ON THE INTERPLAY BETWEEN SEVERAL COMPLEX VARIABLES, GEOMETRIC MEASURE THEORY, AND HARMONIC ANALYSIS

IRINA MITREA, MARIUS MITREA and MEI-CHI SHAW*

Dedicated with gratitude and respect to Professor Cabiria Andreian Cazacu

Abstract

The goal of this paper is to initiate a program aimed at exploring the interplay between several complex variables, geometric measure theory and harmonic analysis. Here, the main emphasis is the study of the boundary behavior of the Bochner-Martinelli integral operator in uniformly rectifiable domains.

1 Introduction

As is well-known, there are deep, fascinating connections between Complex Analysis (CA), Geometric Measure Theory (GMT) and Harmonic Analysis (HA) in the complex plane (see, e.g., J. Garnett's book [?] and the references therein). Indeed, this is an area of mathematics which has a long and distinguished tradition, which continues to undergo tremendous transformations thanks to spectacular advances in recent years (cf. G. David's characterization of the L^2 boundedness of the Cauchy operator in terms of Ahlfors regularity, and X. Tolsa's results on analytic capacity, just to name a few). However, this very fruitful interplay between CA, GMT and HA seems to have been much less explored in the *higher dimensional setting*, in which case CA is replaced by Several Complex Variables (SCV).

The main point of the present article is to elaborate on this idea by considering the Bochner-Martinelli integral operator (thought of as the higher dimensional version of the Cauchy operator), from the perspective of Calderón-Zygmund theory, in a class of domains which are essentially optimal from the point of view of Geometric Measure Theory.

---

*2000 *Mathematics Subject Classification.* Primary 26B20, 32W05, 32W10, 49Q15; Secondary 31B10, 35C15.
Key words: Differential forms, uniformly rectifiable domains, the Bochner-Martinelli integral operator

Classically, the Bochner-Martinelli integral operator acting on scalar functions defined on a C^1-smooth submanifold Σ of $\mathbb{C}^n$ is given by

$$\mathcal{B}f(z) := \int_\Sigma f(\zeta)K(z,\zeta), \quad z \in \mathbb{C}^n \setminus \Sigma, \tag{1.1}$$

where, if $d[\overline{\zeta}]_j = d\overline{\zeta}_1 \wedge ... \wedge d\overline{\zeta}_n$ with $d\overline{\zeta}_j$ omitted,

$$K(z,\zeta) = c_n \sum_{j=1}^n (-1)^j \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} d[\overline{\zeta}]_j \wedge d\zeta. \tag{1.2}$$

However, this commonly held point of view is no longer practical if Σ is lacking regularity. To find an alternative formula, we note that the pull-back of the differential form $d[\overline{\zeta}]_j \wedge d\zeta$ under the embedding $\iota : \Sigma \hookrightarrow \mathbb{C}^n$ is

$$\iota^*\Big(d[\overline{\zeta}]_j \wedge d\zeta\Big) = c_n \sum_{j=1}^n (-1)^j (\nu^c)_j \, d\sigma, \tag{1.3}$$

where, with $\nu = (\nu_1, ..., \nu_{2n})$ denoting the (real) outward unit normal, $\nu^c := (\nu_j + i\,\nu_{j+n})_j$ is the so-called complex normal, and σ is the surface measure on Σ. Thus, in some sense, the analysis implicit in (**??**) brings to light the geometry of Σ in a much more transparent fashion than (**??**) (admittedly, an elegant formula but which nonetheless obscures the geometric nature of Σ).

The true virtue of this seemingly mundane observation is that the concept of unit normal and surface measure make sense in much greater generality (than that of a smooth surface) and, hence, it allows us to consider the Bochner-Martinelli integral operator in certain settings which are utterly rough. We shall amply elaborate on this in § **??**. For now, we wish to point out that if $\Omega \subset \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is an open set of locally finite perimeter, then some concept of unit normal continues to make sense, and one can take $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ (with $\mathcal{H}^{2n-1}$ denoting the $(2n-1)$-dimensional Hausdorff measure in $\mathbb{R}^{2n}$) to play the role of the surface measure on the topological boundary of Ω. Thus, already in this very general setting, some concept of Bochner-Martinelli integral operator is meaningful. By only mildly strengthening the geometric measure theoretic hypotheses on Ω (concretely, by demanding that Ω is a uniformly rectifiable domain), we can then prove a large number of properties which have long been known to hold in a smooth setting. In this paper, due to obvious space limitations, we choose to focus on nontangential maximal function estimates and boundary behavior theory, while deferring to a future occasion a more in-depth analysis.

The structure of the paper is as follows. In § **??** we collect a number of results and definitions from GMT. Next, in § **??**, we review some recent results from [**?**] where, building on earlier work of many other people, the authors have succeeded in further extending and refining the scope of the traditional

Calderón-Zygmund theory of singular integrals in Lipschitz domains. Finally, in § **??** we present our main results. This includes Theorem **??**, which summarizes some of the basic properties of the Bochner-Martinelli integral operator in uniformly rectifiable (UR) domains in $\mathbb{C}^n$.

Acknowledgments. The first two-named authors would like to take this opportunity to express their respect and deep gratitude to Professor Cabiria Andreian Cazacu for her early guidance, teaching and support, as well as for being such a great source of inspiration throughout the years.

This work has been supported by the USA NSF-DMS grants 0513173, 0547944, and FRG 0456306. The first-named author has also been supported in part by the Ruth I. Michler Memorial Prize.

2 Review of geometric measure theory

This section is devoted to presenting a brief summary of a number of definitions and results from Geometric Measure Theory which are relevant for the current work.

To proceed, let $\Omega \subset \mathbb{R}^n$ be a fixed domain of locally finite perimeter. Essentially, this is the largest class of domains for which some version of the classical Gauss-Green formula continues to hold. In particular, there exists a suitably-defined concept of outward unit normal to $\partial\Omega$, which we denote by $\nu = (\nu_1, ..., \nu_n)$, and the role of the surface measure on $\partial\Omega$ is played by $\sigma := \mathcal{H}^{n-1}\lfloor\,\partial\Omega$. Here and elsewhere, $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure. By $L^p(\partial\Omega, d\sigma)$, $0 < p \leq \infty$, we shall denote the Lebesgue scale of p-th power, σ-measurable functions on $\partial\Omega$. There are several excellent accounts on these topics, including the monographs by H. Federer [**?**], W. Ziemer [**?**], and L. Evans and R. Gariepy [**?**].

Next, recall that the measure-theoretic boundary $\partial_*\Omega$ of a set $\Omega \subseteq \mathbb{R}^n$ is defined by

$$\partial_*\Omega := \Big\{X \in \partial\Omega : \limsup_{r\to 0} \frac{|B_r(X) \cap \Omega|}{r^n} > 0,\ \limsup_{r\to 0} \frac{|B_r(X) \setminus \Omega|}{r^n} > 0\Big\} \tag{2.1}$$

where $|E|$ stands for the Lebesgue measure of $E \subseteq \mathbb{R}^n$. As is well-known, if Ω has locally finite perimeter, then the outward unit normal is defined σ-a.e. on $\partial_*\Omega$. In particular, if

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \tag{2.2}$$

then ν is defined σ-a.e. on $\partial\Omega$.

Following [**?**], we shall call a nonempty open set $\Omega \subset \mathbb{R}^n$ a *UR (uniformly rectifiable) domain* provided $\partial\Omega$ is uniformly rectifiable (in the sense of Definition **??**) and (**??**) holds. Let us emphasize that, by definition, a UR domain Ω has locally finite perimeter as well as an Ahlfors regular boundary. We remind the reader that a closed set $\Sigma \subset \mathbb{R}^n$ is called *Ahlfors regular* provided there exist $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 R^{n-1} \leq \mathcal{H}^{n-1}(B(X,R) \cap \Sigma) \leq C_2 R^{n-1}, \tag{2.3}$$

for each $X \in \Sigma$ and $R \in (0,\infty)$ (if Σ is compact, we require (**??**) only for $R \in (0,1]$). The constants C_1, C_2 intervening in (**??**) will be referred to as the Ahlfors regularity constants of $\partial\Omega$. For further use, let us point out that, as is apparent from definitions,

$$\begin{aligned} &\Omega \subset \mathbb{R}^n \text{ is an UR domain with } \partial\Omega = \partial\overline{\Omega} \\ &\quad \Longrightarrow \mathbb{R}^n \setminus \overline{\Omega} \text{ is an UR domain, with the same boundary.} \end{aligned} \tag{2.4}$$

Following G. David and S. Semmes [**?**] we make the following.

Definition 2.1 *Call $\Sigma \subset \mathbb{R}^n$ uniformly rectifiable provided it is Ahlfors regular and the following holds. There exist ε, $M \in (0,\infty)$ (called the UR constants of Σ) such that for each $x \in \Sigma$, $R > 0$, there is a Lipschitz map $\varphi : B^{n-1}_R \to \mathbb{R}^n$ (where B^{n-1}_R is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$, such that*

$$\mathcal{H}^{n-1}(\Sigma \cap B_R(X) \cap \varphi(B^{n-1}_R)) \geq \varepsilon R^{n-1}. \tag{2.5}$$

If Σ is compact, this is required only for $R \in (0,1]$.

It is also relevant to recall that a two-sided NTA domain with an Ahlfors regular boundary is a UR domain; cf. [[**?**], § 3].

We now turn to the notion of the non-tangential maximal operator, applied to functions on an open set $\Omega \subset \mathbb{R}^n$. To define this, fix $\kappa > 0$ and for each boundary point $Z \in \partial\Omega$ introduce the non-tangential approach region

$$\Gamma(Z) := \Gamma_\kappa(Z) := \{X \in \Omega : \, |X - Z| < (1+\kappa)\, \mathrm{dist}\,(X, \partial\Omega)\}. \tag{2.6}$$

It should be noted that, under the current hypotheses, it could happen that $\Gamma(Z) = \emptyset$ for points $Z \in \partial\Omega$. This point will be discussed further below.

Next, for $u : \Omega \to \mathbb{R}$, we define the non-tangential maximal function of u by

$$\mathcal{N}u(Z) := \mathcal{N}_\kappa u(Z) := \sup\{|u(X)| : \, X \in \Gamma_\kappa(Z)\}, \qquad Z \in \partial\Omega. \tag{2.7}$$

Here and elsewhere in the sequel, we make the convention that $\mathcal{N}u(Z) = 0$ whenever $Z \in \partial\Omega$ is such that $\Gamma(Z) = \emptyset$.

The following result, established in [**?**], shows that the choice of the parameter κ plays a relatively minor role when measuring the size of the nontangential maximal function in $L^p(\partial\Omega, d\sigma)$.

Proposition 2.2 *Assume $\Omega \subset \mathbb{R}^n$ is open and has an Ahlfors regular boundary. Then for every $\kappa, \kappa' > 0$ and $0 < p < \infty$ there exist $C_0, C_1 > 0$ such that*

$$C_0 \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega, d\sigma)} \leq \|\mathcal{N}_{\kappa'} u\|_{L^p(\partial\Omega, d\sigma)} \leq C_1 \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega, d\sigma)}, \tag{2.8}$$

for each function u.

We conclude this section by recording the following version of Green's theorem from [**?**].

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$ be an open set which is either bounded or has an unbounded boundary. Assume that $\partial\Omega$ is Ahlfors regular and satisfies (**??**) (thus, in particular, Ω is of locally finite perimeter). As before, set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the measure theoretic outward unit normal to $\partial\Omega$. Then Green's formula*

$$\int\limits_{\Omega} \operatorname{div} v \, dX = \int\limits_{\partial\Omega} \langle \nu, v|_{\partial\Omega} \rangle \, d\sigma \tag{2.9}$$

holds for each vector field $v \in C^0(\Omega)$ that satisfies

$$\begin{array}{l} \operatorname{div} v \in L^1(\Omega), \quad \mathcal{N} v \in L^1(\partial\Omega, d\sigma) \cap L^p_{loc}(\partial\Omega, d\sigma) \textit{ for some } p \in (1,\infty), \\ \textit{and the pointwise nontangential trace } v|_{\partial\Omega} \textit{ exists } \sigma\textit{-a.e. on } \partial\Omega. \end{array} \tag{2.10}$$

An explanation is in order here. Generally speaking, given a domain $\Omega \subset \mathbb{R}^n$, a number $\kappa > 0$ and a function $u : \Omega \to \mathbb{R}$, we set

$$u\Big|_{\partial\Omega}(Z) := \lim_{\substack{X \to Z \\ X \in \Gamma_\kappa(Z)}} u(X), \qquad Z \in \partial\Omega, \tag{2.11}$$

whenever the limit exists. For this definition to be pointwise σ-a.e. meaningful, it is necessary that

$$Z \in \overline{\Gamma_\kappa(Z)} \quad \text{for } \sigma\text{-a.e.} \quad Z \in \partial\Omega. \tag{2.12}$$

We shall call a domain Ω satisfying (**??**) above *weakly accessible* and it has been proved in [**?**] that any domain as in the statement of Theorem **??** is weakly accessible.

3 Calderón-Zygmund theory in uniformly rectifiable domains

The purpose of this section is to review results pertaining to the Calderón-Zygmund theory in uniformly rectifiable domains, from [**?**]. To get started, consider a function satisfying

$$\begin{array}{l} k \in C^N(\mathbb{R}^n \setminus \{0\}) \text{ and for each } X \in \mathbb{R}^n \setminus \{0\}, \\ k(-X) = -k(X), \;\; k(\lambda X) = \lambda^{1-n} k(X) \;\; \forall\, \lambda > 0, \end{array} \tag{3.1}$$

and define the singular integral operator

$$\mathcal{T}f(X) := \int_{\partial\Omega} k(X-Y)f(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{3.2}$$

as well as

$$T_* f(X) := \sup_{\varepsilon>0} |T_\varepsilon f(X)|, \qquad X \in \partial\Omega, \quad \text{where} \tag{3.3}$$

$$T_\varepsilon f(X) := \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k(X-Y)f(Y)\,d\sigma(Y), \qquad X \in \partial\Omega. \tag{3.4}$$

The following result was established in Proposition 4 bis of [**?**].

Proposition 3.1 *Assume $\Omega \subset \mathbb{R}^n$ is a UR domain. Take $p \in (1,\infty)$. There exist $N \in \mathbb{Z}_+$ and $C \in (0,\infty)$, each depending only on p along with the Ahlfors regularity and UR constants of $\partial\Omega$, with the following property. If k satisfies (**??**), then*

$$\|T_* f\|_{L^p(\partial\Omega,d\sigma)} \leq C\|k|_{S^{n-1}}\|_{C^N}\|f\|_{L^p(\partial\Omega,d\sigma)} \tag{3.5}$$

for each $f \in L^p(\partial\Omega, d\sigma)$.

Next, we let $L^{1,\infty}(\partial\Omega,d\sigma)$ denote the weak-L^1 space on $\partial\Omega$, i.e. the collection of all σ-measurable functions f on $\partial\Omega$ for which

$$\|f\|_{L^{1,\infty}(\partial\Omega,d\sigma)} := \sup_{\lambda>0}\Big[\lambda\,\sigma(\{X \in \partial\Omega : |f(X)| > \lambda\})\Big] < \infty. \tag{3.6}$$

Corresponding to the case $p = 1$ in (**??**), the following result has been proved in [**?**].

Proposition 3.2 *In the context of Proposition **??**, there also holds*

$$\|T_* f\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \leq C(\Omega,k)\|f\|_{L^1(\partial\Omega,d\sigma)} \tag{3.7}$$

for each $f \in L^1(\partial\Omega, d\sigma)$.

Proposition **??** can be further complemented with the following nontangential maximal function estimate from [**?**].

Proposition 3.3 *In the setting of Proposition **??**, for each $\kappa > 0$ there exists a finite constant $C > 0$, depending only on p, κ, as well as the Ahlfors regularity and UR constants of $\partial\Omega$ such that, with $\mathcal{N} = \mathcal{N}_\kappa$, one has*

$$\|\mathcal{N}(\mathcal{T}f)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|k|_{S^{n-1}}\|_{C^N}\|f\|_{L^p(\partial\Omega,d\sigma)}. \tag{3.8}$$

Moreover, corresponding to $p = 1$,

$$\|\mathcal{N}(\mathcal{T}f)\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \leq C(\Omega,k,\kappa)\|f\|_{L^1(\partial\Omega,d\sigma)}. \tag{3.9}$$

To state the main result of this section, we let "hat" denote the Fourier transform in $\mathbb{R}^n$.

Theorem 3.4 *Let Ω be a UR domain, and let k be as in (**??**). Also, recall the operators $\mathcal{T}$ and T_ε associated with this kernel as in (**??**), (**??**). Then, for each $p \in [1,\infty)$, $f \in L^p(\partial\Omega, d\sigma)$, the limit*

$$Tf(X) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(X) \tag{3.10}$$

exists for a.e. $X \in \partial\Omega$. Also, the induced operators

$$T : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma), \quad p \in (1,\infty), \tag{3.11}$$

$$T : L^1(\partial\Omega, d\sigma) \longrightarrow L^{1,\infty}(\partial\Omega, d\sigma), \tag{3.12}$$

are bounded. Finally, the jump-formula

$$\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{T}f(Z) = \tfrac{1}{2\sqrt{-1}}\hat{k}(\nu(X))f(X) + Tf(X) \tag{3.13}$$

is valid at a.e. $X \in \partial\Omega$, whenever $f \in L^p(\partial\Omega, d\sigma)$, $1 \le p < \infty$.

A proof of this theorem can be found in [**?**].

4 The Bochner-Martinelli integral operator

This section contains the main results of this paper. We begin by discussing a number of standard conventions and by reviewing notation used throughout the section. For more on background and related issues, the reader is referred to the monographs [**?**], [**?**], and [**?**].

For starters, the relationship between the complex variables $z_j \in \mathbb{C}^n$, and the real ones, $(x_j, y_j) \in \mathbb{R}\times\mathbb{R}$, $1 \le j \le n$, under the natural identification $\mathbb{C}^n \equiv (\mathbb{R}\times\mathbb{R}) \otimes \cdots \otimes (\mathbb{R}\times\mathbb{R}) \equiv \mathbb{R}^{2n}$ can be described by

$$\begin{aligned}
&z_j = x_j + iy_j, \quad dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j,\\
&dx_j = 2^{-1}(dz_j + d\bar{z}_j), \quad dy_j = (-i)2^{-1}(dz_j - d\bar{z}_j),\\
&\partial_{z_j} = 2^{-1}(\partial_{x_j} - i\partial_{y_j}), \quad \partial_{\bar{z}_j} = 2^{-1}(\partial_{x_j} + i\partial_{y_j}),\\
&\partial_{x_j} = \partial_{z_j} + \partial_{\bar{z}_j}, \quad \partial_{y_j} = i(\partial_{z_j} - \partial_{\bar{z}_j}), \qquad 1 \le j \le n.
\end{aligned} \tag{4.1}$$

Consequently, the exterior derivative operator in $\mathbb{R}^{2n}$ can be written as

$$\begin{aligned}
d &= \sum_{j=1}^n \partial_{x_j} dx_j \wedge \cdot + \sum_{j=1}^n \partial_{y_j} dy_j \wedge \cdot\\
&= \sum_{j=1}^n \partial_{z_j} dz_j \wedge \cdot + \sum_{j=1}^n \partial_{\bar{z}_j} d\bar{z}_j \wedge \cdot = \partial + \bar{\partial},
\end{aligned} \tag{4.2}$$

where, as customary, we have set

$$\bar{\partial} := \sum_{j=1}^{n} \partial_{\bar{z}_j} d\bar{z}_j \wedge \cdot \quad \text{and} \quad \partial := \sum_{j=1}^{n} \partial_{z_j} dz_j \wedge \cdot \tag{4.3}$$

for the standard d-bar operator and its complex conjugate, respectively. Thus,

$$\partial \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0, \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0. \tag{4.4}$$

For any two ordered arrays I, J, the generalized Kronecker symbol ε^I_J is given by

$$\varepsilon^I_J := \begin{cases} \det\left((\delta_{i,j})_{i \in I, j \in J}\right), & \text{if } |I| = |J|, \\ 0, & \text{otherwise,} \end{cases} \tag{4.5}$$

where $\delta_{j,k} := 1$ if $j = k$, and zero if $j \neq k$. We shall employ an inner product on forms defined by the requirement that

$$\langle dz^I \wedge d\bar{z}^J, dz^A \wedge d\bar{z}^B \rangle = 2^{|I|+|J|} \varepsilon^I_A \varepsilon^J_B, \qquad \forall\, I, J, A, B. \tag{4.6}$$

The power of 2 is an artifact of $dz_j = dx_j + idy_j$ having length $2^{1/2}$ (rather than being of unit length). Thus, in particular, if $0 \leq \alpha, \beta \leq n$, then

$$\begin{gathered} \langle f, g \rangle = 2^{\alpha+\beta} \sum_{|I|=\alpha, |J|=\beta} f_{I,J} \overline{g_{I,J}}, \qquad \text{whenever} \\ f = \sum_{|I|=\alpha, |J|=\beta} f_{I,J}\, dz^I \wedge d\bar{z}^J \text{ and } g = \sum_{|I|=\alpha, |J|=\beta} g_{I,J}\, dz^I \wedge d\bar{z}^J. \end{gathered} \tag{4.7}$$

The volume element in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ is given by

$$\begin{aligned} dV &= dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n \\ &= (-i2)^{-n} dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n \\ &= (i2)^{-n} (-1)^{n(n-1)/2} d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n \wedge dz_1 \wedge \ldots \wedge dz_n. \end{aligned} \tag{4.8}$$

For further reference, let us recall that if $u = \sum_{|I|=\alpha, |J|=\beta} u_{I,J}\, dz^I \wedge d\bar{z}^J$ then the complex conjugate of u is

$$\bar{u} := \sum_{|I|=\alpha, |J|=\beta} \overline{u_{I,J}}\, d\bar{z}^I \wedge dz^J = (-1)^{\alpha\beta} \sum_{|I|=\alpha, |J|=\beta} \overline{u_{I,J}}\, dz^J \wedge d\bar{z}^I. \tag{4.9}$$

Hence, $\overline{u \wedge v} = \bar{u} \wedge \bar{v}$, $\bar{\bar{u}} = u$, and $\bar{u} \in \Lambda^{\beta,\alpha}\mathbb{C}^n$ if $u \in \Lambda^{\alpha,\beta}\mathbb{C}^n$. Here and elsewhere, we denote by $\Lambda^{\beta,\alpha}\mathbb{C}^n$ the space of complex coefficient differential forms of type (α, β).

Going further, let $*$ be the Hodge star operator in $\mathbb{R}^{2n}$, which can be characterized as the unique isomorphism $* : \Lambda^\ell \mathbb{R}^{2n} \to \Lambda^{2n-\ell}\mathbb{R}^{2n}$ such that

$$u \wedge (*\bar{u}) = |u|^2\, dV. \tag{4.10}$$

In particular, $*1 = dV$. In fact, it can be checked that

$$\begin{aligned} & I, J, K \text{ increasing, mutually disjoint subsets of } \{1, ..., n\} \Longrightarrow \\ & * \left(dz^I \wedge d\bar{z}^J \wedge (dz \wedge d\bar{z})^K \right) = i^n 2^{M-n} (-1)^{M(M-1)/2} \\ & \qquad\qquad \cdot dz^I \wedge d\bar{z}^J \wedge (dz \wedge d\bar{z})^{\{1,...,n\} \setminus (I \cup J \cup K)}, \end{aligned} \tag{4.11}$$

where $M := |I| + |J| + 2|K|$. Above, we have set

$$(dz \wedge d\bar{z})^K := (dz_{k_1} \wedge d\bar{z}_{k_1}) \wedge \cdots \wedge (dz_{k_\ell} \wedge d\bar{z}_{k_\ell}) \tag{4.12}$$

if $K = (k_1, ..., k_\ell)$.

Using the Hodge-star operator, we define the interior product between a 1-form θ and an ℓ-form u by setting

$$\theta \vee u := *(\theta \wedge *u). \tag{4.13}$$

For further reference as well as for the convenience of the reader, some basic, elementary properties of these objects are summarized in the following lemma.

Lemma 4.1 *For arbitrary one-forms θ, η, and any ℓ-form u, ℓ-form ω, $(\ell+1)$-form w, and $(2n-\ell)$-form v, the following are true:*

(1) $**u = (-1)^\ell u$, $\langle u, *v \rangle = (-1)^\ell \langle *u, v \rangle$ *and* $\langle *u, *v \rangle = \langle u, v \rangle$;

(2) $\theta \wedge (\theta \wedge u) = 0$ *and* $\theta \vee (\theta \vee u) = 0$;

(3) $\theta \wedge (\eta \vee u) + \eta \vee (\theta \wedge u) = \langle \theta, \bar{\eta} \rangle u$ *and* $\langle \theta \wedge u, w \rangle = \langle u, \bar{\theta} \vee w \rangle$;

(4) $*(\theta \wedge u) = (-1)^\ell \theta \vee (*u)$ *and* $*(\theta \vee u) = (-1)^{\ell-1} \theta \wedge (*u)$.

(5) $*\bar{u} = \overline{*u}$ *and* $u \wedge *\bar{\omega} = \langle u, \omega \rangle \, dV$.

Moreover, if θ is normalized such that $\langle \theta, \theta \rangle = 1$, then also:

(6) $u = \theta \wedge (\bar{\theta} \vee u) + \bar{\theta} \vee (\theta \wedge u)$ *and* $|u|^2 = |\theta \wedge u|^2 + |\bar{\theta} \vee u|^2$;

(7) $|\bar{\theta} \wedge (\theta \vee u)| = |\theta \vee u|$ *and* $|\theta \vee (\bar{\theta} \wedge u)| = |\bar{\theta} \wedge u|$.

Finally, $* : \Lambda^{\alpha,\beta}\mathbb{C}^n \longrightarrow \Lambda^{n-\beta,n-\alpha}\mathbb{C}^n$ *and, if* $\theta \in \Lambda^{1,0}\mathbb{C}^n$, $\eta \in \Lambda^{0,1}\mathbb{C}^n$, *then*

$$\begin{aligned} \theta\wedge &: \Lambda^{\alpha,\beta}\mathbb{C}^n \longrightarrow \Lambda^{\alpha+1,\beta}\mathbb{C}^n, & \theta\vee &: \Lambda^{\alpha,\beta}\mathbb{C}^n \longrightarrow \Lambda^{\alpha,\beta-1}\mathbb{C}^n, \\ \eta\wedge &: \Lambda^{\alpha,\beta}\mathbb{C}^n \longrightarrow \Lambda^{\alpha,\beta+1}\mathbb{C}^n, & \eta\vee &: \Lambda^{\alpha,\beta}\mathbb{C}^n \longrightarrow \Lambda^{\alpha-1,\beta}\mathbb{C}^n. \end{aligned} \tag{4.14}$$

For later, technical purposes, it will be important to point out that, as seen from definitions,

$$\begin{aligned} & w = \sum_{|J|=\beta+1} w_J \, d\bar{z}^J \text{ and } \eta = \sum_{j=1}^n \eta_j \, d\bar{z}_j \Longrightarrow \\ & \qquad \bar{\eta} \vee w = 2 \sum_{|J|=\beta+1} \sum_{|I|=\beta} \sum_{j=1}^n \varepsilon_J^{jI} \bar{\eta}_j w_J \, d\bar{z}^I. \end{aligned} \tag{4.15}$$

This can be proved based on (**??**), (**??**) and a straightforward calculation.

Next, if we set

$$\vartheta := -*\partial*, \qquad \bar{\vartheta} := -*\bar{\partial}*, \tag{4.16}$$

then ϑ maps (α,β)-forms into $(\alpha,\beta-1)$-forms, and

$$\vartheta\circ\vartheta = 0, \quad \bar{\vartheta}\circ\bar{\vartheta} = 0, \quad \vartheta\circ\bar{\vartheta} + \bar{\vartheta}\circ\vartheta = 0. \tag{4.17}$$

Suppose now that Ω is an open set of locally finite perimeter in $\mathbb{R}^{2n}$ with outward unit normal $\nu = (\nu_1, \nu_2, ..., \nu_{2n-1}, \nu_{2n})$. We further identify this vector with the 1-form

$$\nu = \nu_1 dx_1 + \nu_2\, dy_1 + \cdots \nu_{2n-1}\, dx_n + \nu_{2n}\, dy_n. \tag{4.18}$$

The complex unit normal is defined as

$$\nu^c := (\nu_1 + i\nu_2, ..., \nu_{2n-1} + i\nu_{2n}) \in \mathbb{C}^n, \tag{4.19}$$

and we set

$$\nu^{1,0} := \sum_{j=1}^{n} \overline{(\nu^c)_j}\, dz_j \in \Lambda^{1,0}\mathbb{C}^n, \quad \nu^{0,1} := \sum_{j=1}^{n} (\nu^c)_j\, d\overline{z}_j \in \Lambda^{0,1}\mathbb{C}^n. \tag{4.20}$$

It follows that

$$\begin{aligned} \nu^{1,0} = \overline{\nu^{0,1}}, &\qquad \nu^{1,0} + \nu^{0,1} = 2\nu, \\ \langle \nu^{1,0}, \nu^{0,1}\rangle = 0, &\qquad |\nu^{1,0}| = |\nu^{0,1}| = 2^{1/2}. \end{aligned} \tag{4.21}$$

Below, we discuss a basic integration by parts formula in a very general setting. This is particularly well-suited for extending a great many integration representation formulas from classical complex analysis, a point on which we shall elaborate later.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set, and define $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Assume that $\partial\Omega$ is Ahlfors regular and satisfies*

$$\mathcal{H}^{2n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{4.22}$$

Thus, in particular, Ω is of locally finite perimeter and, if ν denotes the measure theoretic outward unit normal to $\partial\Omega$, then ν is defined σ-a.e. on $\partial\Omega$.

In this setting, the following integration by parts formula holds

$$\begin{aligned} \int_\Omega \langle \bar{\partial} u, v\rangle\, dV - \int_\Omega \langle u, \vartheta v\rangle\, dV &= \int_{\partial\Omega} \Big\langle \nu^{0,1} \wedge u|_{\partial\Omega}\,,\, v|_{\partial\Omega} \Big\rangle\, d\sigma \\ &= \int_{\partial\Omega} \Big\langle u|_{\partial\Omega}\,,\, \nu^{1,0} \vee v|_{\partial\Omega} \Big\rangle\, d\sigma, \end{aligned} \tag{4.23}$$

for any differential forms $u \in C^0(\Omega, \Lambda^{\alpha,\beta})$*, and* $v \in C^0(\Omega, \Lambda^{\alpha,\beta+1})$ *satisfying*

$$\begin{aligned} &\langle \bar\partial u, v\rangle,\ \langle u, \vartheta v\rangle \in L^1(\Omega), \quad \mathcal{N}u \in L^p(\partial\Omega, d\sigma), \quad \mathcal{N}v \in L^q(\partial\Omega, d\sigma) \\ &\text{the nontangential traces } u\big|_{\partial\Omega},\ v\big|_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \end{aligned} \tag{4.24}$$

for some $1 < p, q < \infty$ *with* $1/p + 1/q < 1$.

Proof. This is proved much as the standard version of (**??**), corresponding to a smooth domain and differential forms which are smooth up to the boundary, with Theorem **??** playing the role of the classical Green's formula. □

For each $\beta = 0, 1, .., n$, consider now the double form

$$\begin{aligned} \Gamma_\beta(\zeta, z) &:= \frac{(n-2)!}{\beta\,!\,2^{\beta+1}\pi^n}\frac{1}{|\zeta - z|^{2n-2}}\left(\sum_{j=1}^n d\bar\zeta_j \otimes dz_j\right)^\beta \\ &= 2^{-\beta}E_n(\zeta, z)\sum_{|I|=\beta} d\bar\zeta^I \otimes dz^I, \end{aligned} \tag{4.25}$$

where we have set

$$E_n(\zeta, z) := \begin{cases} -\dfrac{1}{2\pi}\log|\zeta - z|^2, & \text{for } n = 1, \\ \dfrac{(n-2)!}{2\pi^n}\,|\zeta - z|^{2-2n}, & \text{for } n \geq 2. \end{cases} \tag{4.26}$$

Hence, $\Gamma_\beta(z, \zeta) = \overline{\Gamma_\beta(\zeta, z)}$. Since the surface area of the unit ball in $\mathbb{R}^{2n}$ is given by $\omega_{2n-1} = 2\pi^n/(n-1)!$, it follows that E_n is -2 times the standard fundamental solution for the real Laplacian

$$\Delta := \sum_{j=1}^n (\partial_{x_j}^2 + \partial_{y_j}^2) \quad \text{in} \quad \mathbb{R}^{2n}. \tag{4.27}$$

Next, with

$$\Box := \bar\partial\vartheta + \vartheta\bar\partial = -2\sum_{k=1}^n \partial_{z_k}\partial_{\bar z_k} = -\tfrac{1}{2}\Delta \tag{4.28}$$

denoting the complex Laplacian in $\mathbb{C}^n$, and $\delta_z(\zeta)$ the Dirac Distribution in $\mathbb{R}^n$ with mass at z, we have

$$\Box_\zeta\Gamma_\beta(\zeta, z) = 2^{-\beta}\delta_z(\zeta)\sum_{|I|=\beta} d\bar\zeta^I \otimes dz^I, \tag{4.29}$$

$$\vartheta_\zeta\Gamma_\beta(\zeta, z) = \partial_z\Gamma_{\beta-1}(\zeta, z), \tag{4.30}$$

$$\partial_\zeta\overline{\Gamma_\beta(\zeta, z)} = \vartheta_z\overline{\Gamma_{\beta+1}(\zeta, z)}. \tag{4.31}$$

Then the Bochner-Martinelli kernel for $(0,\beta)$-forms in $\mathbb{C}^n$, $0 \leq \beta \leq n$, is defined as the double differential form

$$K_{n\beta}(\zeta,z) := -*\,\partial_\zeta\overline{\Gamma_\beta(\zeta,z)}. \tag{4.32}$$

If $\partial\Omega$ is a C^1-smooth submanifold of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, then the Bochner-Martinelli integral operator is defined on a $(0,\beta)$-form f on $\partial\Omega$ as

$$\mathcal{B}_\beta f(z) := \int_{\partial\Omega} \iota_\zeta^*\Big(f(\zeta)\wedge K_{n\beta}(\zeta,z)\Big), \qquad z\in\mathbb{C}^n\setminus\partial\Omega, \tag{4.33}$$

where $\iota : \partial\Omega \hookrightarrow \mathbb{C}^n$ is the canonical inclusion. Since, generally speaking,

$$\iota^*(u\wedge *\bar{\omega}) = \langle \nu\wedge u, \omega\rangle\Big|_{\partial\Omega}\, d\sigma, \tag{4.34}$$

where $d\sigma$ is the surface measure on $\partial\Omega$, and ν is the outward unit normal to Ω, an equivalent way of defining the Bochner-Martinelli integral operator on a $(0,\beta)$-form f on the boundary of a C^1-smooth domain Ω is

$$\mathcal{B}_\beta f(z) = -\int_{\partial\Omega} \langle \nu(\zeta)\wedge f(\zeta), \bar{\partial}_\zeta\Gamma_\beta(\zeta,z)\rangle\, d\sigma, \qquad z\in\mathbb{C}^n\setminus\partial\Omega. \tag{4.35}$$

As explained before, it is this expression which we find most suitable for extending the Bochner-Martinelli integral operator to situations when Ω is lacking smoothness in a traditional sense.

We are going to be particularly interested in the scenario when the topological boundary of Ω is not necessarily a submanifold of $\mathbb{C}^n$. Specifically, we make the following definition.

Definition 4.3 *Let $\Omega\subset\mathbb{R}^{2n}$ be a bounded open set of locally finite perimeter. Set $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ and denote by ν the measure theoretic outward unit normal to $\partial\Omega$. In this setting, introduce the Bochner-Martinelli integral operator $\mathcal{B}_\beta$ as in (**??**) and also consider*

$$B_\beta f(z) := -\lim_{\varepsilon\to 0^+} \int\limits_{\substack{\zeta\in\partial\Omega\\ |z-\zeta|>\varepsilon}} \langle \nu(\zeta)\wedge f(\zeta), \bar{\partial}_\zeta\Gamma_\beta(\zeta,z)\rangle\, d\sigma, \qquad z\in\partial\Omega, \tag{4.36}$$

and

$$B_{b,*}f(z) := \sup_{\varepsilon>0}\left| \int\limits_{\substack{\zeta\in\partial\Omega\\ |z-\zeta|>\varepsilon}} \langle \nu(\zeta)\wedge f(\zeta), \bar{\partial}_\zeta\Gamma_\beta(\zeta,z)\rangle\, d\sigma\right|, \qquad z\in\partial\Omega. \tag{4.37}$$

At this point, we are well-positioned to state and prove the theorem below, which constitutes the main result of this paper. To state it, we introduce

$$L^p(\partial\Omega,\Lambda^{\alpha,\beta}) := L^p(\partial\Omega,d\sigma)\otimes\Lambda^{\alpha,\beta}\mathbb{C}^n, \tag{4.38}$$

i.e., the space of differential forms of type (α,β) with coefficients from $L^p(\partial\Omega,d\sigma)$.

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be a UR domain, and fix $\beta \in \{0, ..., n\}$. Also, recall the operators (**??**), (**??**), (**??**). Then, for each $p \in [1, \infty)$, $f \in L^p(\partial\Omega, \Lambda^{0,\beta})$, the limit in (**??**) exists for σ-a.e. $z \in \partial\Omega$. Also,*

$$B_\beta,\, B_{\beta,*} : L^p(\partial\Omega, \Lambda^{0,\beta}) \longrightarrow L^p(\partial\Omega, \Lambda^{0,\beta}), \quad p \in (1, \infty), \tag{4.39}$$

$$B_\beta,\, B_{\beta,*} : L^1(\partial\Omega, \Lambda^{0,\beta}) \longrightarrow L^{1,\infty}(\partial\Omega, \Lambda^{0,\beta}), \tag{4.40}$$

are bounded. Also, for every $p \in (1, \infty)$, there exists a finite constant $C = C(\Omega) > 0$ with the property that for every $f \in L^p(\partial\Omega, \Lambda^{0,\beta})$,

$$\|\mathcal{N}(\mathcal{B}_\beta f)\|_{L^p(\partial\Omega, d\sigma)} \leq C \|f\|_{L^p(\partial\Omega, \Lambda^{0,\beta})}. \tag{4.41}$$

Moreover, if $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^{2n} \setminus \bar{\Omega}$, then the jump-formulas

$$\mathcal{B}_\beta f \Big|_{\partial\Omega_\pm} = \pm\tfrac{1}{4}\, \nu^{1,0} \vee (\nu^{0,1} \wedge f) + B_\beta f \tag{4.42}$$

is valid at σ-a.e. point $z \in \partial\Omega$, whenever $f \in L^p(\partial\Omega, \Lambda^{0,\beta})$, $1 \leq p < \infty$. Furthermore, if f is complex tangential (i.e., $\nu^{1,0} \vee f = 0$ σ-a.e. on $\partial\Omega$) then in fact

$$\mathcal{B}_\beta f \Big|_{\partial\Omega_\pm} = (\pm\tfrac{1}{2} I + B_\beta) f, \tag{4.43}$$

where I is the identity operator.

Proof. The claims in the first part of the theorem, up to and including (**??**), can be proved by invoking the results from § **??**; we leave the details to the interested reader. What we shall be focusing on is establishing the jump-formulas (**??**). To set the stage, we note that

$$\begin{aligned} \bar{\partial}_\zeta \Gamma_\beta(\zeta, z) &= \Big(\sum_{j=1}^n \partial_{\bar{\zeta}_j}\, d\bar{\zeta}_j \wedge \cdot\Big) 2^{-\beta} E_n(\zeta, z) \sum_{|I|=\beta} d\bar{\zeta}^I \otimes dz^I \\ &= 2^{-\beta} \sum_{j=1}^n \sum_{|I|=\beta} \partial_{\bar{\zeta}_j} [E_n(\zeta, z)] (d\bar{\zeta}_j \wedge d\bar{\zeta}^I) \otimes dz^I. \end{aligned} \tag{4.44}$$

Moreover, if $f(\zeta) = \sum_{|I|=\beta} f_I(\zeta)\, d\zeta^I$, then

$$\begin{aligned} \nu \wedge f &= 2^{-1} \nu^{0,1} \wedge f + 2^{-1} \nu^{1,0} \wedge f \qquad (4.45) \\ &= 2^{-1} \sum_{|J|=\beta+1} (\nu^{0,1} \wedge f)_J\, d\bar{\zeta}^J + 2^{-1} \sum_{|I|=\beta} \sum_{j=1}^n \overline{(\nu^c)_j} f_I\, d\zeta_j \wedge d\bar{\zeta}^I, \end{aligned}$$

which gives a decomposition of $\nu \wedge f$ as a sum of two differential forms of type $(0, \beta + 1)$ and $(1, \beta)$, respectively. Given that $\bar{\partial}_\zeta \Gamma_\beta(\zeta, z)$ is a double form of type $((0,\beta+1),(\beta,0))$, we may therefore write

$$\begin{aligned}
&-\langle \nu(\zeta)\wedge f(\zeta), \bar\partial_\zeta \Gamma_\beta(\zeta,z)\rangle \qquad (4.46)\\
&\qquad = -\sum_{|J|=\beta+1}\sum_{|I|=\beta}\sum_{j=1}^{n} \varepsilon_J^{jI}(\nu^{0,1}\wedge f)_J(\zeta)\overline{\partial_{\bar\zeta_j}[E_n(\zeta,z)]}\,d\bar z^I\\
&\qquad = -\sum_{|J|=\beta+1}\sum_{|I|=\beta}\sum_{j=1}^{n} \varepsilon_J^{jI}(\nu^{0,1}\wedge f)_J(\zeta)\partial_{\zeta_j}[E_n(\zeta,z)]\,d\bar z^I.
\end{aligned}$$

Also, for every $j\in\{1,...,n\}$, Theorem **??** gives that, at σ-a.e. boundary points z,

$$\begin{aligned}
&\text{the jump induced by the kernel } -\partial_{\zeta_j}[E_n(\zeta,z)] \text{ is}\\
&\pm\tfrac{1}{2}\,\overline{(\nu^c)}_j(z) \text{ times the corresponding integral density.}
\end{aligned} \qquad (4.47)$$

Thanks to (**??**) we may therefore conclude that at σ-a.e. $z\in\partial\Omega$,

$$\begin{aligned}
&\pm\sum_{|J|=\beta+1}\sum_{|I|=\beta}\sum_{j=1}^{n} \varepsilon_J^{jI}(\nu^{0,1}\wedge f)_J(z)\overline{(\nu^c)}_j(z)\,d\bar z^I\\
&\qquad\qquad = \pm\tfrac{1}{4}\,\nu^{1,0}(z)\vee(\nu^{0,1}(z)\wedge f(z)), \qquad (4.48)
\end{aligned}$$

which finishes the proof of the jump-formulas in (**??**).

As for (**??**), it suffices to observe that if f is complex tangential, then

$$\nu^{1,0}\vee(\nu^{0,1}\wedge f) = -\nu^{0,1}\wedge(\nu^{1,0}\vee f)+\langle\nu^{1,0},\overline{\nu^{0,1}}\rangle f = 0+2f = 2f. \qquad (4.49)$$

by Lemma **??** and (**??**). □

References

[1] S.-C. Chen and M.-C. Shaw, *Partial Differential Equations in Several Complex Variables*, AMS/IP Studies in Advanced Mathematics, Vol. 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

[2] G. David, *Opérateurs d'intégrale singulière sur les surfaces régulières*, Ann. Scient. Ecole Norm. Sup., 21 (1988), 225–258.

[3] G. David and S. Semmes, *Singular Integrals and Rectifiable Sets in* $\mathbb{R}^n$*: Beyond Lipschitz Graphs*, Astérisque No. 193, 1991.

[4] H. Federer, *Geometric Measure Theory*, reprint of the 1969 edition, Springer-Verlag, 1996.

[5] J.B. Garnett, *Bounded Analytic Functions*, Revised first edition, Graduate Texts in Mathematics, 236, Springer, New York, 2007.

[6] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[7] S. Hofmann, M. Mitrea and M. Taylor, *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains*, preprint (2007).

[8] I. Lieb and J. Michel, *The Cauchy-Riemann Complex. Integral Formulae and Neumann Problem*, Aspects of Mathematics, E34, Friedr. Vieweg & Sohn, Braunschweig, 2002.

[9] R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Graduate Texts in Mathematics, Vol. 108, Springer-Verlag, New York, 1986.

[10] W. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.

`Irina Mitrea`
Department of Mathematics
University of Virginia
Charlottesville, VA 22904-4137, USA
`e-mail`: *im3p@virginia.edu*

M. Mitrea
Department of Mathematics
University of Missouri-Columbia
202 Mathematical Sciences Building
Columbia, MO 65211, USA
`e-mail`: marius@math.missouri.edu

M.-C. Shaw
Department of Mathematics
University of Notre Dame
244 Hayes-Healy
Notre Dame, IN 46556, USA
`e-mail`: shaw.1@nd.edu