

THE CLOSED RANGE PROPERTY FOR $\bar{\partial}$ ON DOMAINS WITH PSEUDOCONCAVE BOUNDARY

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Dedicated to Professor Linda P. Rothschild

1. Introduction

In the Hilbert space approach, the closed range property for an unbounded closed operator characterizes the range of the operator. Thus it is important to know whether the range of an unbounded operator is closed. When the unbounded operator is the Cauchy-Riemann equation, the Hilbert space approach has been established by the pioneering work of Kohn [Ko1] for strongly pseudoconvex domains and by Hörmander [Hör] for pseudoconvex domain in \mathbb{C}^n or a Stein manifold. The following L^2 existence and regularity theorems for $\bar{\partial}$ on pseudoconvex domains in \mathbb{C}^n (or a Stein manifold) are well known.

Theorem (Hörmander [Hör1]). *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain. For any $f \in L^2_{(p,q)}(\Omega)$, where $0 \leq p \leq n$ and $1 \leq q < n$, such that $\bar{\partial}f = 0$ in Ω , there exists $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar{\partial}u = f$ and*

$$\int_{\Omega} |u|^2 \leq \frac{e\delta^2}{q} \int_{\Omega} |f|^2$$

where δ is the diameter of Ω .

This implies that the range of $\bar{\partial}$ is equal to the kernel of $\bar{\partial}$, which is closed since $\bar{\partial}$ is a closed operator. It also follows that the harmonic forms are trivial for $1 \leq q \leq n$. Furthermore, if the boundary $b\Omega$ is smooth, we also have the following global boundary regularity results for $\bar{\partial}$.

Theorem (Kohn [Ko2]). *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary $b\Omega$. For any $f \in W^s_{(p,q)}(\Omega)$, where $s \geq 0$, $0 \leq p \leq n$ and $1 \leq q < n$, such that $\bar{\partial}f = 0$ in Ω , there exists $u \in W^s_{(p,q)}(\Omega)$ satisfying $\bar{\partial}u = f$.*

In this paper we study the $\bar{\partial}$ -equation on domains with pseudoconcave boundary. When the domain is the annulus between two pseudoconvex domains in \mathbb{C}^n , the closed range property and boundary regularity for $\bar{\partial}$ were established in the author's earlier work [Sh1] for $0 < q < n - 1$ and $n \geq 3$. In this paper, we will study the critical case when $q = n - 1$ on the annulus Ω . In this case the space of harmonic forms is infinite dimensional. We also show that in the case when $0 < q < n - 1$, the space of harmonic forms is trivial.

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This improves the earlier results in [Sh1] where only finite dimensionality for the harmonic forms has been established. We first study the closed range property for the case when the annulus is between two strictly pseudoconvex domains in Section 2. This simpler case warrants special attention since it already illuminates the difference between $q < n - 1$ and $q = n - 1$. Then we study the case when the annulus is between two weakly pseudoconvex domains in Section 3. Special attention is given to the case when $n = 2$ and $q = 1$. In Section 4 we survey some known existence and regularity results for $\bar{\partial}$ on pseudoconcave domains with Lipschitz domains in the complex projective space $\mathbb{C}P^n$ when $n \geq 3$. The closed range property for $\bar{\partial}$ for $(0, 1)$ -forms with L^2 coefficients on pseudoconcave domains in $\mathbb{C}P^n$ is still an open problem (see Conjecture 1 and Conjecture 2 at the end of the paper). Very little is known on the pseudoconcave domains in $\mathbb{C}P^2$.

The author would like to thank professor Lars Hörmander who first raised this question to the author on the closed range property for $\bar{\partial}$ on the annulus for the critical degree. This paper is greatly inspired by his recent paper [Hör2]. She would also like to thank professor Emil Straube for his comments on the proof of Theorem 3.2.

2. The $\bar{\partial}$ -equation on the annulus between two strictly pseudoconvex domains in \mathbb{C}^n

Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between two strictly pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ with smooth boundary. In this section, we study the L^2 existence for $\bar{\partial}$ on Ω . We first prove the L^2 existence theorem of the $\bar{\partial}$ -Neumann operator for the easier case when $q < n - 1$. Let $\bar{\partial}^*$ be the Hilbert space adjoint of $\bar{\partial}$. As before, we formulate the $\bar{\partial}$ -Laplacian $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and the harmonic space of (p, q) -forms $\mathcal{H}_{(p,q)}$ is defined as the kernel of \square .

Theorem 2.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between two strictly pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ with smooth boundary.*

The $\bar{\partial}$ -Neumann operator $N(p, q)$ exists and $N(p, q) : L^2_{(p,q)}(\Omega) \rightarrow W^1_{(p,q)}(\Omega)$, where $0 \leq p \leq n$, $1 \leq q \leq n - 2$. The space of harmonic space $\mathcal{H}_{(p,q)}$ is finite dimensional.

Proof. Recall that condition $Z(q)$ means that the Levi form has at least $n - q$ positive eigenvalues or $q + 1$ negative eigenvalues. From our assumption, the Levi form for the boundary $b\Omega_2$ has $n - 1$ negative eigenvalues at each boundary point. Thus it satisfies condition $Z(q)$ for $0 \leq q < n - 1$. The Levi form on $b\Omega_1$ satisfies condition $Z(q)$ for $0 < q < n$. Thus we have the estimates (see [Hör1], [FK] or [CS])

$$\|f\|_{\frac{1}{2}}^2 \leq C(\|\bar{\partial}f\|^2 + \|\bar{\partial}^*f\|^2 + \|f\|^2), \quad f \in L^2_{(p,q)}(\Omega) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*).$$

This gives the existence of the $\bar{\partial}$ -Neumann operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \rightarrow W^1_{(p,q)}(\Omega)$ and the harmonic space $\mathcal{H}_{(p,q)}$ is finite dimensional.

Theorem 2.2. *Let Ω be the same as in Theorem 2.1. For any $\bar{\partial}$ -closed $f \in L^2_{(p,q)}(\Omega) \cap \text{Ker}(\bar{\partial})$, where $0 \leq p \leq n$, $1 \leq q \leq n - 2$, there exists $u \in W^{\frac{1}{2}}_{(p,q-1)}(\Omega)$ with $\bar{\partial}u = f$. The space of harmonic (p, q) -forms $\mathcal{H}_{(p,q)}$ is trivial when $1 \leq q \leq n - 2$.*

Proof. From Theorem 2.1, for any $f \in L^2_{(p,q)}(\Omega) \cap \text{Ker}(\bar{\partial})$ and $f \perp \mathcal{H}_{(p,q)}$, there exists a $u \in W^{\frac{1}{2}}_{(p,q-1)}(\Omega)$ such that $\bar{\partial}u = f$.

To show that the harmonic forms are trivial, notice that from the regularity of the $\bar{\partial}$ -Neumann operator in Theorem 2.1, the harmonic forms are smooth up to the boundary. Also to prove Theorem 2.2, it suffices to prove the *a priori* estimates for smooth $\bar{\partial}$ -closed forms f since they are dense.

Let $N_{(p,q)}^2$ denote the $\bar{\partial}$ -Neumann operator on the strongly pseudoconvex domain Ω_2 . For any smooth $\bar{\partial}$ -closed $f \in C_{(p,q)}^\infty(\bar{\Omega})$, we first extend f from Ω smoothly to \tilde{f} in Ω_1 . We have that

$$\|\bar{\partial}\tilde{f}\|_{W_{(p,q+1)}^{-1}(\Omega_2)} \leq C\|\tilde{f}\|_{L_{(p,q)}^2(\Omega_1)} \leq C\|f\|_{L_{(p,q)}^2(\Omega)}.$$

Let $v = -\star\bar{\partial}N_{(n-p,n-q-1)}^{\Omega_2}\star\bar{\partial}\tilde{f}$. We have $\bar{\partial}v = \bar{\partial}\tilde{f}$ in the distribution sense in Ω_1 if we extend v to be zero outside Ω_2 since $q < n - 1$ (see Theorem 9.1.2 in the book by Chen-Shaw [CS]). Also v satisfies

$$\|v\|_{W^{-\frac{1}{2}}(\Omega_2)} \leq C\|\bar{\partial}\tilde{f}\|_{W_{(p,q+1)}^{-1}(\Omega_2)}.$$

Setting $F = \tilde{f} - v$, we have that $\bar{\partial}F = 0$ in Ω_1 and $F = f$ on Ω . Thus we have extended f as a $\bar{\partial}$ -closed form F in Ω_1 and

$$\|F\|_{W_{(p,q)}^{-\frac{1}{2}}(\Omega_1)} \leq C\|f\|_{L_{(p,q)}^2(\Omega)}.$$

Thus $F = \bar{\partial}U$ with $U \in W^{\frac{1}{2}}(\Omega_1)$. Setting $u = U|_\Omega$, we have $\bar{\partial}u = f$ with

$$\|u\|_{W_{(p,q-1)}^{\frac{1}{2}}(\Omega)} \leq C\|f\|_{L_{(p,q)}^2(\Omega)}.$$

For general $\bar{\partial}$ -closed $f \in L_{(p,q)}^2(\Omega)$, we approximate f by smooth forms to obtain a solution $u \in W_{(p,q-1)}^{\frac{1}{2}}(\Omega)$ satisfying $\bar{\partial}u = f$.

If $f \in \mathcal{H}_{(p,q)}(\Omega)$, we have $\bar{\partial}f = \bar{\partial}^*f = 0$ in Ω . This means that $f = \bar{\partial}u$ and

$$\|f\|^2 = (\bar{\partial}u, \bar{\partial}u) = (u, \bar{\partial}^*f) = 0.$$

Thus we have $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$. \square

Next we discuss the case when $q = n - 1$, where condition $Z(q)$ is not satisfied on $b\Omega_2$.

Theorem 2.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between two strictly pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ with smooth boundary, where $n \geq 3$. For any $0 \leq p \leq n$, the range of $\bar{\partial} : L_{(p,n-2)}^2(\Omega) \rightarrow L_{(p,n-1)}^2(\Omega)$ is closed and the $\bar{\partial}$ -Neumann operator $N_{(p,n-1)}$ exists on $L_{(p,n-1)}^2(\Omega)$. For any $f \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{(p,n-1)})^\perp$, we have*

$$(2.1) \quad \|f\|_{\frac{1}{2}}^2 \leq C(\|\bar{\partial}f_1\|^2 + \|\bar{\partial}^*f_2\|^2).$$

Moreover, for any $f \in L_{(p,n-1)}^2(\Omega)$,

$$\|N_{(p,n-1)}f\|_{\frac{1}{2}} \leq C\|f\|,$$

$$\|\bar{\partial}N_{(p,n-1)}f\|_{\frac{1}{2}} + \|\bar{\partial}^*N_{(p,n-1)}f\|_{\frac{1}{2}} \leq C\|f\|.$$

Proof. We note that for any domain, one can always solve $\bar{\partial}$ for the top degree and condition $Z(n)$ is a void condition. When $n \geq 3$, Ω satisfies both condition $Z(n-2)$ and $Z(n)$, thus from Theorem (3.1.19) in [FK], the $\bar{\partial}$ -equation has closed range and $\bar{\partial}$ -Neumann operator exists for $(p, n-1)$ -forms. For $f \in L^2_{(p,n-1)}(\Omega)$, we have

$$f = \bar{\partial}\bar{\partial}^*N_{(p,n-1)}f + \bar{\partial}^*\bar{\partial}N_{(p,n-1)}f + H_{(p,n-1)}f$$

where $H_{(p,n-1)}f$ is the projection onto the harmonic space $\mathcal{H}_{(p,n-1)}$.

We note that for any $(p, n-1)$ -form $f \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{(p,n-1)})^\perp$, we write $f = f_1 + f_2$ with $f_1 \perp \text{Ker}(\bar{\partial})$ and $f_2 \perp \text{Ker}(\bar{\partial}^*)$. Then $\bar{\partial}f = \bar{\partial}f_1$ and $\bar{\partial}^*f = \bar{\partial}^*f_2$ and we have (see Proposition 3.1.18 in [FK])

$$(2.2) \quad f = \bar{\partial}^*N_{(p,n)}\bar{\partial}f_1 + \bar{\partial}N_{(p,n-2)}\bar{\partial}^*f_2.$$

Using the regularity for $N_{(p,n)}$ and $N_{(p,n-2)}$, we have

$$(2.3) \quad \begin{aligned} \|f\|_{\frac{1}{2}}^2 &\leq 2(\|\bar{\partial}^*N_{(p,n)}\bar{\partial}f_1\|_{\frac{1}{2}}^2 + \|\bar{\partial}N_{(p,n-2)}\bar{\partial}^*f_2\|_{\frac{1}{2}}^2) \\ &\leq C(\|\bar{\partial}f_1\|^2 + \|\bar{\partial}^*f_2\|^2) \\ &= C((\bar{\partial}f, \bar{\partial}f) + (\bar{\partial}^*f, \bar{\partial}^*f)). \end{aligned}$$

If we assume that, in addition, f is in $\text{Dom}(\square)$, then

$$(2.4) \quad \begin{aligned} \|f\|_{\frac{1}{2}}^2 &\leq C((\bar{\partial}f, \bar{\partial}f) + (\bar{\partial}^*f, \bar{\partial}^*f)) = C(\square f, f) \\ &\leq C\|\square f\|\|f\|. \end{aligned}$$

Setting $f = N_{(p,n-1)}\phi$ for some $\phi \in L^2_{(p,n-1)}(\Omega)$, we have from (2.3) that

$$(2.5) \quad \|N\phi\|_{\frac{1}{2}}^2 \leq C\|\square N\phi\|\|N\phi\| \leq C^2\|\phi\|^2.$$

Thus the operator $N_{(p,n-1)}$ is a bounded operator from L^2 to $W^{\frac{1}{2}}$. To show that $\bar{\partial}N_{(p,n-1)}$ and $\bar{\partial}^*N_{(p,n-1)}$ is bounded from L^2 to $W^{\frac{1}{2}}$, we use the fact that (2.1) holds for both $q = n-2$ and $q = n$. Substituting f in (2.1) by $\bar{\partial}N_{(p,n-1)}f$ and $\bar{\partial}^*N_{(p,n-1)}f$, then we have

$$(2.6) \quad \|\bar{\partial}^*Nf\|_{\frac{1}{2}}^2 \leq C\|\bar{\partial}\bar{\partial}^*Nf\|^2, \quad f \in L^2_{(p,n-1)}(\Omega),$$

$$(2.7) \quad \|\bar{\partial}Nf\|_{\frac{1}{2}}^2 \leq C\|\bar{\partial}^*\bar{\partial}Nf\|^2, \quad f \in L^2_{(p,n-1)}(\Omega).$$

Adding (2.6) and (2.7), we get for any $f \in L^2_{(p,n-1)}(\Omega)$,

$$(2.8) \quad \|\bar{\partial}^*Nf\|_{\frac{1}{2}}^2 + \|\bar{\partial}Nf\|_{\frac{1}{2}}^2 \leq C(\|\bar{\partial}\bar{\partial}^*Nf\|^2 + \|\bar{\partial}^*\bar{\partial}Nf\|^2) \leq C\|\square Nf\|^2 \leq C\|f\|^2.$$

This proves the theorem.

Corollary 2.4. *For any $0 \leq p \leq n$ and $n \geq 3$, the range of $\bar{\partial} : L^2_{(p,n-2)}(\Omega) \rightarrow L^2_{(p,n-1)}(\Omega)$ is closed. For any $f \in L^2_{(p,n-1)}(\Omega)$ with $\bar{\partial}f = 0$ and $f \perp \mathcal{H}_{(p,n-1)}(\Omega)$, there exists $u \in W^{\frac{1}{2}}_{(p,n-2)}(\Omega)$ satisfying $\bar{\partial}u = f$.*

Corollary 2.5. *The space of harmonic $(p, n-1)$ -forms $\mathcal{H}_{(p,n-1)}$ is infinite dimensional.*

Proof. From Corollary 2.3 and Theorem 2.2, we have that the range of $\square_{(p,n-1)}(\Omega)$ is closed. The infinite dimensionality of the null space of $\square_{(p,n-1)}(\Omega)$ is proved in Theorem 3.1 in Hörmander [Hör2]. We will give another proof in Corollary 3.4 in the next section.

Remark: (1). The case for the closed range property on the annulus domains when $n = 2$ and $q = 1$ is more involved (see Theorem 3.3 in the next section).

(2). When the domain Ω is the annulus between two concentric balls or ellipsoids, the infinite dimensional space $\mathcal{H}_{(p,n-1)}$ has been computed explicitly by Hörmander (see Theorem 2.2 in [Hör2]). For integral formula when $1 \leq q < n-1$ in this case, see Section 3.5 in the book by Range [Ra] (see also the paper by Hortmann [Hor]).

3. The $\bar{\partial}$ -equation on the annulus between two weakly pseudoconvex domains in \mathbb{C}^n

We recall the following existence and estimates for $\bar{\partial}$ in the annulus between two pseudoconvex domains (see Shaw [Sh1])

Theorem 3.1. *Let $\Omega \subset\subset \mathbb{C}^n$, $n \geq 3$, be the annulus domain $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between two pseudoconvex domains Ω_1 and Ω_2 with smooth boundary and $\Omega_2 \subset\subset \Omega_1$. Let ϕ_t be a smooth function which is equal to $t|z|^2$ near $b\Omega_1$ and $-t|z|^2$ near $b\Omega_2$ where $t > 0$. Then for $0 \leq p \leq n$ and $1 \leq q < n-1$, the $\bar{\partial}$ -Neumann operator $N^t_{(p,q)}$ with weights ϕ_t exists for sufficiently large $t > 0$ on $L^2_{(p,q)}(\Omega)$ and the harmonic space $\mathcal{H}_{(p,q)}(\Omega)$ is finite dimensional. For any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$(3.1) \quad f = \bar{\partial}\bar{\partial}_t^* N^t_{(p,q)} f + \bar{\partial}_t^* \bar{\partial} N^t_{(p,q)} f + H^t_{(p,q)} f$$

where $H^t_{(p,q)} f$ is the projection of f into $\mathcal{H}_{(p,q)}(\Omega)$.

Furthermore, for each $s \geq 0$, there exists T_s such that $N_{(p,q)}$, $\bar{\partial}_t^* N^t_{(p,q)}$, $\bar{\partial} N^t_{(p,q)}$ and the weighted Bergman projection $B^t_{(p,0)} = I - \bar{\partial}_t^* N^t_{(p,1)} \bar{\partial}$ are exact regular on $W^s_{(p,q)}(\Omega)$ for $t > T_s$.

The existence and regularity of $N^t_{(p,q)}$ was proved in [Sh1]. The exact regularity for the related operators $\bar{\partial}_t^* N^t_{(p,q)}$, $\bar{\partial} N^t_{(p,q)}$ and the weighted Bergman projection $B_t = I - \bar{\partial}_t^* N^t_{(p,1)} \bar{\partial}$ are proved following the same arguments as in Theorem 6.1.4 in [CS]. We will show that $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$ using the following $\bar{\partial}$ -closed extension of forms.

Theorem 3.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be the same as in Theorem 3.1 with $n \geq 3$. For any $f \in L^2_{(p,q)}(\Omega)$, where $0 \leq p \leq n$ and $0 \leq q < n-1$, such that $\bar{\partial}f = 0$ in Ω , there exists $F \in W^{-1}_{(p,q)}(\Omega_1)$ such that $F|_{\Omega} = f$ and $\bar{\partial}F = 0$ in Ω_1 in the distribution sense.*

If $1 \leq q < n-1$, there exists $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar{\partial}u = f$ in Ω .

Proof. From Theorem 3.1, we have that smooth $\bar{\partial}$ -closed forms are dense in $L^2_{(p,q)}(\Omega) \cap \text{Ker}(\bar{\partial})$ (see Corollary 6.1.6 in [CS]). We may assume that f is smooth and it suffices to prove *a priori* estimates. Let \tilde{f} be the smooth extension of f so that $\tilde{f}|_{\Omega} = f$. The rest of the proof is similar to the proof of Theorem 2.2. We will only indicate the necessary changes. Let $v \in W^1_{(p,q+1)}(\Omega_2)$ and \tilde{v} be an $W^1(\Omega_1)$ extension of v to Ω_1 and \tilde{v} has compact support in Ω_1 . Since f is $\bar{\partial}$ -closed on Ω , we have

$$|(\bar{\partial}\tilde{f}, v)_{(\Omega_2)}| = |(\bar{\partial}\tilde{f}, v)_{(\Omega_1)}| = |(\tilde{f}, \vartheta\tilde{v})_{(\Omega_1)}| \leq \|\tilde{f}\|_{(\Omega_1)} \|\vartheta\tilde{v}\|_{(\Omega_1)} \leq C\|f\|_{(\Omega_2)} \|v\|_{W^1(\Omega_2)}.$$

It follows that $\bar{\partial}\tilde{f}$ is in $W^{-1}(\Omega_2)$, defined as the dual of $W^1(\Omega_2)$ and

$$\|\bar{\partial}\tilde{f}\|_{W^{-1}_{(p,q+1)}(\Omega_2)} \leq C\|\tilde{f}\|_{L^2_{(p,q)}(\Omega_2)}.$$

Let $N^{\Omega_2}_{(n-p,n-q-1)}$ be the weighted $\bar{\partial}$ -Neumann operator with weights $t|z|^2$ for some large t on Ω_2 . Here we omit the dependence on t to avoid too many indices. It follows from Kohn [Ko2] that $N^{\Omega_2}_{(n-p,n-q-1)}$ is exact regular on $W^s(\Omega_2)$ for each $s > 0$ if we choose sufficiently large t . We define $T\tilde{f}$ by $T\tilde{f} = -\star\bar{\partial}N^{\Omega_2}_{(n-p,n-q-1)}(\star\bar{\partial}\tilde{f})$ on Ω_2 , where \star is the Hodge star operator with respect to the weighted L^2 space. From Theorem 9.1.2 in [CS], $T\tilde{f}$ satisfies $\bar{\partial}T\tilde{f} = \bar{\partial}\tilde{f}$ in Ω_1 in the distribution sense if we extend $T\tilde{f}$ to be zero outside Ω_2 . Furthermore, from the exact regularity of the weighted $\bar{\partial}$ -Neumann operator and $\bar{\partial}N^{\Omega_2}_{(n-p,n-q-1)}$ in the dual of $W^1(\Omega_2)$. we have that

$$\|T\tilde{f}\|_{W^{-1}(\Omega_2)} \leq C\|\bar{\partial}\tilde{f}\|_{W^{-1}_{(p,q+1)}(\Omega_2)} \leq C\|\tilde{f}\|_{L^2_{(p,q)}(\Omega_2)}.$$

Since $T\tilde{f}$ is in the dual of $W^1(\Omega_2)$, the extension of $T\tilde{f}$ by zero to Ω_1 is continuous from $W^{-1}(\Omega_2)$ to $W^{-1}(\Omega_1)$. Define

$$(3.2) \quad F = \begin{cases} f, & x \in \bar{\Omega}, \\ \tilde{f} - T\tilde{f}, & x \in \Omega_2. \end{cases}$$

Then $F \in W^{-1}_{(p,q)}(\Omega_1)$ and F is a $\bar{\partial}$ -closed extension of f .

It follows that $F = \bar{\partial}U$ for some $U \in W^{-1}_{(p,q-1)}(\Omega_1)$ where we can take U to be the canonical solution $\bar{\partial}_t^* N_t^{\Omega_1} F$ with respect to the weight $t|z|^2$ with large $t > 0$. It follows that $U \in L^2(\Omega_1, \text{loc})$ from the interior regularity.

If $1 < q < n - 1$, we can actually have that $U \in L^2(\Omega_1)$ from the boundary regularity for $\bar{\partial} \oplus \bar{\partial}_t^*$. Let ζ be a cut-off function which is supported in a tubular neighborhood V of $b\Omega_1$ such that $\zeta = 1$ in a neighborhood of $b\Omega_1$. We first show *a priori* estimates assuming U is in $L^2(\Omega_1)$. Then we have from the Hörmander's weighted estimates

$$(3.3) \quad \|\zeta U\|_{t(\Omega_1)} \leq C(\|\bar{\partial}(\zeta U)\|_{t(\Omega_1)} + \|\bar{\partial}_t^*(\zeta U)\|_{t(\Omega_1)}) < \infty$$

since $\bar{\partial}U = f$ and $\bar{\partial}_t^* U = 0$ are in $L^2(\Omega \cap V)$ and $(\bar{\partial}\zeta)U$ is in $L^2(\Omega_1)$. The constant in (3.3) depends only on the diameter of Ω_1 . To pass from *a priori* estimates to the real estimates, we approximate Ω_1 from inside by strongly pseudoconvex domains with smooth boundary.

We refer the reader to the paper by Boas-Shaw [BS] or Michel-Shaw [MS] (see also the proof of Theorem 4.4.1 in [CS]) for details.

When $q = 1$, we have to modify the solution. Let $F_1 = \bar{\partial}(\zeta U) = \bar{\partial}(\zeta)U + \zeta\bar{\partial}U \in L^2(\Omega_1)$. We write $F = \bar{\partial}(\zeta U) + \bar{\partial}((1 - \zeta)U) = F_1 + F_2$. Let $U_1 = \bar{\partial}^* N^{\Omega_1} F_1$. Then $U_1 \in L^2(\Omega_1)$ and $\bar{\partial}U_1 = F_1$. Since F_2 is a $\bar{\partial}$ -closed form with compact support in Ω_1 , we can solve $\bar{\partial}U_2 = F_2$ in \mathbb{C}^n by convolution with the Bochner-Martinelli kernel (or solve the $\bar{\partial}$ -equation on a large ball containing $\bar{\Omega}_1$). This gives that $U_2 \in L^2(\mathbb{C}^n)$. Setting $u = U_1 + U_2$ and restricting u to Ω , we have $u \in L^2_{(p,0)}(\Omega)$ satisfying $\bar{\partial}u = f$ in Ω . Notice that the latter method can also be applied to the case when $1 \leq q < n - 1$. \square

For $q = n - 1$, there is an additional compatibility condition for the $\bar{\partial}$ -closed extension of $(p, n - 1)$ -forms.

Theorem 3.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between two pseudoconvex domains Ω_1 and Ω_2 with smooth boundary and $\Omega_2 \subset\subset \Omega_1$, $n \geq 2$. For any $\bar{\partial}$ -closed $f \in L^2_{(p,n-1)}(\Omega)$, where $0 \leq p \leq n$, the following conditions are equivalent:*

- (1) *There exists $F \in W^{-1}_{(p,n-1)}(\Omega_1)$ such that $F|_{\Omega} = f$ and $\bar{\partial}F = 0$ in Ω_1 in the distribution sense.*
- (2) *The restriction of f to $b\Omega_2$ satisfies the compatibility condition*

$$(3.4) \quad \int_{b\Omega_2} f \wedge \phi = 0, \quad \phi \in W^1_{(n-p,0)}(\Omega_2) \cap \text{Ker}(\bar{\partial}).$$

- (3) *There exists $u \in L^2_{(p,n-2)}(\Omega)$ satisfying $\bar{\partial}u = f$ in Ω .*

Proof. We remark that any h in $W^1(\Omega_2)$ has a trace in $W^{\frac{1}{2}}(b\Omega_2)$ and any $\bar{\partial}$ -closed $(p, n - 1)$ -form with $L^2(\Omega)$ coefficients has a well-defined complex tangential trace in $W^{-\frac{1}{2}}(b\Omega_2)$ (see e.g. [LM]). Thus the pairing between f and ϕ in (2) is well-defined.

Since the weighted $\bar{\partial}$ -Neumann operator $N^2_{(p,1)}$ on Ω_2 with weights $t|z|^2$ is exact regular on $W^s(\Omega_2)$ for sufficiently large t , the Bergman projection is bounded from $W^s_{(p,0)}(\Omega_2)$ to itself for any $s \geq 0$ (see Corollary 6.1.6 in [CS]). We have that the space $W^s_{(p,0)}(\Omega_2) \cap \text{Ker}(\bar{\partial})$ is dense in $L^2_{(p,0)}(\Omega_2) \cap \text{Ker}(\bar{\partial})$. In particular, $W^1_{(p,0)}(\Omega_2) \cap \text{Ker}(\bar{\partial})$ is infinite dimensional.

We first show that (3) implies (2). Since $\bar{\partial}f = 0$ and f is in $L^2(\Omega)$, the tangential part of f has a distribution trace $f_b \in W^{-\frac{1}{2}}(b\Omega)$ on the boundary. Let $u_\epsilon \rightarrow u$ and $\bar{\partial}u_\epsilon \rightarrow \bar{\partial}u = f$ in L^2 where $u_\epsilon \in C^1(\bar{\Omega})$. Let $\bar{\partial}_b$ be the tangential Cauchy-Riemann equations induced by restricting $\bar{\partial}$ to $b\Omega$. For any $h \in W^1_{(n-p,0)}(\Omega_2) \cap \text{Ker}(\bar{\partial})$, the restriction of h is in $W^{\frac{1}{2}}_{(n-p,0)}(b\Omega_2)$.

Let $f \in L^2_{(p,n-1)}(\Omega)$ be a $\bar{\partial}$ -closed form with $\bar{\partial}f = 0$ in Ω . Suppose that f is $\bar{\partial}$ -exact for some $u \in L^2_{(p,n-2)}(\Omega)$. Let ζ be a cut-off function with $\zeta = 1$ on $b\Omega_2$ and $\zeta = 0$ on $b\Omega_1$. Then f must satisfy the compatibility condition

$$\int_{b\Omega_2} f \wedge h = \int_{b\Omega} \bar{\partial}(\zeta u) \wedge h = \lim_{\epsilon} \int_{b\Omega} \bar{\partial}(\zeta u_\epsilon) \wedge h = \lim_{\epsilon} \int_{b\Omega} \zeta u_\epsilon \wedge \bar{\partial}h = 0.$$

This proves that (3) implies (2). To show that (2) implies (1), we will modify the arguments in the proof of Theorem 3.2.

Using the same notation as before, we first approximate f by smooth forms f_ν on Ω such that $f_\nu \rightarrow f$ and $\bar{\partial}f_\nu \rightarrow 0$ in $L^2(\Omega)$. Since $\bar{\partial}f_\nu$ is top degree, we can always solve $\bar{\partial}g_\nu = \bar{\partial}f_\nu$ with smooth g_ν and $g_\nu \rightarrow 0$. Thus we may assume that f can be approximated by smooth $\bar{\partial}$ -closed forms f_ν in $L^2(\Omega)$ and denote the smooth extension of f_ν by \tilde{f}_ν . We define $T\tilde{f}_\nu$ by $T\tilde{f}_\nu = -\star\bar{\partial}N_{(n-p,0)}^{\Omega_2}(\star\bar{\partial}\tilde{f}_\nu)$ on Ω_2 . From the proof of Theorem 9.1.3 in [CS], the space

$$\bar{\partial}T\tilde{f}_\nu = \bar{\partial}f_\nu - B_{(n-p,0)}^{\Omega_2}(\star\bar{\partial}\tilde{f}_\nu).$$

We claim that $B_{(n-p,0)}^{\Omega_2}(\star\bar{\partial}\tilde{f}_\nu) \rightarrow B_{(n-p,0)}^{\Omega_2}(\star\bar{\partial}f) = 0$. Let $\phi \in W_{(n-p,0)}^1(\Omega_2) \cap \text{Ker}(\bar{\partial})$. Then from (2), we have

$$(\phi, \star\bar{\partial}\tilde{f}_\nu) = \int_{\Omega_2} \phi \wedge \bar{\partial}\tilde{f}_\nu = \int_{b\Omega_2} \phi \wedge f_\nu \rightarrow \int_{b\Omega_2} \phi \wedge f = 0.$$

From the regularity of the weighted $\bar{\partial}$ -Neumann operator, we have $W_{(n-p,0)}^1(\Omega_2) \cap \text{Ker}(\bar{\partial})$ is dense in $L_{(n-p,0)}^2(\Omega_2) \cap \text{Ker}(\bar{\partial})$ (see Corollary 6.1.6 in [CS]). This gives that $\bar{\partial}T\tilde{f} = \bar{\partial}\tilde{f}$ in Ω_2 since $B_{(n-p,0)}^{\Omega_2}(\star\bar{\partial}\tilde{f}) = 0$. Furthermore, $\bar{\partial}T\tilde{f} = \bar{\partial}\tilde{f}$ in Ω_1 in the distribution sense if we extend $T\tilde{f}$ to be zero outside Ω_2 .

Define F similarly as before,

$$F = \begin{cases} f, & x \in \bar{\Omega}, \\ \tilde{f} - T\tilde{f}, & x \in \Omega_2. \end{cases}$$

Then $F \in W_{(p,q)}^{-1}(\Omega_1)$ and F is a $\bar{\partial}$ -closed extension of f . This proves that (2) implies (1).

To show that (1) implies (3), one can solve $F = \bar{\partial}U$ for some $U \in L_{(p,q-1)}^2(\Omega_1)$. Let $u = U$ on Ω , we have $u \in L_{(p,q-1)}^2(\Omega)$ satisfying $\bar{\partial}u = f$ in Ω . Thus (1) implies (3). \square

Corollary 3.4. *Let Ω be the same as in Theorem 3.3. Then $\bar{\partial}$ has closed range in $L_{(p,n-1)}^2(\Omega)$ and the $\bar{\partial}$ -Neumann operator $N_{(p,n-1)}$ exists on $L_{(p,q)}^2(\Omega)$. The space of harmonic $(p, n-1)$ -forms $\mathcal{H}_{(p,n-1)}$ is of infinite dimension.*

Proof. That $\bar{\partial}$ has closed range follows from Condition (2) in Theorem 3.3. The Bergman space $W_{(n-p,0)}^1(\Omega_2)$ is infinite dimensional. Each will yield a $\bar{\partial}$ -closed $(p, n-1)$ -form F on Ω which is not $\bar{\partial}$ -exact. Since the space $\mathcal{H}_{(p,n-1)}(\Omega)$ is isomorphic to the quotient space of L^2 $\bar{\partial}$ -closed $(p, n-1)$ -forms over the closed subspace space of $\bar{\partial}$ -exact forms, we conclude that $\mathcal{H}_{(p,n-1)}(\Omega)$ is infinite dimensional.

We summarize the closed range property and regularity for $\bar{\partial}$ on the annulus between two pseudoconvex domains in \mathbb{C}^n in the following theorem.

Theorem 3.5. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ between two pseudoconvex domains Ω_1 and Ω_2 with smooth boundary and $\Omega_2 \subset\subset \Omega_1$. Then the $\bar{\partial}$ -Neumann operator $N_{(p,q)}$ exists on $L_{(p,q)}^2(\Omega)$ for $0 \leq p \leq n$ and $1 \leq q \leq n-1$. For any $f \in L_{(p,q)}^2(\Omega)$, we have*

$$f = \bar{\partial}\bar{\partial}^*N_{(p,q)}f + \bar{\partial}^*\bar{\partial}N_{(p,q)}f, \quad 1 \leq q \leq n-2.$$

$$f = \bar{\partial}\bar{\partial}^*N_{(p,n-1)}f + \bar{\partial}^*\bar{\partial}N_{(p,n-1)}f + H_{(p,n-1)}f, \quad q = n-1$$

where $H_{(p,n-1)}$ is the projection operator onto the harmonic space $\mathcal{H}_{(p,q)}(\Omega)$ which is infinite dimensional.

Suppose that $f \in W_{(p,q)}^s(\Omega)$, where $s \geq 0$ and $1 \leq q \leq n-1$. We assume that $\bar{\partial}f = 0$ in Ω for $q \leq n-1$ and if $q = n-1$, we assume furthermore that f satisfies the condition

$$\int_{b\Omega_2} f \wedge \phi = 0, \quad \phi \in W_{(n-p,0)}^1(\Omega_2) \cap \text{Ker}(\bar{\partial}).$$

Then there exists $u \in W_{(p,q)}^s(\Omega)$ satisfying $\bar{\partial}u = f$.

Proof. Since $\bar{\partial}$ has closed range in $L_{(p,q)}^2(\Omega)$ for all degrees, we have that the $\bar{\partial}$ -Neumann operator exists (without weights). The proof is exactly the same as the proof of Theorem 4.4.1 in [CS]. The regularity for $\bar{\partial}$ follows from Theorem 3.1 for $q < n-1$ and the earlier work of [Sh1]. When $q = n-1$, we can trace the proof of Theorem 3.3 to see that there exists a $\bar{\partial}$ -closed form $F \in W_{(p,q)}^{s-1}(\Omega_1)$ which is equal to f on Ω . Thus one can find a solution $u \in W_{(p,n-2)}^s(\Omega)$ satisfying $\bar{\partial}u = f$.

Remark: All the results in this section can be extended to annulus between pseudoconvex domains in a Stein manifold with trivial modification.

4. The $\bar{\partial}$ -equation on weakly pseudoconcave domains in $\mathbb{C}P^n$

Much of the results in Section 2 can be applied to the strongly pseudoconcave domains or complements of finite type pseudoconvex domains in $\mathbb{C}P^n$ without much change. For the $\bar{\partial}$ -equation on a weakly pseudoconcave domain in $\mathbb{C}P^n$, we cannot use the weight function methods used in Section 3 since $\mathbb{C}P^n$ is not Stein. We have the following results obtained in the recent papers [CSW] and [CS2] for pseudoconcave domains in $\mathbb{C}P^n$ Lipschitz boundary. Related results for $\bar{\partial}$ on the pseudoconcave domains in $\mathbb{C}P^n$, see the paper by Henkin-Jordan [HI].

We recall that a domain is called Lipschitz if the boundary is locally the graph of a Lipschitz function. For some basic properties of Lipschitz domains, see the preliminaries in [Sh4].

Theorem 4.1. *Let Ω^+ be a pseudoconcave domain in $\mathbb{C}P^n$ with Lipschitz boundary, where $n \geq 3$. For any $f \in W_{(p,q)}^{1+\epsilon}(\Omega^+)$, where $0 \leq p \leq n$, $1 \leq q < n-1$, $p \neq q$ and $0 < \epsilon < \frac{1}{2}$, such that $\bar{\partial}f = 0$ in Ω^+ , there exists $u \in W_{(p,q-1)}^{1+\epsilon}(\Omega^+)$ with $\bar{\partial}u = f$ in Ω^+ . If $b\Omega^+$ is C^2 , the statement is also true for $\epsilon = 0$.*

The proof of Theorem 4.1 depends on the $\bar{\partial}$ -closed extension of forms, which in term depends on the following $\bar{\partial}$ -Cauchy problem.

Proposition 4.2. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with Lipschitz boundary, $n \geq 3$. Suppose that $f \in L_{(p,q)}^2(\delta^{-t}, \Omega)$ for some $t > 0$, where $0 \leq p \leq n$ and $1 \leq q < n$. Assuming that $\bar{\partial}f = 0$ in $\mathbb{C}P^n$ with $f = 0$ outside Ω , then there exists $u_t \in L_{(p,q-1)}^2(\delta^{-t}, \Omega)$ with $u_t = 0$ outside Ω satisfying $\bar{\partial}u_t = f$ in the distribution sense in $\mathbb{C}P^n$. If $b\Omega$ is C^2 , the statement is also true for $t = 0$.*

Proof. Following Takeuchi (see [Ta] or [CS1]), the weighted $\bar{\partial}$ -Neumann operators N_t exists for forms in $L_{(n-p,n-q)}^2(\delta^t, \Omega)$. Let $\star_{(t)}$ denote the Hodge-star operator with respect to the

weighted norm $L^2(\delta^t, \Omega)$. Then

$$\star_{(t)} = \delta^t \star = \star \delta^t$$

where \star is the Hodge star operator with the unweighted L^2 norm. Since $f \in L^2_{(p,q)}(\delta^{-t}, \Omega)$, we have that $\star_{(-t)}f \in L^2_{(p,q)}(\delta^t, \Omega)$. Let u_t be defined by

$$(4.1) \quad u_t = -\star_{(t)} \bar{\partial} N_t \star_{(-t)} f.$$

Then $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$, since $\bar{\partial} N_t \star_{(-t)} f$ is in $\text{Dom}(\bar{\partial}_t^*) \subset L^2_{(n-p, n-q+1)}(\delta^t, \Omega)$. Since $\bar{\partial}_t^* = \delta^{-t} \partial \delta^t = -\star_{(-t)} \bar{\partial} \star_{(t)}$, we have $\star_{(-t)} f \in \text{Dom}(\bar{\partial}_t^*)$ and $\bar{\partial}_t^* \star_{(-t)} f = 0$ in Ω . This gives

$$(4.2) \quad \bar{\partial}_t^* N_t \star_{(-t)} f = N_t \bar{\partial}_t^* \star_{(-t)} f = 0.$$

From (3.2), we have $\bar{\partial} u_t = f$

First notice that $\star_{(-t)}(-1)^{p+q} \bar{\partial} N_t \star_{(-t)} f = \bar{\partial} N_t \star_{(-t)} f \in \text{Dom}(\bar{\partial}_t^*)$. We also have $\bar{\partial}_t^* \star_{(-t)} u = (-1)^{p+q} \star_{(-t)} f$ in Ω . Extending u_t to be zero outside Ω , one can show that $\bar{\partial} u_t = f$ in $\mathbb{C}P^n$ using that the boundary is Lipschitz.

If $b\Omega^+$ is C^2 , the $\bar{\partial}$ -Neumann operator exists without weights. The above arguments can be applied to the case when $t = 0$. For details, see [CSW] and [CS2].

Proposition 4.3. *Let $\Omega \subset \subset \mathbb{C}P^n$ be a pseudoconvex domain with Lipschitz boundary and let $\Omega^+ = \mathbb{C}P^n \setminus \bar{\Omega}$. For any $f \in W^{1+\epsilon}_{(p,q)}(\Omega^+)$, where $0 \leq p \leq n$, $0 \leq q < n-1$ and $0 < \epsilon < \frac{1}{2}$, such that $\bar{\partial} f = 0$ in Ω^+ , there exists $F \in W^\epsilon_{(p,q)}(\mathbb{C}P^n)$ with $F|_{\Omega^+} = f$ and $\bar{\partial} F = 0$ in $\mathbb{C}P^n$ in the distribution sense.*

If $b\Omega$ is C^2 , the statement is also true for $\epsilon = 0$.

Proof. Since Ω has Lipschitz boundary, there exists a bounded extension operator from $W^s(\Omega^+)$ to $W^s(\mathbb{C}P^n)$ for all $s \geq 0$ (see e.g. [Gr]). Let $\tilde{f} \in W^{1+\epsilon}_{(p,q)}(\mathbb{C}P^n)$ be the extension of f so that $\tilde{f}|_{\Omega^+} = f$ with $\|\tilde{f}\|_{W^{1+\epsilon}(\mathbb{C}P^n)} \leq C\|f\|_{W^{1+\epsilon}(\Omega^+)}$. Furthermore, we can choose an extension such that $\bar{\partial} \tilde{f} \in W^\epsilon(\Omega) \cap L^2(\delta^{-2\epsilon}, \Omega)$.

We define $T\tilde{f}$ by $T\tilde{f} = -\star_{(2\epsilon)} \bar{\partial} N_{2\epsilon}(\star_{(-2\epsilon)} \bar{\partial} \tilde{f})$ in Ω . From Proposition 4.2, we have that $T\tilde{f} \in L^2(\delta^{-2\epsilon}, \Omega)$. But for a Lipschitz domain, we have that $T\tilde{f} \in L^2(\delta^{-2\epsilon}, \Omega)$ is comparable to $W^\epsilon(\Omega)$ when $0 < \epsilon < \frac{1}{2}$. This gives that $T\tilde{f} \in W^\epsilon(\Omega)$ and $T\tilde{f}$ satisfies $\bar{\partial} T\tilde{f} = \bar{\partial} \tilde{f}$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside Ω .

Since $0 < \epsilon < \frac{1}{2}$, the extension by 0 outside Ω is a continuous operator from $W^\epsilon(\Omega)$ to $W^s(\mathbb{C}P^n)$ (see e.g. [Gr]). Thus we have $T\tilde{f} \in W^\epsilon(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & x \in \bar{\Omega}^+, \\ \tilde{f} - T\tilde{f}, & x \in \Omega. \end{cases}$$

Then $F \in W^\epsilon_{(p,q)}(\mathbb{C}P^n)$ and F is a $\bar{\partial}$ -closed extension of f .

From Proposition 4.3, Theorem 4.1 follows easily.

A Lipschitz (or C^1) hypersurface is said to be Levi-flat if it is locally foliated by complex manifolds of complex dimension $n-1$. A C^2 hypersurface M is called Levi-flat if its Levi-form vanishes on M . Any C^k Levi-flat hypersurface, $k \geq 2$ is locally foliated by complex

manifolds of complex dimension $n - 1$. The foliation is of class C^k if the hypersurface is of class C^k , $k \geq 2$ (see Barrett-Fornaess [BF]). The proof in [BF] also gives that if a real C^1 hypersurface admits a continuous foliation by complex manifolds, then the foliation is actually C^1 . One of the main applications of the $\bar{\partial}$ -equation on pseudoconcave domains in $\mathbb{C}P^n$ is the following result (see [CS2]).

Theorem 4.4. *There exist no Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ for $n \geq 3$.*

It was first proved in Lins-Neto [LNe] that there exist no real analytic Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$. Siu [Si] proved the nonexistence of smooth (or $\frac{3n}{2} + 7$) Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$. The proof for the nonexistence of C^2 Levi-flat hypersurfaces in $\mathbb{C}P^2$ in [CSW] is still incomplete. The main missing ingredient is the lack of closed range property for $\bar{\partial}$ on pseudoconcave domains. In the case for an annulus domains in \mathbb{C}^n , notice that we have used the regularity of the weighted $\bar{\partial}$ -Neumann operator on pseudoconvex domains proved by Kohn in the proof of Theorem 3.3. We end the section with the following three open questions.

Conjecture 1. *Let $\Omega^+ \subset \subset \mathbb{C}P^n$ be a pseudoconcave domain with C^2 -smooth boundary (or Lipschitz) $b\Omega^+$, $n \geq 2$. Then $\bar{\partial} : L^2_{(p,q-1)}(\Omega) \rightarrow L^2_{(p,q)}(\Omega)$ has closed range for $0 \leq p \leq n$ and $1 \leq q \leq n - 1$.*

Conjecture 2. *Let Ω^+ be a C^2 pseudoconcave domain in $\mathbb{C}P^n$, $n \geq 2$. For any $0 \leq p \leq n$, the space of harmonic $(p, n - 1)$ -forms $\mathcal{H}_{(p,n-1)}$ is infinite dimensional and for any $f \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{(p,n-1)})^\perp$, we have*

$$\|f\|^2 \leq C(\|\bar{\partial}f_1\|^2 + \|\bar{\partial}^*f_2\|^2).$$

Both Conjecture 1 and Conjecture 2 will imply the nonexistence of Levi-flat hypersurfaces in $\mathbb{C}P^2$.

Conjecture 3. *Let Ω be a pseudoconvex domain in $\mathbb{C}P^n$ with C^2 boundary, where $n \geq 2$. Then the range of $\bar{\partial}_b : L^2_{(p,q-1)}(b\Omega) \rightarrow L^2_{(p,q)}(b\Omega)$ is closed in the $L^2_{(p,q)}(b\Omega)$ space for all $0 \leq p \leq n$ and $1 \leq q \leq n - 1$.*

When Ω is a smooth pseudoconvex domain in \mathbb{C}^n , Conjecture 3 is proved in Shaw [Sh2], Boas-Shaw [BS] and Kohn [Ko3] (see also Chapter 9 in Chen-Shaw [CS] and also Harrington [Ha] for C^1 pseudoconvex boundary). If Ω is Lipschitz pseudoconvex in \mathbb{C}^n and we assume that there exists a plurisubharmonic defining function in a neighborhood of $\bar{\Omega}$, L^2 existence for $\bar{\partial}_b$ has been established in [Sh3].

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