_{-x}) \setminus \{o\}\right) P(\mathrm{d}\varphi)}{\lambda|B|}. \quad (2)$$

Note that the nearest neighbor distribution function D can be expressed via P_o, or $P_0^!$, as

$$\begin{aligned} D(r) &= 1 - P_o(\varphi \in \mathbb{N} : \varphi(b(o,r) = 1)) \\ &= 1 - P_o^!(\varphi \in \mathbb{N} : \varphi(b(o,r) = 0)). \end{aligned}$$

Definition: The *J-function* is defined as $J(r) = (1 - D(r)) / (1 - H_s(r))$, where $D(r)$ is the nearest neighbor distribution, and $H_s(r)$ is the *spherical contact distribution*. For the uniform PPP, $J(r) = 1$.

4 Palm distribution of a PPP

Theorem 4.1 (Slivnyak) *For a PPP with distribution P, the Palm distribution is*

$$P_x = P * \delta_{\delta_x}, \tag{3}$$

where δ_{δ_x} denotes the distribution of a PP that consists of a singleton $\{x\}$, and '$*$' denotes convolution of distributions (which corresponds to superposition of the PPs).

Note that (3) can be interpreted as

$$P_x(Y) = \mathbb{P}(\Phi \in Y || x) = \mathbb{P}(\Phi \cup \{x\} \in Y), \text{ for } Y \text{ in } \mathcal{N},$$

or equivalently as

$$\int_{\mathbb{N}} f(\varphi) P_x(\mathrm{d}\varphi) = \int_{\mathbb{N}} f(\varphi \cup \{x\}) P(\mathrm{d}\varphi)$$

for all measurable non-negative functions f. Written in terms of the reduced Palm distribution, Slivnyak's theorem states that for a PPP, $P_x^! = P \ \forall x$.

Proof: We need to show that the void probabilities for the two processes are equal, i.e., $P*\delta_{\delta_x}(V_K) = P_x(V_K) \ \forall$ compact K, where $V_K = \{\varphi \in \mathbb{N} : \varphi(K) = 0\}$.

Suppose A is an arbitrary bounded Borel set. Then,

$$\begin{aligned} \int_A P * \delta_{\delta_x}(V_K)\Lambda(\mathrm{d}x) &= \int_{A\setminus K} P(V_K)\Lambda(\mathrm{d}x) = P(V_K)\Lambda(A \setminus K) \\ &= \mathbb{E}(\mathbf{1}\{\Phi(K) = 0\})\mathbb{E}\Phi(A \setminus K) \\ &= \mathbb{E}(\Phi(A \setminus K)\mathbf{1}\{\Phi(K) = 0\}) \\ &= \mathcal{C}((A \setminus K) \times V_K). \end{aligned}$$

But $\mathcal{C}((A \cap K) \times V_K) = \mathbb{E}(\Phi(A \cap K)\mathbf{1}\{\Phi(K) = 0\}) = 0$. Therefore,

$$\int_A P * \delta_{\delta_x}(V_K)\Lambda(\mathrm{d}x) = \mathcal{C}(A \times V_K) = \int_A P_x(V_K)\Lambda(\mathrm{d}x),$$

which establishes (3). □

Remarks:

- Clearly, P_o is never a stationary distribution, but $P_o^!$ can be stationary (Ex: for a stationary PPP). If the PP is motion-invariant, then its Palm distribution is isotropic.
- Consider a BPP on a set W with n nodes, Φ_n. We have $P_x^![\Phi_n] = P[\Phi_{n-1}]$.

5 Second moment measures for stationary PPs

For $\Phi \subset \mathbb{R}^d$, stationarity implies

$$\mathbb{E}[\Phi(A + v)\Phi(B + v)] = \mathbb{E}[\Phi(A)\Phi(B)], \quad \forall v \in \mathbb{R}^d. \tag{4}$$

Thus $\mu^{(2)}$ and $\alpha^{(2)}$ are invariant under simultaneous shifts. Applying a transformation $T(x, y) = (x, y-x)$, the simultaneous shift becomes a shift only in the first coordinate.

The image of $\alpha^{(2)}$ under the transformation is a measure μ, which is transation-invariant in the first coordinate. Therefore, $\mu = \lambda v_d \otimes \lambda\mathcal{K}$, where $\mathcal{K}$ is a measure on $\mathbb{R}^d$ called the *reduced second moment measure* of Φ.

Using the refined Campbell theorem, we have

$$\begin{aligned}\alpha^{(2)}(B_1 \times B_2) &= \mathbb{E}\left(\sum_{x_1,x_2\in\Phi}^{\neq} \mathbf{1}_{B_1}(x_1)\mathbf{1}_{B_2}(x_2)\right) \\ &= \int_{\mathbb{N}} \sum_{x\in\varphi} \mathbf{1}_{B_1}(x)\varphi(B_2 \setminus \{x\})P(\mathrm{d}\varphi) \\ &= \lambda \int_{\mathbb{R}^d}\int_{\mathbb{N}} \mathbf{1}_{B_1}(x)\varphi((B_2 - x)\setminus\{o\})P_o(\mathrm{d}\varphi)\mathrm{d}x. \qquad (5)\end{aligned}$$

We want to express this as

$$\alpha^{(2)}(B_1 \times B_2) = \lambda^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x)\mathbf{1}_{B_2}(x+h)\mathcal{K}(\mathrm{d}h)\mathrm{d}x = \lambda^2 \int_{B_1} \mathcal{K}(B_2 - x)\mathrm{d}x. \qquad (6)$$

Definition: Given that there is a point of the process at the origin, $\lambda\mathcal{K}(B)$ is the mean number of points in $B \setminus \{o\}$. Accordingly,

$$\lambda\mathcal{K}(B) = \int_{\mathbb{N}} \varphi(B\setminus\{o\})P_o(\mathrm{d}\varphi) = \int_{\mathbb{N}} \varphi(B)P_o^!(\mathrm{d}\varphi).$$

If $\exists$ a second-moment density $\rho^{(2)}(x,y)$, it only depends on the difference of the coordinates, $y - x$, and we can write

$$\lambda^2\mathcal{K}(B) = \int_B \rho^{(2)}(u)\mathrm{d}u.$$

6 Ripley's K-function

The K-function is defined as $K(r) = \mathcal{K}(b(o,r)), r \geq 0$. So, $\lambda K(r)$ is the mean number of points y of the process that satisfy $0 < \|y - x\| < r$ for a "typical" point x of the process.

For a uniform PPP $\Phi \subset \mathbb{R}^d$, $K(r) = c_d r^d$. Also, for all stationary PPs, $K(r) \sim c_d r^d, r \to \infty$.

Lemma 6.1 (Invariance of K under thinning) *Suppose Φ is a stationary process and Φ' is obtained by independently thinning the points of Φ. Then, the K-functions of Φ and Φ' are identical.*

7 Main Take-Aways

- The concept of reduced Palm distributions is useful for tackling several problems in wireless communications and networking. For example, it might be of interest to compute the mean interference at a specific node in the network. One then needs to condition of the fact that the node is present in the network, but however, it does not contribute towards the interference.

- The K-function has a meaningful interpretation for motion-invariant processes. Also, in general, it does not characterize PPs.

8 Sources

A. Baddeley, "Spatial Point Processes and their Applications", pp. 35-39.

D. Stoyan, W. S. Kendall and J. Mecke, "Stochastic Geometry and its Applications", pp. 113-120.