LECTURES ON METHODS OF PARTIAL DIFFERENTIAL EQUATIONS IN SEVERAL COMPLEX VARIABLES [1]

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1. The Levi problem in several complex variables

1.1. The Cauchy-Riemann Equation in $\mathbb{C}$.

Let $\mathbb{C}$ be the complex Euclidian space with coordinate $z = x+iy$. We can define the *Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

and we say that a function h is *holomorphic* in a domain D of $\mathbb{C}$, and we write $h \in \mathcal{O}(D)$, if and only if

$$\frac{\partial h}{\partial \bar z} = 0. \tag{1.1}$$

Equation (1.1) is called the *homogeneous Cauchy-Riemann equation* in $\mathbb{C}$. The behavior of holomorphic function and the study of the homogeneous Cauchy-Riemann equation is closely related to the solvability and regularity of the *inhomogeneous Cauchy-Riemann equation*

$$\frac{\partial u}{\partial \bar z} = f, \tag{1.2}$$

with f a given function.

We have the following result.

Theorem 1.1. *Let D be a bounded domain in $\mathbb{C}$ and let $f \in C^k(\bar D)$, for $k \geq 1$. Then the function defined by*

$$u(z) := \frac{1}{2\pi i}\iint_D \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar\zeta,$$

is in $C^k(D)$ and satisfies (1.2). Moreover, if f is only in $C(\bar{D})$, then $u(z)$ defined as before satisfies (1.2) in the distribution sense.

Remark 1.2. Observe that the function

$$u(z) = \frac{1}{\pi} \cdot \frac{1}{z},$$

is a fundamental solution to (1.1).

This can be derived by differentiating the fundamental solution for $\triangle$. Since

$$\frac{1}{2\pi} \triangle \log |z| = \frac{2}{\pi} \frac{\partial^2}{\partial \bar{z} \partial z} \log |z| = \delta_0,$$

where δ_0 is the Dirac delta function centered at 0, we have

$$\frac{2}{\pi} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \log |z| = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} \frac{1}{z} = \delta_0.$$

This implies that $1/\pi z$ is a fundamental solution for $\partial/\partial\bar{z}$.

Let D be a bounded domain in $\mathbb{C}$ with C^1 boundary. Let f be a continous function f on $\overline{D}$. We define $u = Gf$ to be the unique solution to the Dirichlet problem with $\triangle u = f$ with $u = 0$ on bD. It follows that $\partial^2 u/\partial\bar{z}\partial z = f$ in $\mathbb{C}$ in the distribution sense. Thus we have $f = \bar{\partial}(\partial G f)$ and the function $\partial G f = u$ satisfies $\bar{\partial} u = f$.

1.2. **The Cauchy-Riemann equations in $\mathbb{C}^n$.**

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidian space. We denote coordinates by $z = (z_1, \dots, z_n)$, where $z_j = x_j + iy_j$, $1 \le j \le n$. We can define the *Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),$$

for $1 \le j \le n$. It is easy to see that the foolowing decomposition holds

$$\mathbb{C}T\mathbb{C}^n = T^{1,0}(\mathbb{C}^n) \oplus T^{0,1}(\mathbb{C}^n),$$

where

$$T^{1,0}(\mathbb{C}^n) = \text{span} \left\{ \frac{\partial}{\partial z_j} \right\}, \qquad T^{0,1}(\mathbb{C}^n) = \text{span} \left\{ \frac{\partial}{\partial \bar{z}_j} \right\}.$$

Definition 1.3. Let f a C^1 function defined on an open subset D of $\mathbb{C}^n$. We say that f is *holomorphic* in D and we write $f \in \mathcal{O}(D)$ if $f(z)$ is holomorphic in each variable z_j, when the others are fixed, i.e. f satisfies

$$\frac{\partial f}{\partial \bar{z}_j} = 0,$$

for all $1 \le j \le n$.

The *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial \bar{z}_j} = f_j, \qquad j = 1, \dots, n, \quad n \ge 2. \tag{1.3}$$

This system is overdetermined (one unknown function with n equations). In order for (1.3) to be solvable, f must satisfy the following *compatibility conditions*

$$\frac{\partial f_k}{\partial \bar{z}_j} = \frac{\partial f_j}{\partial \bar{z}_k}, \qquad 1 \leq j, k \leq n. \tag{1.4}$$

As in the 1-dimensional case, we can state the following for the solvability of Cauchy Riemann equation.

Theorem 1.4. *Let $f_j \in C_0^\infty(D)$, $1 \leq j \leq n$, where D is a relatively compact subset of $\mathbb{C}^n$, $n \geq 2$. Suppose that f_j satisfy* (1.4)*. Then there is a function $u \in C_0^\infty(D)$ which satisfies* (1.3)*.*

Proof. Set

$$u(z) = \frac{1}{2\pi i} \iint_D \frac{f_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta}.$$

Then u has compact support and we have

$$\frac{\partial u}{\partial \bar{z}_1} = f_1, \qquad \frac{\partial u}{\partial \bar{z}_j} = f_j, \quad j = 2, \ldots, n.$$

□

One of the most striking difference between the 1-dimensional and the n-dimensional case is the so called *Hartogs phenomenon*, which states that, if a bounded domain D in $\mathbb{C}^n$, $n \geq 2$, has connected boundary, then any holomorphic function f defined in an open neighborhood of the boundary bD can be holomorphically extended to all D. This is not true in $\mathbb{C}$, as one can check easily with the function $f(z) = 1/z$, which is holomorphic in $\mathbb{C} \setminus \{0\}$ and cannot be extended in any way as an entire function.
More precisely, we can state the following.

Theorem 1.5 (Hartogs)**.** *Let D be an open set in $\mathbb{C}^n$ and let K be a compact subset in D, such that $D \setminus K$ is connected. If $f \in \mathcal{O}(D \setminus K)$, then f can be extended holomorphically to all D.*

Proof. Let $\chi \in C_0^\infty(D)$ a cut-off function such that $\chi \equiv 1$ on some open neighborhood of K. Then

$$\bar{\partial}((1 - \chi)f) = -\bar{\partial}\chi \cdot f,$$

has compact support and satisfies compatibility condition in $\mathbb{C}^n$.

Then, from the previous theorem, there is u with compact support such that

$$\bar{\partial}u = -\bar{\partial}(\chi f).$$

Finally, the function

$$h = (1 - \chi)f - u,$$

is the desired extension. □

1.3. **Domains of holomorphy.**

Definition 1.6. A domain D in $\mathbb{C}^n$ is called a *domain of holomorphy* if is not possible to find two nonempty open sets D_1, D_2 in $\mathbb{C}^n$ such that
(*i*) D_1 is connected, $D_1 \subsetneq D$, $D_2 \subset D \cap D_1$;
(*ii*) for any $f \in \mathcal{O}(D)$ there is $\tilde{f} \in \mathcal{O}(D_1)$ such that $f|_{D_2} = \tilde{f}$.

Equivalently, D is a domain of holomorphy if and only if for any $p \in bD$, there exists a holomorphic function $f \in \mathcal{O}(D)$ such that f is singular at p.

An important notion related to domain in $\mathbb{C}^n$ is that of *pseudoconvexity.*

Definition 1.7. Let D be a domain of $\mathbb{C}^n$ with C^2 boundary, i.e. there is a neighborhood U of D and a function $\rho : D \to \mathbb{R}$ of class C^2, such that

$$D \cap U = \{z \in \mathbb{C}^n | \rho(z) < 0\}, \qquad bD \cap U = \{z \in \mathbb{C}^n | \rho(z) = 0\},$$

$$|\nabla \rho(z)| \neq 0, \quad z \in bD.$$

We say that D is *pseudoconvex* if

$$\mathcal{L}_p(\rho; a) : (= \partial\bar{\partial}\rho|_p(a, \bar{a})) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) a_j \bar{a}_k \geq 0, \qquad \forall\ p \in bD,$$

for all vector $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ ($a \in T_p^{0,1}(bD)$) such that

$$\sum_{j=1}^{n} a_j \frac{\partial \rho}{\partial z_j} = 0.$$

$\mathcal{L}_p(\rho; a)$ is called the *Levi form* of ρ at p. A domain D is called strictly pseudoconvex if the Levi form is positive definite.

A function ρ is called (strictly) plurisubharmonic at p if $\mathcal{L}_p(\rho; a) \geq 0$ (> 0) for all vectors $a \in \mathbb{C}^n$.

A domain D is pseudoconvex if and only if it is union of strictly pseudoconvex domains $\{D_j\}$ with C^2 boundary such that $D = \cup_j D_j$ and $D_j \subset D_{j+1}$ for each j.

Theorem 1.8. *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary bD. Then there exists a C^2 defining function for D which is strictly plurisubharmonic in a neighborhood of bD.*

It is easy to see that convex domains are domains of holomorphy.

It is also easy to show that every strictly pseudoconvex domain is *locally* biholomorphic to a strictly convex domain. Thus locally for each $p \in bD$, one can find a neighborhood U_p and a holomorphic function $f_p : U_p \cap D$ so that f_p cannot be extended holomorphically past p. Can one find a f_p holomorphic in D?

1.4. **The Levi Problem and the $\bar{\partial}$ Equation.** Let D be a pseudoconvex domain in $\mathbb{C}^n$ with $n \geq 2$. One of the major problems in complex analysis is to show that a pseudoconvex domain D is a domain of holomorphy. Near each boundary point $p \in bD$, one must find a holomorphic function $f(z)$ on D which cannot be continued holomorphically near p. This problem is called the Levi problem for D at p. It involves the construction of a holomorphic function with certain specific local properties.

If the domain D is strongly pseudoconvex with C^∞ boundary bD and $p \in bD$, one can construct a *local* holomorphic function f in an open neighborhood U of p, such that f is holomorphic in $U \cap D$, $f \in C(\overline{D} \cap U \setminus \{p\})$ and $f(z) \to \infty$ as $z \in D$ approaches p. In fact f can be easily obtained as follows: let r be a strictly plurisubharmonic defining function for D and we assume that $p = 0$. Let

$$F(z) = -2\sum_{i=1}^{n} \frac{\partial r}{\partial z_i}(0) z_i - \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial z_j}(0) z_i z_j.$$

$F(z)$ is holomorphic, and it is called the Levi polynomial of r at 0. Using Taylor's expansion at 0, there exists a sufficiently small neighborhood U of 0 and $C > 0$ such that for any $z \in \overline{D} \cap U$,

$$\mathrm{Re} F(z) = -r(z) + \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + O(|z|^3) \geq C|z|^2.$$

Thus, $F(z) \neq 0$ when $z \in \overline{D} \cap U \setminus \{0\}$. Setting

$$f = \frac{1}{F},$$

it is easily seen that f is locally a holomorphic function which cannot be extended holomorphically across 0.

Global holomorphic functions cannot be obtained simply by employing smooth cut-off functions to patch together the local holomorphic data, since the cut-off functions are no longer holomorphic. Let χ be a cut-off function such that $\chi \in C_0^\infty(U)$ and $\chi = 1$ in a neighborhood of 0. We note that χf is not holomorphic in D. However, if χf can be corrected by solving a $\bar{\partial}$-equation, then the Levi problem will be solved.

Let us consider the (0,1)-form g defined by

$$g = \bar{\partial}(\chi f) = (\bar{\partial}\chi) f.$$

This form g can obviously be extended smoothly up to the boundary. It is easy to see that g is a $\bar{\partial}$-closed form in D and $g \in C^\infty_{(0,1)}(\overline{D})$. If we can find a solution $u \in C^\infty(\overline{D})$ such that

$$\bar{\partial} u = g \quad \text{in} \quad D,$$

then we define for $z \in D$,

$$h(z) = \chi(z) f(z) - u(z).$$

It follows that h is holomorphic in D, $h \in C^\infty(\overline{D} \setminus \{0\})$ and h is singular at 0. Thus one can solve the Levi problem for strongly pseudoconvex domains provided one can solve equation (3.6.1) with solutions smooth up to the boundary.

It has been known that domains of holomorphy are pseudoconvex. The converse whether a pseudoconvex domain is a domain of holomorphy is the so called *Levi problem*: given a domain D is it possible, for each point $p \in bD$, to find a holomorphic function f over D such that it cannot be continued holomorphically near p? The answer to this question is given in the following theorem.

Theorem 1.9. *Let D be a domain in $\mathbb{C}^n$. The following are equivalent:*
(i) D is pseudoconvex;
(ii) D is a domain of holomorphy;
(iii) for each $f \in C^\infty(D)$ with $\bar\partial f = 0$ there is $u \in C^\infty(D)$ such that $\bar\partial u = f$.

It is well known that (ii) implies (i).
To show that (i) implies (ii) is the so-called Levi problem.
We will show that (i) implies (iii) and (iii) implies (ii).
The Levi problem has been solved by
(1) (1942) Oka for $n = 2$ and later for all $n \geq 2$.
(2) (1958) Grauert using Sheaf methods.
(3) (1962) Kohn by the $\bar\partial$-Neumann problem.
(4) (1965) Hörmander by the weighted $\bar\partial$-Neumann problem.

2. L^2 theory for $\bar\partial$ on pseudoconvex domains in $\mathbb{C}^n$

By the Hilbert space theory of unbounded linear operator, if H_1 and H_2 are two Hilbert spaces, $T : H_1 \to H_2$ is a closed, densely defined linear operator, then

$$H_2 = \overline{\mathcal{R}(T)} \oplus \mathrm{Ker}(T^*),$$

and

$$H_1 = \overline{\mathcal{R}(T^*)} \oplus \mathrm{Ker}(T).$$

If we can show that $\mathcal{R}(\bar\partial)$ is closed and then, if $\bar\partial u = f$ can be solved, we must have $f \perp \mathrm{Ker}(\bar\partial^*)$, because if g is such that $\bar\partial^* g = 0$, then

$$(f, g) = (\bar\partial u, g) = (u, \bar\partial^* g) = 0.$$

For a closed densely defined linear operator T, $\mathcal{R}(T)$ is closed if and only if

$$\|Tu\| \geq C\|u\|, \qquad u \in \mathrm{Dom}(T) \cap (\mathrm{Ker}(T))^\perp$$

or if and only if

$$\|T^*u\| \geq C\|u\|, \qquad u \in \mathrm{Dom}(T^*) \cap (\mathrm{Ker}(T^*))^\perp.$$

2.1. **The $\bar{\partial}$-Neumann problem.** Let D be a domain in $\mathbb{C}^n$. Let f be a function of class C^1 and Let $(z_1, \dots, z_n)$ be the complex coordinates for $\mathbb{C}^n$, we get

$$df = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right).$$

We define

$$\bar{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j, \qquad \partial f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j$$

so that $d = \partial + \bar{\partial}$. This means that the differential of a function can be decomposed in the sum of a $(1,0)$-form ∂f and a $(0,1)$-form $\bar{\partial} f$. A function f is holomorphic if and only if $\bar{\partial} f = 0$.
We define the space of (p,q)-forms $\Lambda^{p,q}(D)$ as the subspace of $(p+q)$-forms which can be written as

$$f = \sum_{|I|=p, |J|=q} f_{IJ} dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \dots, i_p)$ and $J = (j_1, \dots, j_q)$ are multiindices of length p and q respectively, $dz^I = dz^{i_1} \wedge \dots \wedge dz^{i_p}$, $d\bar{z}^J = d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$.

Since d is a complex, it is easy to see that the sequence

$$\dots \xrightarrow{\bar{\partial}} \Lambda^{p,q-1}(D) \xrightarrow{\bar{\partial}} \Lambda^{p,q}(D) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(D) \xrightarrow{\bar{\partial}} \dots$$

is a complex, i.e. $\bar{\partial}^2 = 0$. It is called the *Cauchy-Riemann complex.*

Let $L^2_{(p,q)}(D)$ and $C^\infty_{(p,q)}(D)$ denote the (p,q)-form with L^2 or C^∞ coefficients respectively, and let $(\ ,\ )$ denotes the usual L^2 inner product. We can define the formal adjoint of $\bar{\partial}$ under the L^2-norm as

$$\vartheta : C^\infty_{(p,q)}(D) \to C^\infty_{(p,q-1)}(D),$$

$$(\vartheta f, g) = (f, \bar{\partial} g),$$

for every $g \in C^\infty_{(p,q)}(D)$ with compact support.

We denote the (maximal) L^2 closure of $\bar{\partial}$ with the same symbol. Then we can define its L^2 adjoint $\bar{\partial}^*$ as Hilbert space adjoint by

$$\bar{\partial}^* : L^2_{(p,q)}(D) \to L^2_{(p,q-1)}(D),$$

in the following sense: f is in $\mathrm{Dom}(\bar{\partial}^*)$, the domain of $\bar{\partial}^*$, if there is a $g \in L^2_{(p,q-1)}(D)$ such that, for every $\psi \in \mathrm{Dom}(\bar{\partial}) \cap L^2_{(p,q-1)}(D)$ we have

$$(f, \bar{\partial}\psi) = (g, \psi).$$

Then we define $\bar{\partial}^* f = g$.

We have that $\bar{\partial}$ is an unbounded operator and $\mathrm{Dom}(\bar{\partial}) \subsetneq L^2_{(p,q)}(D)$. Moreover, if D is a bounded domain, then $C^\infty_{(p,q)}(\bar{D}) \subset \mathrm{Dom}(\bar{\partial})$. Finally, $\bar{\partial}$ is a closed, densely defined linear operator. The same is true for $\bar{\partial}^*$ too.

If $f \in C^1_{(p,q)}(\bar{D}) \cap \mathrm{Dom}(\bar{\partial}^*)$, then

$$(f, \bar{\partial} u) = (\bar{\partial}^* f, u), \qquad \forall\ u \in \mathrm{Dom}(\bar{\partial}).$$

If $u \in C^1_{(p,q-1)}(\bar{D})$ and D is a C^1 domain with defining function ρ, then

$$(f, \bar{\partial} u) = (\bar{\partial}^* f, u) = (\vartheta f, u) + \int_{b\Omega} \langle f \lrcorner \bar{\partial}\rho, u\rangle d\sigma.$$

Since compactly supported forms in $C^\infty_{(p,q)}(\bar{D})$ are dense in $L^2_{(p,q)}(D)$, it follows that if $f \in C^1_{(p,q)}(\bar{D}) \cap \mathrm{Dom}(\bar{\partial}^*)$, we must have

$$\bar{\partial}^* f = \vartheta f,$$

and

$$f \lrcorner \bar{\partial}\rho = 0 \text{on } bD.$$

More explicitly, when $f \in C^1_{(0,1)}(\bar{D})$, this condition becomes

$$\sum_{j=1}^n f_j(\zeta) \frac{\partial \rho}{\partial \bar{\zeta}_j}(\zeta) = 0, \qquad \forall\ \zeta \in bD.$$

We can now define the Laplacian of the $\bar{\partial}$-complex. It is the operator

$$\Box := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D),$$

with the domain of definition

$$\mathrm{Dom}(\Box) := \{f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) : \bar{\partial} f \in \mathrm{Dom}(\bar{\partial}^*), \bar{\partial}^* f \in \mathrm{Dom}(\bar{\partial})\}.$$

The operator $\Box$ is a linear, closed, densely defined, self adjoint operator.

As in the case to be in $\mathrm{Dom}(\bar{\partial}^*)$, in order to belong to $\mathrm{Dom}(\Box)$, a (p, q)-form f must satisfies the $\bar{\partial}$-Neumann boundary conditions

$$f \lrcorner \bar{\partial}\rho = 0, \qquad \bar{\partial} f \lrcorner \bar{\partial}\rho = 0,$$

on bD. The boundary value problem

$$\begin{cases} \Box u = f & \text{in } D \\ u \lrcorner \bar{\partial}\rho = 0 & \text{in } bD \\ \bar{\partial} u \lrcorner \bar{\partial}\rho = 0 & \text{in } bD \end{cases}$$

is the $\bar{\partial}$-*Neumann problem.*

Example.

Let D be a bounded domain with $0 \in bD$ and assume that there is a neighborhood U of 0 such that

$$D \cap U = \{(z_1, \ldots, z_n) \in U : \Im(z_n) < 0\}.$$

If we take $u \in C^2_{(0,1)}(\bar{D})$, then we can write $u = \sum_{j=1}^n u_j d\bar{z}_j$ and $u \in \mathrm{Dom}(\Box)$ if and only if

$$(1) \qquad u_n = 0, \qquad \text{on } bD \cap U,$$

$$(2) \qquad \frac{\partial u_j}{\partial \bar{z}_n} = 0, \qquad \text{on } bD \cap U, \quad j = 1, \dots, n-1.$$

In fact the condition $u \lrcorner \bar{\partial}\rho = 0$ is equivalent to

$$\sum_{j=1}^n u_j \frac{\partial \rho}{\partial \bar{z}_j} = 0 \Leftrightarrow u_n = 0,$$

on bD, while the condition $\bar{\partial} u \lrcorner \bar{\partial}\rho$ can be rewritten as

$$\frac{\partial u_j}{\partial \bar{z}_n} - \frac{\partial u_n}{\partial \bar{z}_j} = 0 \Leftrightarrow \frac{\partial u_j}{\partial \bar{z}_n} = 0, \quad j = 1, \dots, n-1,$$

on bD.

Actually, to obtain L^2 existence results, we need to work in the weighted L^2 spaces. If $\varphi \in C^2(\bar{D})$ we define

$$L^2(D, \varphi) := \left\{ f : \|f\|^2_\varphi := \int_D |f|^2 e^{-\varphi} dV < \infty \right\},$$

the *weighted* L^2 *space* with *weight* φ. Observe that $\bar{\partial}^*_\varphi$, the Hilbert space adjoint of $\bar{\partial}$ in $L^2(D, \varphi)$, is related to $\bar{\partial}^*$ by

$$\bar{\partial}^*_\varphi = e^{\varphi} \left(\bar{\partial}^* e^{-\varphi} \right) = \bar{\partial}^* + A_0(\varphi),$$

where $A_0(\varphi)$ is a zeroth order operator which depends on φ. We set up the weighted $\bar{\partial}$-Neumann problem $\Box_\phi = \bar{\partial}\bar{\partial}^*_\phi + \bar{\partial}^*_\phi \bar{\partial}$ as before.

2.2. L^2 existence for $\bar{\partial}$ and $\bar{\partial}$-Neumann operator.

From the preliminary lemma, $\mathcal{R}(\Box)$ is closed if and only if

$$\|u\| \le C \|\Box u\|, \qquad \forall\ u \in \mathrm{Dom}(\Box) \cap (\mathrm{Ker}(\Box))^\perp.$$

In order to prove the L^2 existence both for $\bar{\partial}$ and $\bar{\partial}$-Neumann operator, we need the following *a priori* identity.

Proposition 2.1 (Morrey-Kohn). *Let $D \subset\subset \mathbb{C}^n$ be a domain with C^2 boundary and defining function ρ. For each $f \in C^1_{(p,q)}(\bar{D}) \cap \mathrm{Dom}(\bar{\partial}^*)$ we have*

$$(2.1) \qquad \|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 = \sum_{j=1}^n \left\| \frac{\partial f}{\partial \bar{z}_j} \right\|^2 + \sum_{j,k=1}^n \int_{b\Omega} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} f_{I,jK} \bar{f}_{J,kK} d\sigma.$$

Under this assumptions, we can rewrite a more general form for the identity of Morrey-Kohn-Hörmander:

$$\begin{aligned}\|\bar\partial f\|_\varphi^2 + \|\bar\partial_\varphi^* f\|_\varphi^2 &= \sum_{j=1}^n \left\| \frac{\partial f}{\partial \bar z_j} \right\|_\varphi^2 + \sum_{j,k=1}^n \int_{b\Omega} \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} f_{I,jK} \bar f_{J,kK} e^{-\varphi} d\sigma \\ &\quad + \sum_{j,k=1}^n \int_\Omega \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} f_{I,jK} \bar f_{J,kK} e^{-\varphi} dV. \end{aligned} \tag{2.2}$$

Since D is pseudoconvex, we have

$$\sum_{j,k=1}^n \int_{b\Omega} \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} f_{I,jK} \bar f_{J,kK} e^{-\varphi} d\sigma \geq 0. \tag{2.3}$$

If we choose $\varphi = |z|^2$, the last term becomes

$$\int_\Omega |f|^2 e^{-\varphi} dV = \|f\|_\varphi^2.$$

Then, if D is pseudoconvex, we get

$$(\Box f, f)_\varphi = \|\bar\partial_\varphi^* f\|_\varphi^2 + \|\bar\partial f\|_\varphi^2 \geq \|f\|_\varphi^2,$$

which is the *a priori* estimate.

A crucial fact to prove the actual estimate from the *a priori* estimate is the following lemma.

Lemma 2.2 (A Density Lemma). *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Then $C^1_{(p,q)}(\bar D) \cap \mathrm{Dom}(\bar\partial_\varphi^*)$ is dense in $\mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial_\varphi^*)$ in the graph norm $f \mapsto \|f\|_\varphi + \|\bar\partial f\|_\varphi + \|\bar\partial^* f\|_\varphi$.*

Proof. We shall give only the main steps of the proof, which is essentially a variation of the Friedrichs' Lemma[2]. By a partition of unity, we may assume that the domain is star-shaped and $0 \in D$.

(**1**) The space $C^\infty_{(p,q)}(\bar D)$ *is dense in* $\mathrm{Dom}(\bar\partial)$ *in the graph norm* $f \mapsto \|f\|_\varphi + \|\bar\partial f\|_\varphi$.

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[2] **Friedrichs' lemma**: Let $\chi \in C_0^\infty(\mathbb{R}^N)$ a function with support in the unit ball and such that

$$\int_{\mathbb{R}^N} \chi dV = 1,$$

and define $\chi_\epsilon(x) = \epsilon^{-N}\chi(x/\epsilon)$. If $v \in L^2(\mathbb{R}^N)$ is with compact support and u is a C^1 function in a neighborhood of the support of v, we have

$$uD_j(v * \chi_\epsilon) - (uD_j v) * \chi_\epsilon \to 0, \quad \text{as } \epsilon \to 0,$$

where $D_j = \partial/\partial x_j$ and $uD_j v$ is defined in the sense of distribution.

Using convolution, we have that if $f \in L^2_{p,q}(D) \cap \mathrm{Dom}(\bar\partial)$ and $f^\epsilon = f(\frac{1}{1+\epsilon}z)$, then

$$\left.\begin{array}{l} f^\epsilon * \chi_\epsilon \to f \\ \\ \bar\partial(f^\epsilon * \chi_\epsilon) \to \bar\partial f \end{array}\right\} \qquad \epsilon \to 0.$$

The first sequence converges since it follows easily from the Young's inequality:

$$\|f^\epsilon * \chi_\epsilon\| \le \|f^\epsilon\| \to \|f\|.$$

The second sequence converges from

$$\bar\partial(f^\epsilon * \chi_\epsilon) = \frac{1}{1+\epsilon}(\bar\partial f)(\frac{1}{1+\epsilon}z).$$

In other words, the weak maximal closure of $\bar\partial$ is equal to the strong maximal closure.

(**2**) $C^\infty_{(p,q)}(\bar D)$ *with compact support in* D *is dense in* $\mathrm{Dom}(\bar\partial^*_\varphi)$.

We assume that $\varphi = 0$. This follows from the fact that $\bar\partial$ is the maximal closure, its adjoint $\bar\partial^*$ is minimal . We extend f to $\tilde f$ by setting $\tilde f = 0$ outside D . Then we claim that $\vartheta \tilde f = \widetilde{\vartheta f} \in L^2(\mathbb{C}^n)$ in the distribution sense. To see this, we have for any $v \in C^\infty_{(p,q-1)}(\overline{D})$,

$$(\tilde f, \bar\partial v)_{\mathbb{C}^n} = (f, \bar\partial v)_D = (\bar\partial^* f, v)_D = (\vartheta f, v)_D = (\widetilde{\vartheta} f, v)_{\mathbb{C}^n}.$$

From (1), $C^\infty_{(p,q-1)}(\overline{D})$ is dense in $\mathrm{Dom}(\bar\partial)$, we have proved the claim.

We can approximate f by $f^{-\epsilon} = \tilde f(\frac{z}{1-\epsilon})$ and then regularize, we get $f^{-\epsilon} * \chi_\epsilon \in C^\infty_0(D)$. Using again Friedrichs' lemma,

$$\left.\begin{array}{l} f^{-\epsilon} * \chi_\epsilon \to f \\ \\ \bar\partial^*(f^{-\epsilon} * \chi_\epsilon) \to \bar\partial^* f \end{array}\right\} \qquad \epsilon \to 0.$$

This shows that the Hilbert space adjoint $\bar\partial^*$ is the same as the strong minimal extension of ϑ.

(**3**) $C^1_{(p,q)}(\bar D) \cap \mathrm{Dom}(\bar\partial^*_\varphi)$ *is dense in* $\mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*_\varphi)$.

This last part is essentially proved using the fact that $f \in C^1_{(p,q)}(\bar D) \cap \mathrm{Dom}(\bar\partial^*_\varphi)$ if and only if $f \lrcorner \bar\partial\rho = 0$ on bD, ρ being a defining function for D. By this and the fact that

$$(\bar\partial f, g) = (f, \vartheta g) + \int_{b\Omega} \langle f \wedge \bar\partial\rho, g\rangle d\sigma = (f, \vartheta g) + \int_{b\Omega} \langle f, g \lrcorner \bar\partial\rho\rangle d\sigma.$$

We first decompose f by $f = f_\nu + f_\tau$ where $f_\nu = f \lrcorner \bar\partial\rho \wedge \bar\partial\rho$ is the complex normal part and f_τ is the complex tangent component. We first approximate

f by

$$f^{(\epsilon)} = f_\tau^\epsilon + f_\nu^{-\epsilon} = f_\tau(\frac{1}{1+\epsilon}z) + \tilde{f}_\nu(\frac{1}{1-\epsilon}z). \tag{2.4}$$

Then we regularize again as in Friederichs' lemma, we obtain

$$\bar{\partial}(f^{(\epsilon)} * \chi_\epsilon) - (\bar{\partial} f_\tau^\epsilon + \bar{\partial} f_\nu^{-\epsilon}) * \chi_\epsilon \to 0$$

since the complex normal component is not in the Cauchy data of $\bar{\partial}$.

On the other hand, we get

$$\vartheta(f^{(\epsilon)} * \chi_\epsilon) - (\vartheta f_\tau^\epsilon + \vartheta f_\nu^{-\epsilon}) * \chi_\epsilon \to 0$$

since $f \in \mathrm{Dom}(\bar{\partial}^*)$. This gives that

$$\left.\begin{aligned} & f^{(\epsilon)} * \chi_\epsilon \to f \\ & \bar{\partial}(f^{(\epsilon)} * \chi_\epsilon) \to \bar{\partial} f \\ & \bar{\partial}^*(f^{(\epsilon)} * \chi_\epsilon) \to \bar{\partial}^* f \end{aligned}\right\} \qquad \epsilon \to 0.$$

This proves the density lemma. $\square$

Using the density lemma, we have, for any $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}_\varphi^*)$

$$\|f\|_\varphi^2 \le \|\bar{\partial} f\|_\varphi^2 + \|\bar{\partial}_\varphi^* f\|_\varphi^2 = (\Box_\varphi f, f)_\varphi \le \|\Box_\varphi f\|_\varphi \cdot \|f\|_\varphi,$$

so that

$$\|f\|_\varphi \le \|\Box_\varphi f\|,$$

which implies that $\Box_\varphi$ has closed range and $\mathrm{Ker}(\Box_\varphi) = 0$.

Now we can prove the L^2 weighted existence of $\bar{\partial}$-Neumann weighted operator. We have the following.

Proposition 2.3. *Let D a bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. For each $1 \le q \le n$ there is a bounded operator $N_\varphi : L^2_{(p,q)}(D, \varphi) \to L^2_{(p,q)}(D, \varphi)$ such that:*

(1) *$f = \bar{\partial}\bar{\partial}_\varphi^* N_\varphi f \oplus \bar{\partial}_\varphi^* \bar{\partial} N_\varphi f$, for any $f \in L^2_{(p,q)}(D, \varphi)$;*

(2) *if $\bar{\partial} f = 0$ then $u_\varphi := \bar{\partial}_\varphi^* N_\varphi f$ is the solution for $\bar{\partial} u = f$ such that*

$$\|u_\varphi\|_\varphi \le \frac{1}{q}\|f\|_\varphi.$$

Theorem 2.4 (L^2 existence theorem for $\bar{\partial}$)**.** *Let $D \subset\subset \mathbb{C}^n$ a bounded pseudoconvex domain. For every $f \in L^2_{(p,q)}(D)$, $1 \le q \le n$, with $\bar{\partial} f = 0$, one can find $u \in L^2_{(p,q-1)}(D)$ such that $\bar{\partial} u = f$ and*

$$\|u\| \le \sqrt{\frac{e}{q}}\delta\|f\|,$$

where $\delta = \mathrm{diam}(D)$.

Choosing $\varphi = t|z|^2$, we get

$$te^{-t\delta^2}\|u\|^2 \leq \frac{1}{q}\|f\|^2. \tag{2.5}$$

This proves Hörmander's theorem with $t = \delta^{-2}$.

Summing up the previous results and passing to the unweighted L^2 spaces, we have the following result due to Hörmander.

Theorem 2.5 (Hörmander)**.** *Let $D \subset\subset \mathbb{C}^n$ a bounded pseudoconvex domain. Then $\Box$ has closed range and $\mathcal{R}(\Box) = L^2_{(p,q)}(D)$, $1 \leq q \leq n$. Moreover, there is a bounded operatore $N : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D)$ such that*

(i) $\Box N = N\Box = I$ on $\mathrm{Dom}(\Box)$;

(ii) $\alpha = \bar{\partial}\bar{\partial}^* N\alpha \oplus \bar{\partial}^*\bar{\partial}N\alpha$, for any $\alpha \in L^2_{(p,q)}(D, \varphi)$;

(iii) if $\alpha \in L^2_{(p,q)}(D)$ such that $\bar{\partial}\alpha = 0$ then $u = \bar{\partial}N\alpha$ is the canonical solution to $\bar{\partial}u = \alpha$ and

$$\|u\| \leq \sqrt{\frac{e}{q}}\delta\|\alpha\|;$$

(iv) we have the following estimates for N:

$$\|Nf\| \leq \frac{e}{q}\delta^2\|f\|,$$

$$\|\bar{\partial}Nf\| \leq \sqrt{\frac{e}{q}}\delta\|f\|,$$

$$\|\bar{\partial}^* Nf\| \leq \sqrt{\frac{e}{q}}\delta\|f\|,$$

for any $f \in L^2_{(p,q)}(D)$.

Theorem 2.6. *Let D be a pseudoconvex domain in $\mathbb{C}^n$. For any $f \in C^\infty_{(p,q)}(D)$, where $0 \leq p \leq n$ and $1 \leq q < n$, such that $\bar{\partial}f = 0$ in D, there exists $u \in C^\infty_{(p,q-1)}(D)$ satisfying $\bar{\partial}u = f$.*

From Hörmander's theorem, we have

Corollary 2.7. *Let D be a pseudoconvex domain in $\mathbb{C}^n$. Then D is a domain of holomorphy.*

2.3. Global Regularity for $\bar{\partial}$ on pseudoconvex domains.

If the boundary bD is smooth, we also have the following global boundary regularity results for $\bar{\partial}$.

Theorem 2.8 (Kohn)**.** *Let $D \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary bD. For any $f \in C^\infty_{(p,q)}(\overline{D})$, where $0 \leq p \leq n$ and $1 \leq q < n$, such that $\bar{\partial}f = 0$ in D, there exists $u \in C^\infty_{(p,q-1)}(\overline{D})$ satisfying $\bar{\partial}u = f$.*

Proof. For any $s > 0$, there exists $T_s >> 1$ such that $N_t \in W^s(D)$ for all $t > T_s$. From the Sobolev embedding theorem, there exists $u_k \in C^k(\overline{D})$ for each $k \in \mathbb{N}$. By a Mittag-Leffler procedure, one can extract a solution $u_\infty \in C^\infty(\overline{D})$. □

Theorem 2.9. *Let* D *be a bounded pseudoconvex domain in* $\mathbb{C}^n$, $n \geq 2$. *For each* $0 \leq p \leq n$, $1 \leq q \leq n$ *and* $t > 0$, *there exists a bounded operator* $N_t : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D)$, *such that*
Range$(N_t) \subset$ *Dom*$(\Box_t)$. $N_t \Box_t = \Box_t N_t = I$ *on Dom*$(\Box_t)$.
For any $f \in L^2_{(p,q)}(D)$, $f = \overline{\partial}\overline{\partial}^*_t N_t f \oplus \overline{\partial}^*_t \overline{\partial} N_t f$.
$\overline{\partial} N_t = N_t \overline{\partial}$ *on Dom*$(\overline{\partial})$, $1 \leq q \leq n-1$,
$\overline{\partial}^*_t N_t = N_t \overline{\partial}^*_t$ *on Dom*$(\overline{\partial}^*_t)$, $2 \leq q \leq n$.
The following estimates hold: For any $f \in L^2_{(p,q)}(D)$,

$$tq \parallel N_t f \parallel_{(t)} \leq \parallel f \parallel_{(t)},$$

$$\sqrt{tq} \parallel \overline{\partial} N_t f \parallel_{(t)} \leq \parallel f \parallel_{(t)},$$

$$\sqrt{tq} \parallel \overline{\partial}^* N_t f \parallel_{(t)} \leq \parallel f \parallel_{(t)} .$$

If $f \in L^2_{(p,q)}(D)$ *and* $\overline{\partial} f = 0$ *in* D, *then for each* $t > 0$, *there exists a solution* $u_t = \overline{\partial}^*_t N_t f$ *satisfying* $\overline{\partial} u_t = f$ *and the estimate*

$$tq \parallel u_t \parallel^2_{(t)} \leq \parallel f \parallel^2_{(t)} .$$

We have chosen $t = \delta^{-2}$, where δ is the diameter of D, to obtain the best constant for the bound of the $\bar{\partial}$-Neumann operator without weights. Our next theorem gives the regularity for N_t in the Sobolev spaces when t is large.

Theorem 2.10. *Let* D *be a smooth bounded pseudoconvex domain in* $\mathbb{C}^n$, $n \geq 2$. *For every nonnegative integer* k, *there exists a constant* $S_k > 0$ *such that the weighted* $\bar{\partial}$*-Neumann operator* N_t *maps* $W^k_{(p,q)}(D)$ *boundedly into itself whenever* $t > S_k$, *where* $0 \leq p \leq n$, $1 \leq q \leq n$.

An operator is called exactly regular on $W^k_{(p,q)}(D)$, $k \geq 0$, if it maps the Sobolev space $W^k_{(p,q)}(D)$ continuously into forms with $W^k(D)$ coefficients. The following theorem shows that all the related operators of N_t are also exactly regular if N_t is exactly regular.

Theorem 2.11. *Let* D *be a smooth bounded pseudoconvex domain in* $\mathbb{C}^n$, $n \geq 2$. *For every nonnegative integer* k, *there exists a constant* $S_k > 0$ *such that for every* $t > S_k$ *the operators* $\overline{\partial} N_t$, $\overline{\partial}^*_t N_t$, $\overline{\partial}\overline{\partial}^*_t N_t$ *and* $\overline{\partial}^*_t \overline{\partial} N_t$ *are exactly regular on* $W^k_{(p,q)}(D)$, *where* $0 \leq p \leq n$, $1 \leq q \leq n$. *Furthermore, there exists a constant* $S'_k > 0$ *such that for* $t > S'_k$, *the weighted Bergman projection* $P_{t,(p,0)}$ *maps* $W^k_{(p,0)}(D)$ *boundedly into itself.*

3. Boundary regularity for the $\bar{\partial}$-Neumann problem

3.1. **$\bar{\partial}$-Neumann problem on strictly pseudoconvex Lipschitz domains.** We are interested in the following questions:

1. Can one solve equation (6.0.1) with a smooth solution $u \in C^{\infty}_{(p,q-1)}(\overline{D})$ if f is in $C^{\infty}_{(p,q)}(\overline{D})$?

2. Does the canonical solution $\overline{\partial}^* N f$ belong to $W^s_{(p,q-1)}(D)$ if f is in $W^s_{(p,q)}(D)$?

We derive the 1/2-subelliptic estimate for the $\bar{\partial}$-Neumann problem operator when Ω is a strictly pseudoconvex Lipschitz domain. We use $W^s(\Omega)$ to denote the Sobolev space, $s \in \mathbb{R}$, and by $\|\cdot\|_s$ its norm.

A bounded pseudoconvex Lipschitz domain Ω in $\mathbb{C}^n$ is said to have a plurisubharmonic Lipschitz defining function ρ if ρ is a global defining function in Ω (or on the boundary of $b\Omega$ and ρ is plurisubharmonic in Ω. Recall that a continuous function is plurisubharmonic if it is subharmonic in every complex line. Plurisubharmonicity is well-defined for continuous functions or even upper semicontinuous functions. We next define strictly (or strongly) pseudoconvex domains with Lipschitz boundaries.

Definition 3.1. A bounded Lipschitz domain Ω in $\mathbb{C}^n$ is called strictly pseudoconvex (with Lipschitz boundary $b\Omega$) if there exists an exhaustion $\{\Omega_\nu\}$ of Ω such that
(1) The sequence $\{\Omega_\nu\}$ is an increasing sequence of relatively compact subsets of Ω and $\Omega = \underset{\nu}{\cup}\Omega_\nu$.
(2) Each Ω_ν has a C^∞ plurisubharmonic defining function η_ν such that η_ν is uniformly bounded in $\overline{\Omega}$ and

$$\sum_{i,j=1}^{n} \frac{\partial^2 \eta_\nu}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \geq c_0 |a|^2 \quad \text{for } z \in \Omega_\nu \cap U \text{ and } a \in \mathbb{C}^n,$$

where U is a neighborhood of bD and $c_0 > 0$ is a constant independent of ν.
(3) There exist positive constants c_1, c_2 such that $c_1 \leq |\nabla \eta_\nu| \leq c_2$ on $\Omega_\nu \cap U$, where c_1, c_2 are independent of ν.

We also have the following definition.

Definition 3.2. A bounded Lipschitz domain D in $\mathbb{C}^n$ is called strictly pseudoconvex if it has a strictly plurisubharmonic Lipschitz defining function, i.e., there exists a Lipschitz function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that ρ is locally a Lipschitz graph and
(1) $\rho < 0$ in D, $\rho > 0$ outside $\overline{D}$ and $C_1 < |d\rho| < C_2$ on bD almost everywhere, where C_1, C_2 are positive constants,
(2) $\rho - C|z|^2$ is plurisubharmonic for some $C > 0$ in $U \cap \Omega$ where U is a neighborhood of $b\Omega$.

Lemma 3.3. *Let D be a bounded Lipschitz domain in $\mathbb{C}^n$. Then D is strictly pseudoconvex in the sense of Definition ***2.1.1 and Definition ***2.1.2 are equivalent.*

Proof. We first assume that D is pseudoconvex in the sense of Definition 2.1.2. We will construct a sequence of subdomains D_ν and ρ_ν. The proof is similar to the proof of Lemma 0.3. Let D_ν and η_ν be similarly defined as in Lemma 0.3. It remains to prove that η_ν satisfies (2) in Definition 2.1.1. But this follows easily from the the fact that suitable regularizations of a plurisubharmonic function are plurisubharmonic.

Let D be strictly pseudoconvex in the sense of Definition 2.1.1. Since the sequence η_ν is uniformly bounded in Λ^1 on each compact subset of D, we have from the Ascoli Theorem that there is a subsequence convergent to some limit function $\eta \in \Lambda^1(D)$. Then η is a Lipschitz defining function which satisfies (1) and (2) in Definition 2.1.2. Thus the two definitions are equivalent.

□

Some examples of strictly pseudoconvex domains with Lipschitz boundaries are given below.

1. Piecewise smooth strictly pseudoconvex domains.

A bounded domain Ω in $\mathbb{C}^n$ is said to have a piecewise smooth strictly pseudoconvex boundary $\partial\Omega$ defined by C^2-differentiable functions if there exists a finite open covering $U_1, \dots, U_k$ of an open neighborhood U of $\partial\Omega$ and C^2 strictly plurisubharmonic functions $\rho_j : U_j \to \mathbb{R}$, $j = 1, \dots, k$ such that

(i) $\Omega \cap U = \{x \in U |$ for each $1 \le i \le k$, either $x \notin U_i$ or $\rho_i(x) < 0\}$,

(ii) for $1 \le i_1 < i_2 < \cdots < i_\ell \le k$, the 1-forms $d\rho_{i_1}, \dots, d\rho_{i_l}$ are linearly independent over $\mathbb{R}$ at every point of $\bigcap_{v=1}^{\ell} U_{i_\nu} \cap \partial\Omega$.

Then Ω is a strongly pseudoconvex domain with Lipschitz boundary in the sense of Definition 2.1.2. To see this, first note that Ω is a Lipschitz domain from the assumption of transversal intersection. Let ρ_i be a C^2 strictly plurisubharmonic defining function for Ω_i and $\Omega = \{z \in \mathbb{C}^n | \rho_i(z) < 0,\ i = 1, \dots, k\}$. Set $\rho(z) = \max\{\rho_1(z), \dots, \rho_k(z)\}$. If ρ_i is only locally defined, we define for $z \in U$,

$$\tilde\rho(z) = \underset{i, z\in U_i}{\longrightarrow} \max\{\rho_i(z)\}$$

and extend $\tilde\rho$ to Ω by $\rho(z) = \max\{\tilde\rho(z), -\delta_0\}$ where $\delta_0 > 0$ is sufficiently small. Then ρ is a plurisubharmonic function. Let $W = \{z \in \Omega | -\delta_0 < \rho(z) < 0\}$.

Since each ρ_i is strictly plurisubharmonic, there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho_\ell}{\partial z_i \partial \bar z_j} a_i \bar a_j \ge c_0 |a|^2 \text{ for } z \in \overline{\Omega} \cap U_\ell \text{ and } a \in \mathbb{C}^n,\ 1 \le \ell \le k.$$

It follows that $r_0(z) = \rho(z) - c_0|z|^2$ is a plurisubharmonic function on W since for each $1 \leq \ell \leq k$, $\rho_\ell - c_0|z|^2$ is a plurisubharmonic function on U_ℓ. Thus $\tilde{\rho}(z) - c_0|z|^2 = \underset{i, z\in U_i}{\rightarrow} \max\{\rho_i(z) - c_0|z|^2\} = r_0(z)$ is plurisubharmonic on W.

Since $|\nabla\rho| = |\nabla\rho_i|$ for some i on the smooth part of $\partial\Omega$, we have $|\nabla\rho| \leq C_2$ a.e. on $\partial\Omega$. To show that $|\nabla\rho|$ is bounded from below, we note that $\partial\Omega$ is Lipschitz and satisfies the exterior cone condition. There exist a finite covering $\{V_\mu\}_{1\leq\mu\leq k}$ of $\partial\Omega$, a finite set of unit vectors $\{\xi_\mu\}_{1\leq\mu\leq k}$ and $c_1 > 0$ such that the inner product $\langle\nabla\rho, \xi\rangle_\mu \geq c_1 > 0$ a.e. for $z \in V_\mu$, $1 \leq \mu \leq k$. $t_\nu > 0$ are pseudoconvex Thus Ω has a strictly plurisubharmonic Lipschitz defining function ρ and D is a strongly pseudoconvex Lipschitz domain.

2. Strictly convex domains.

Let Ω be a strictly convex domain. By this we mean that there exists a Lipschitz defining function ρ such that $\rho - C|z|^2$ is convex for some $C > 0$ and $C_1 \leq |\nabla\rho| \leq C_2$ a.e. on $b\Omega$. Since Ω is convex, Ω has Lipschitz boundary. It is easy to see that Ω is strongly pseudoconvex with Lipschitz boundary.

Let Ω be a bounded Lischitz domain. We observe, first of all, that the first order system $\bar\partial \oplus \bar\partial^*$ is elliptic in the interior of Ω. More, if $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$, the $f \in W^1(\Omega, \mathrm{loc})$. Then the problem about the regularity of $\bar\partial$-Neumann operator is only on the boundary.

Theorem 3.4. *Let Ω a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary. Then there exists $C > 0$ such that*[3]

$$\|f\|^2_{1/2} \leq C\left(\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2\right), \qquad f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*) \tag{3.1}$$

Now, we can state and prove the main result[4].

Theorem 3.5. *Let Ω be a bounded strictly pseudoconvex domain with Lipschitz boundary. For any $0 \leq p \leq n$, $1 \leq q \leq n-1$, the $\bar\partial$-Neumann operator N can be extended as a bounded operator $N : W^{-1/2}_{(p,q)}(\Omega) \to W^{1/2}_{(p,q)}(\Omega)$ and we have*

$$\|Nf\|^2_{1/2} \leq C\|f\|^2_{-1/2}, \qquad f \in W^{-\frac{1}{2}}_{(p,q)}(\Omega), \tag{3.2}$$

where C is independent on f. Finally, we have

$$\|\bar\partial Nf\|_{1/2} + \|\bar\partial^* Nf\|_{1/2} \leq C\|f\|, \qquad f \in L^2_{(p,q)}(\Omega), \tag{3.3}$$

---

[3] In fact, we can prove before that, under the same assumptions

$$\int_{b\Omega} |f|^2 d\sigma \leq C\left(\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2\right).$$

From this, the result of theorem follows.

[4] A first estimate in this sense was given by Kohn [KO], who showed that $\|Nf\|_{s+1} \leq C_s\|f\|_s$, under the assumption of C^∞ boundary.

Proof. We first prove *a priori* estimates: Suppose that Ω is a strictly pseudoconvex domain with C^2 boundary. For any $f \in C^2_{(p,q)}(\overline{\Omega}) \cap \mathrm{Dom}(\bar{\partial}^*)$, we will prove the estimate holds.

Let ρ be a C^2 strictly plurisubharmonic defining function with $|\nabla\rho| = 1$ on $b\Omega$. By Green's Theorem[5] with $u = -\rho$ and $v = |f|^2/2$, we have

$$\int_\Omega -\rho|\nabla f|^2 dV + \Re\int_\Omega -\rho\bar{f}\Delta f dV + \int_\Omega \Delta\rho\frac{|f|^2}{2}dV = \int_{b\Omega}\frac{\partial\rho}{\partial n}\frac{|f|^2}{2}d\sigma,$$

where $\Delta v = \Re(\bar{f}\Delta f) + |\nabla f|^2$ and $\rho = 0$ on $b\Omega$. Since $\triangle\rho > 0$ by the strictly pseudoconvexity we have

$$\int_\Omega |\rho||\nabla f|^2 dV \leq C\left(\int_{b\Omega}|f|^2 d\sigma + \int_\Omega |f|^2 dV\right) + \Re\int_\Omega -\rho\bar{f}\Delta f dV.$$

Using the fact that $\Delta f = \Box f$ and the Hardy-Littlewood Lemma[6] which give the following estimate

$$\|f\|^2_{1/2} \leq C\left(\int_\Omega |\nabla f|^2 dV + \int_\Omega |f|^2 dV\right),$$

we have,

$$\|f\|^2_{1/2} \leq CQ(f,f) = C(\Box f, f) \leq C\|\Box f\|_{-1/2}\|f\|_{1/2}$$

which implies

$$\|f\|_{1/2} \leq C\|\Box f\|_{-1/2}.$$

Here $Q(f,f) = (\Box f, f)$ is the energy functional. Finally, substituting f with Nf, we get

$$\|Nf\|_{1/2} \leq C\|\Box Nf\|_{-1/2} = C\|f\|_{-1/2}.$$

This proves the *a priori* estimates.

---

[5] Let $u, v \in C^2(\overline{\Omega})$, then

$$\int_\Omega (u\Delta v - v\Delta u)\, dV = \int_{b\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma.$$

[6] We give here a modified version of Hardy-Littlewood Lemma for Sobolev spaces. Let Ω a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta(x)$ the distance function from $x \in \Omega$ to the boundary $b\Omega$. If $u \in L^2(\Omega) \cap W^1(\Omega, \mathrm{loc})$ and there is a constant $0 < \alpha < 1$ such that

$$\int_\Omega \delta(x)^{2-2\alpha}|\nabla u|^2 dV < \infty,$$

then $u \in W^\alpha(\Omega)$. Furthermore, there is a constant C depending only on Ω such that

$$\|u\|^2_\alpha \leq C\left(\int_\Omega \delta(x)^{2-2\alpha}|\nabla u|^2 dV + \int_\Omega |u|^2 dV\right).$$

From the assumption of strong pseudoconvexity for Ω, there exists a sequence of strongly pseudoconvex smooth subdomains Ω_ν satisfying conditions (1)-(3) in Definition (***). We have for any $f \in C^2_{(p,q)}(\overline{\Omega}_\nu) \cap \mathrm{Dom}(\bar\partial^*_\nu)$,

$$\int_{b\Omega_\nu} |f|^2 dS \leq C(\| \bar\partial f \|^2_{\Omega_\nu} + \| \vartheta f \|^2_{\Omega_\nu}),$$

where C is independent of ν. From the *a priori* estimate,

$$\| f \|^2_{\frac{1}{2}(\Omega_\nu)} \leq C(\| \bar\partial f \|^2_{\Omega_\nu} + \| \vartheta f \|^2_{\Omega_\nu}),$$

where C is independent of ν.

Since $C^1_{(p,q)}(\overline{\Omega_\nu}) \cap \mathrm{Dom}(\bar\partial^*_\nu)$ is dense in $\mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ in the graph norm $\| \bar\partial f \|_{\Omega_\nu} + \| \bar\partial^* f \|_{\Omega_\nu}$. The theorem is proved by an approximation argument.

□

Assume that Ω is a strictly pseudoconvex bounded domain with C^∞ boundary. Then we have the following theorem of Kohn.

Theorem 3.6. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary $b\Omega$. Then the $\bar\partial$-Neumann operator $N : W^s(\Omega) \to W^{s+1}(\Omega)$, for every $s \geq 0$.*

Proof. We only sketch the proof for $s = \frac{1}{2}$. If T is a tangential operator, then $Tf \in \mathrm{Dom}(\bar\partial^*)$, and we have

$$\|f\|^2_{3/2} \sim \int_\Omega |\rho||\nabla Tf|^2 dV \leq C\left(\|\bar\partial Tf\|^2 + \|\bar\partial^* Tf\|^2\right) \leq$$
$$\leq C\|\Box Tf\|_{-1/2}\|Tf\|_{1/2} = C\|\Box f\|_{1/2}\|f\|_{3/2}.$$

Then $N : W^{1/2}(\Omega) \to W^{3/2}(\Omega)$ is bounded. □

Furthermore, form the Sobolev embedding theorem[7] we have $N : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$.

3.2. **Regularity for the $\bar\partial$-Neumann problem on pseudoconvex domains.**

When Ω is still with C^∞ boundary, but only pseudoconvex, the $\bar\partial$-Neumann operator is regular for the following type of domains:

1. If there exists a smoth defining function plurisubharmonic on the boundary $b\Omega$. The regularity of N from $W^s(\Omega)$ to itself was proved by Boas and Straube.

2. If Ω is of finite type in the sense of Kohn or D'Angelo. The regularity of N from $W^s(\Omega)$ to $W^{s+\epsilon}$ for some $\epsilon > 0$ was proved by Kohn and Catlin.

When Ω is still with C^∞ boundary, but only pseudoconvex, in general, N is not regular from $W^s(\Omega)$ in itself. This was proved by Barrett for the

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[7] **Sobolev Embedding Theorem**: If Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, then there is an embedding $W^k(\Omega) \hookrightarrow C^m(\overline{\Omega})$ for an integer $0 \leq m < k - N/2$.

worm domain of Diederich and Fornæss [DF].

4. Strong Oka's lemma and the $\bar{\partial}$-Neumann problem

Definition 4.1. An open complex manifold Ω is called Stein if there exists a strictly plurisubharmonic exhaustion function $\phi : \Omega \to (-\infty, \infty)$ such that

(1) ϕ is strictly plurisubharmonic on Ω,
(2) For each $c \in \mathbb{R}$, $\Omega_c = \phi < c$ is relatively compact in Ω.

A well known result in several complex variables is the classical Oka's Lemma.

Lemma 4.2 (Oka)**.** *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, then $-\log \delta$ is plurisubharmonic where δ is some distance function from $z \in \Omega$ to the boundary. Every pseudoconvex domain Ω in $\mathbb{C}^n$ is a Stein manifold.*

Proof. Let δ be a distance function from $z \in \Omega$ to the boundary $b\Omega$. Then $-\log \delta$ is plurisubharmonic (near the boundary). We can modify to make δ by adding $C|z|^2$ to be strictly plurisubharmonic in Ω. The theorem is proved by regularization. □

If Ω is a relatively compact domain in a Stein manifold, Hörmander's L^2 existence theorem and Kohn's results can all be applied to Ω by using ϕ as the weight function.

Definition 4.3. Let M be a complex hermitian manifold with the metric form ω. Let Ω be relatively compact pseudoconvex domain in M. We say that a distance function δ to the boundary $b\Omega$ satisfies the *strong Oka's condition* if it can be extended from a neighborhood of $b\Omega$ to Ω such that δ satisfies

$$i\partial\bar{\partial}(-\log \delta) \geq c_0\omega \qquad \text{in } \Omega \tag{4.1}$$

for some constant $c_0 > 0$.

4.1. Strong Oka's Lemma and bounded plurisubharmonic exhaustion functions.

We first study the relation between the strong Oka's Lemma and the existence of bounded strictly plurisubharmonic functions on a pseudoconvex domain in a complex manifold with C^2 boundary.

For a bounded pseudoconvex domain Ω with C^2 boundary in $\mathbb{C}^n$ or in a Stein manifold, a well known result by Diederich-Fornaess [DF] shows that there exists a Hölder continuous strictly plurisubharmonic exhaustion function with Hölder exponent $0 < \eta < 1$. We first examine some equivalent conditions for the existence of such bounded plurisubharmonic functions based on the following simple observation.

Proposition 4.4. *Let M be a complex hermitian manifold with metric ω and let $\Omega \subset\subset M$ be a pseudoconvex domain with Lipschitz boundary. Then the following two conditions are equivalent:*

(1) There exists some $0 < t_0 \leq 1$ *such that*

$$i\partial\bar{\partial}(-\log\delta) \geq it_0\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}. \tag{4.2}$$

(2) There exists some $0 < t_0 \leq 1$ *such that*

$$i\partial\bar{\partial}(-\delta^{t_0}) \geq 0. \tag{4.3}$$

Furthermore, suppose that the distance function δ *satisfies the strong Oka condition (4.1). Then (2) implies*
(3) There exists some $0 < t_0 \leq 1$ *such that for any* $0 < t < t_0$*, there exists some constant* $C_t > 0$

$$i\partial\bar{\partial}(-\delta^t) \geq C_t\delta^t(\omega + i\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}). \tag{4.4}$$

In particular, $-\delta^t$ *is a Hölder continuous strictly plurisubharmonic exhaustion function for* Ω*.*

Proof. If (1) holds, we have

$$i\partial\bar{\partial}(-\log\delta) = i\frac{\partial\bar{\partial}(-\delta)}{\delta} + \frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2} \geq it_0\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}. \tag{4.5}$$

The above equation is obviously true for C^2 distance function. It is also true for Lipschitz function δ since the term $\partial\delta\wedge\bar{\partial}\delta$ is well defined. Condition (2) is equivalent to

$$i\frac{\partial\bar{\partial}(-\delta)}{\delta} + (1-t_0)\frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2} \geq 0.$$

Comparing this with (4.5), it is easy to see that (1) and (2) are equivalent.

Assume that the distance function δ satisfies the strong Oka condition (4.1). To see that (2) implies (3), we multiply (4.5) by $(1-\epsilon)$ and (4.1) by ϵ. Adding the two inequalities, we conclude that, for any $0 \leq \epsilon \leq 1$, the inequality

$$i\partial\bar{\partial}(-\log\delta) = i\frac{\partial\bar{\partial}(-\delta)}{\delta} + \frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2} \geq c_0\epsilon\omega + (1-\epsilon)t_0\frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2}$$

holds. Hence, for any $0 < t < t_0$, we choose $\epsilon = \epsilon_t$ such that $(1-\epsilon_t)t_0 > t$. Then

$$\begin{aligned} i\partial\bar{\partial}(-\delta^t) &= it\delta^t(\frac{\partial\bar{\partial}(-\delta)}{\delta} + (1-t)\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}) \\ &= it\delta^t(\partial\bar{\partial}(-\log\delta) - t\frac{\partial\delta\wedge\bar{\partial}\delta}{\delta^2}) \\ &\geq C_tt\delta^t(\omega + \frac{i\partial\delta\wedge\bar{\partial}\delta}{\delta^2}) \end{aligned} \tag{4.6}$$

where $C_t = \min(c_0\epsilon_t, (1-\epsilon_t)t_0 - t)$. This gives that (2) implies (3).

□ □

Theorem 4.5. *Let M be a complex hermitian manifold and let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2 boundary $b\Omega$. Let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$ with respect to the hermitian metric such that δ satisfies the strong Oka condition (4.1). Then there exists $0 < t_0 \leq 1$ such that*

$$i\partial\bar{\partial}(-\log \delta) \geq it_0 \frac{\partial\delta \wedge \bar{\partial}\delta}{\delta^2}.$$

Proof. Near a boundary point, we choose a special orthonormal basis $w_1, \cdots, w_n$ for $(1,0)$-forms such that $w_n = \sqrt{2}\partial\delta$. Let $L_1, \cdots, L_n$ be its dual. Let a be any $(1,0)$-vector. We decompose $a = a_\tau + a_\nu$ where $a_\nu = \langle a, L_n\rangle$ is the complex normal component and a_τ is the complex tangential component. We have

$$\begin{aligned} &\langle \partial\bar{\partial}(-\log\delta), a \wedge \bar{a}\rangle \\ = \ &\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\tau\rangle + 2\Re\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\nu\rangle \\ + \ &\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\nu \wedge \bar{a}_\nu\rangle + \frac{|a_\nu|^2}{\delta^2}. \end{aligned} \tag{4.7}$$

From (4.1), we have

$$\langle \partial\bar{\partial}(-\log\delta), a_\tau \wedge \bar{a}_\tau\rangle = \langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\tau\rangle \geq c_0|a_\tau|^2.$$

If δ is C^2 up to the boundary, we have for any $\epsilon > 0$,

$$|\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\nu\rangle| \leq C(\frac{1}{\epsilon}|a_\tau|^2 + \epsilon\frac{|a_\nu|^2}{\delta^2}).$$

Also near the boundary when $\delta(z) < \epsilon$, we have

$$|\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\nu \wedge \bar{a}_\nu\rangle| \leq \frac{C}{\delta}|a_\nu|^2 \leq C\frac{\epsilon}{\delta^2}|a_\nu|^2.$$

Choosing ϵ sufficiently small, we have

$$\langle \partial\bar{\partial}(-\log\delta), a \wedge \bar{a}\rangle \geq \frac{1}{2}\frac{|a_\nu|^2}{\delta^2} - K|a_\tau|^2$$

for some large constant K. Multiplying (4.1) by $\frac{K}{c_0}$ and adding it to the above inequality, we have

$$(\frac{K}{c_0} + 1)\langle \partial\bar{\partial}(-\log\delta), a \wedge \bar{a}\rangle \geq \frac{1}{2}\frac{|a_\nu|^2}{\delta^2}.$$

This proves the theorem with $t_0 = \frac{1}{2(\frac{K}{c_0}+1)}$.

□

Corollary 4.6. *Let Ω, $\delta(x)$ and t_0 be the same as in Theorem 1.2. Then for any $0 < t < t_0$, the function $\tilde{\delta} = -\delta^t$ is a strictly plurisubharmonic bounded exhaustion function on Ω.*

The corollary follows immediately from the equivalence of (1) and (3) in Lemma *** We have the following Diederich-Fornaess Theorem [DF] and the recent generalization by Harrington [**?**].

Theorem 4.7. *Let* $\Omega \subset\subset M$ *be a pseudoconvex domain with Lipschitz boundary in a Stein manifold* M*. Then there exists a Lipschitz defining function* ρ *and some number* $0 < t < 1$ *such that* $\tilde{\delta} = -(-\rho)^t$ *is a strictly plurisubharmonic bounded exhaustion function on* Ω*.*

Proof. We only prove the case when $b\Omega$ is of class C^2. Since M is Stein, M can be embedded in $\mathbb{C}^N$ for some large N. Let $\delta(x) = d(x, b\Omega)$ be the distance function to $b\Omega$ with respect to metric ω induced by the Euclidean metric in $\mathbb{C}^N$. From Oka's Lemma, we have $i\partial\bar{\partial}(-\log\delta) \geq 0$ in a neighborhood U of $b\Omega$.

Let ϕ be a smooth strictly plurisubharmonic function on M. For any $c_0 > 0$, we can choose some large $\lambda > 0$ such that

$$i\partial\bar{\partial}(-\log(\delta e^{-\lambda\phi})) = -i\partial\bar{\partial}\log\delta + \lambda\omega \geq c_0\omega$$

where ω is the metric form induced by the Euclidean metric. Thus the strong Oka condition (4.1) holds. The theorem follows from Corollary ***.

The case for Lipschitz domain is much more involved and we omit the details.

□

Remark. In the proof of Theorem ***, if Ω is in $\mathbb{C}^n$, we can choose $\phi = |z|^2$.

Theorem 4.8. *Let* Ω *be a bounded pseudoconvex domain with Lipschitz boundary in* $\mathbb{C}^n$*. Then the* $\bar{\partial}$*-Neumann operator* N *(as well as* $\bar{\partial}N$*,* $\bar{\partial}^*N$ *and the Bergman projection* P*) is bounded on* $W^s(\Omega)$ *to* $W^s(\Omega)$ *for any* $0 \leq s < \frac{1}{2}t_0$*.*

Remark. 1. If $t_0 = 1$ in Theorem***, the domain Ω has a plurisubharmonic defining function in Ω. It is proved in Bonami-Charpentier [**?**] that we can take $s = \frac{1}{2}$. if Ω has smooth boundary and there exists a plurisubharmonic defining function on $b\Omega$, it follows from Boas-Straube [**?**] that the $\bar{\partial}$-Neumann operator is bounded in the Sobolev space W^s for all $s > 0$. We also mention that for any $\beta > 0$, there exists a smooth bounded pseudoconvex domain Ω_β in $\mathbb{C}^n$ such that the $\bar{\partial}$-Neumann operator is not bounded on W^s (see the paper by Barrett [**?**]).

When the complex manifold is Kähler with positive curvature, we have also the following result of Ohsawa-Sibony [**?**].

Theorem 4.9. *(Ohsawa-Sibony) Let* $\Omega \subset\subset \mathbb{C}P^n$ *be a pseudoconvex domain with* C^2 *boundary* $b\Omega$ *and let* $\delta(x) = d(x, b\Omega)$ *be the distance function to* $b\Omega$ *with the Fubini-Study metric* ω*. Then there exists* $t_0 = t_0(\Omega)$ *with* $0 < t_0 \leq 1$ *such that*

$$i\partial\bar{\partial}(-\delta^{t_0}) \geq 0.$$

This follows easily from Takeuchi's Theorem (see [?], also [?])

$$i\partial\bar{\partial}(-\log\delta) \geq c_0\omega$$

where c_0 can be chosen to be equal to $\frac{1}{2}$.

Remarks:

1. The theorems above show that if either the complex manifold has positive curvature or there is a positive line bundle, then the strong Oka's lemma holds. Both theorems do not hold without the positivity condition on the metric (see the counterexample in [DF] and Theorem 1.2 in [?]).

2. If Ω is a Lipschitz bounded pseudoconvex domain in a Stein manifold, it is proved in Demailly [?] that there exists a bounded strictly plurisubharmonic exhaustion function in Ω (see also Kerzman-Rosay [?] for the C^1 case). A more refined arguments yield a Hölder continuous strictly plurisubharmonic exhaustion function in Ω (see Harrington [?]).

It is not known if this is true for pseudoconvex domains with Lipschitz boundary in $\mathbb{C}P^n$. We also remark that strictly plurisubharmonic bounded exhaustion functions might not exist if the Lipschitz boundary (as a graph) condition is dropped (see [DF]).

4.2. **Strong Oka's lemma and finite type conditions.** Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let ω be the Kähler form. The Oka's lemma characterizes pseudoconvex domains with the condition that the distance function δ satisifes $i\partial\bar{\partial}(-\log\delta)_x \geq c(x)\omega$ where $c(x) \geq 0$ for every $x \in \Omega$. If we can take $c(x)$ as a postive constant, it is called the strong Oka's condition. In this section, we will relate the growth rate of $c(x)$ to more finer properties of psuedoconvexity and the $\bar{\partial}$-Neumnann problem.

If Ω is a strongly pseudoconvex Lipschitz domain, there exists a strictly plurisubharmonic defining function $-\delta$ such that

$$i\partial\bar{\partial}(-\log\delta) \geq i\frac{\partial\bar{\partial}(-\delta)}{\delta} \geq C\delta^{-1}\omega, \tag{4.8}$$

in the sense of currents. In fact, we can use (4.12) as definition for strong pseudoconvexity.

Definition 4.10. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary $b\Omega$. The domain Ω is of finite type if there exist a distance function δ and $c > 0$, $0 < \epsilon \leq 10$ such that

$$i\partial\bar{\partial}(-\log\delta) \geq c\delta^{-2\epsilon}\omega.$$

Definition 4.11. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary $b\Omega$. The domain Ω is said to satisfy condition (C) (for compactness) if there exist a distance function δ and $c(z) > 0$ such that

$$i\partial\bar{\partial}(-\log\delta) \geq c(z)\omega$$

where where $c(z) \to +\infty$ as $z \to b\Omega$

Theorem 4.12. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary $b\Omega$. Suppose that Ω is of finite type, in the sense of *** then the $\bar\partial$-Neumann operator N on $L^2_{(p,q)}(\Omega)$, where $1 \le q \le n-1$, satisfies*

$$N : W^s(\Omega) \to W^{s+2\epsilon'}(\Omega), \quad \epsilon' < \epsilon$$

and

$$\|f\|_{\epsilon'} \le C(\|\bar\partial f\| + \|\bar\partial^* f\|), \quad f \in Dom(\bar\partial) \cap Dom(\bar\partial^*).$$

If Ω satisfies condition (C), then N is compact.

5. The $\bar\partial$-Neumann problem on Kähler manifolds with nonnegative curvature

Let M be a complex manifold with a hermitian metric ω and let $\Omega \subset\subset M$ be a pseudoconvex domain. If M is Stein, the L^2 existence theorems for $\bar\partial$ and the $\bar\partial$-Neumann problem follow from Hörmander's theory. In the case when the manifold M is Kähler with negative curvature, the distance function to a fixed point is a strictly plurisubharmonic function and hence, M is Stein (see [**?**]). In this section we study the L^2 $\bar\partial$ theory when the manifold is not Stein and there is no strictly plurisubharmonic weight function smooth up to the boundary. One has to modify Hörmander's weighted method to establish the L^2 theory for $\bar\partial$.

Suppose that the manifold has positive curvature, like $\mathbb{C}P^n$. Then there exists some distance function δ for Ω which satisfies the strong Oka's condition (0.1). Then we can use $\phi = -\log\delta$ to be the weight function in Hörmander's theory and study the weighted $\bar\partial$-Neumann problem (see [**?**] or [CS]). However, ϕ is not continuous up to the boundary. To establish the L^2 theory without weights, we use an idea by Berndtsson-Charpentier [**?**] and streamlined in [CSW].

Let t be any real number and $\phi \in C^2(\Omega)$. Let $L^2(\delta^t)$ denote the L^2 space with respect to the weight function $e^{-t\phi} = \delta^t$ and

$$\|f\|_{(t)} \equiv \|f\|^2_{L^2(\delta^t)} = \int_\Omega |f|^2 e^{-t\phi} = \int_\Omega \delta^t |f|^2.$$

We use $\bar\partial^*_t$ to denote the adjoint of $\bar\partial$ with respect to the weighted space. Then $\bar\partial^*_t = \delta^{-t}\vartheta\delta^t$ whenever it is defined, where ϑ denotes the formal adjoint with respect to the unweighted L^2-norm. The norm $\|.\|^2_{(t)}$ is equivalent to the Sobolev norm on a sub-space of $W^{-\frac{t}{2}}(\Omega)$ for harmonic functions or solutions to elliptic equations.

Let δ be a distance function which satisfies the strong Oka condition (0.1). Introduce the following two asymmetric weighted norms. These new norms will be used to obtain more refined L^2 estimates.

For any (p,q)-form f on Ω, we decompose f into complex normal and tangential parts by setting $f^\nu = (f \llcorner_{(\bar\partial\delta)_\#}) \wedge \bar\partial\delta$ and $f^\tau = f - f^\nu$..

The above decomposition is well-defined for any (p,q)-form f supported near the boundary and can be extended to the whole domain.

We define the asymmetric weighted norm

$$|f|_A^2 = |f^\tau|^2 + \frac{|f^\nu|^2}{|\delta|^2}$$

and its dual norm

$$|f|_{A'}^2 = |f^\tau|^2 + |f^\nu|^2|\delta|^2.$$

For any $t > 0$, let $L^2_A(\delta^t)$ and $L^2_{A'}(\delta^t)$ denote the weighted L^2 spaces on (p,q)-forms defined by the norm

$$\|u\|^2_{L^2_A(\delta^t)} = \int_\Omega \delta^t|f|_A^2 = \int_\Omega \delta^t(|f^\tau|^2 + \frac{|f^\nu|^2}{|\delta|^2})$$

and

$$\|u\|^2_{L^2_{A'}(\delta^t)} = \int_\Omega \delta^t|f|_{A'}^2 = \int_\Omega \delta^t(|f^\tau|^2 + |f^\nu|^2|\delta|^2).$$

Theorem 5.1. *Let M be a complex Kähler manifold with nonnegative sectional curvature. Let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2-smooth boundary $b\Omega$. Let $\delta(x) = d(x, b\Omega)$ be the distance function such that δ satisfies the strong Oka condition (0.1). For any $f \in L^2_{A'(p,q)}(\Omega)$, where $0 \le p \le n$ and $1 \le q \le n$, such that $\bar\partial f = 0$ in Ω, there exists $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar\partial u = f$ and*

$$\int_\Omega |u|^2 \le C\int_\Omega |f|^2_{A'}.$$

Proof. We first show that for any $t > 0$ and any (p,q)-form $f \in L^2_{A'}(\delta^t)$, $0 \le p \le n$ and $1 \le q \le n$, such that $\bar\partial f = 0$ in Ω, there exists $u \in L^2_{(p,q-1)}(\delta^t)$ satisfying $\bar\partial u = f$ and

$$\|u\|^2_{L^2(\delta^t)} \le \frac{C}{t}\|f\|^2_{L^2_{A'}(\delta^t)}. \tag{5.1}$$

Let $\phi = -t\log\delta$, where $t > 0$. By the Bochner-Hörmander-Kohn-Morrey formula with weight function $\phi = -t\log\delta$, for any (p,q)-form $g \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ with $q \ge 1$ on Ω, we have

$$\|\bar\partial g\|^2_\phi + \|\bar\partial^*_\phi g\|^2_\phi = \|\overline\nabla g\|^2_\phi + (\Theta g, \overline{g})_\phi + ((i\partial\bar\partial\phi)g, \overline{g})_\phi + \int_{b\Omega}\langle(i\partial\bar\partial\rho)g, \overline{g}\rangle e^{-\phi},$$

where $\|\overline\nabla u\|^2_\phi = \int_\Omega \sum_{j=1}^n |D_{\bar L_j}u|^2 e^{-\phi}$, $\{L_1, ..., L_n\}$ is a local unitary frame of $T^{(1,0)}(\Omega)$ and Θ is a curvature form. From our assumption, we have

$$(\Theta u, \overline{u})_\phi \ge 0.$$

Since Ω is pseudoconvex, we have that for any (p,q)-form $g \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*_t)$,

$$\|\bar\partial g\|^2_{(t)} + \|\bar\partial^*_t g\|^2_{(t)} \ge ((i\partial\bar\partial\phi)g, \bar g)_{(t)} \ge Ct\|g\|_{L^2_A(\delta^t)}.$$

Let $(i\partial\bar\partial\phi)'$ denote the dual norm for (p,q)-forms induced by $i\partial\bar\partial\phi$. It follows that for any $f \in L^2_{A'}(\delta^t)$, there exist $u \in L^2(\delta^t)$ satisfying $\bar\partial u = f$ and

$$\int_\Omega |u|^2\delta^t \le \int_\Omega |\bar\partial u|^2_{(i\partial\bar\partial\phi)'}\delta^t \le \frac{1}{Ct}\int_\Omega |\bar\partial u|^2_{A'}\delta^t. \tag{5.2}$$

This proves (4.12). To get rid of t, we use an argument used in [**?**]. Let $f \in L^2_{(p,q)}(\Omega)$. For any $t > 0$, there exists $u \in L^2_{(p,q-1)}(\delta^t)$ satisfying $\bar\partial u = f$, such that u is perpendicular to $\mathrm{Ker}(\bar\partial)$ in $L^2(\delta^t)$ and u satisfies (4.13).

Consider $v = u\delta^{-t}$. Then $v \perp \mathrm{Ker}(\bar\partial)$ in $L^2(\delta^{2t})$. It follows from (4.13) that the following holds:

$$\int_\Omega |u|^2 = \int_\Omega |v|^2\delta^{2t} \le \frac{1}{2C_0t}\int_\Omega |\bar\partial v|^2_{A'}\delta^{2t}. \tag{5.3}$$

Since

$$|\bar\partial v|^2_{A'}\delta^{2t} \le C(|\bar\partial u|^2_{A'} + 2t^2|u|^2), \tag{5.4}$$

choosing t sufficiently small and substituting (4.15) into (4.14), one obtains

$$\int_\Omega |u|^2 \le C_t\int_\Omega |\bar\partial u|^2_{A'}.$$

This proves the theorem. □ □

Theorem 5.2. *Let M be a complex Kähler manifold with nonnegative sectional curvature. Let $\Omega \subset\subset M$ be a pseudoconvex domain with C^2-smooth boundary $b\Omega$. Let $\delta(x) = d(x, b\Omega)$ be the distance function such that δ satisfies the strong Oka condition (0.1). Then $\Box_{(p,q)}$ has closed range and the $\bar\partial$-Neumann operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ exists for every p, q such that $0 \le p \le n, 0 \le q \le n$. Moreover, for any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$f = \bar\partial\bar\partial^* N_{(p,q)}f \oplus \bar\partial^*\bar\partial N_{(p,q)}f, \quad 1 \le q \le n-1.$$
$$f = \bar\partial^*\bar\partial N_{(p,0)}f \oplus Pf, \quad q = 0,$$

where P is the orthogonal projection from $L^2_{(p,0)}(\Omega)$ onto $L^2_{(p,0)}(\Omega) \cap Ker(\bar\partial)$ and

$$N_{(p,0)} = \bar\partial^* N^2_{(p,1)}\bar\partial.$$

Furthermore, there exists $0 < t_0 \le 1$ such that the $\bar\partial$-Neumann operator $N, \bar\partial N, \bar\partial^* N$ and the Bergman projection P are exactly regular on $W^s_{(p,q)}(\Omega)$ for $0 \le s < \frac{1}{2}t_0$ with respect to the $W^s(\Omega)$-Sobolev norms.

The L^2-existence theorem for the $\bar\partial$-Neumann operator N on Ω follows from the L^2-existence of the solution u for the $\bar\partial$-equation proved in Theorem 2.1.

To show that N is regular in W^s for $s < \frac{1}{2}t_0$, let $t = 2s$ in the Bochner-Hörmander-Kohn-Morrey formula with weight function $\phi = -t\log\delta$. After rearranging terms, we have for any (p,q)-form $g \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$,

$$\begin{aligned}\|\bar\partial g\|^2_{(t)} + \|\bar\partial^* g\|^2_{(t)} - 2\Re(\bar\partial^* g, \overline{g \llcorner_{(\bar\partial\delta^t)_\sharp}}) \\ = \|\overline\nabla g\|^2_{(t)} + (\Theta g, \overline{g})_{(t)} - ((i\partial\bar\partial(\delta^t) g, \overline{g}).\end{aligned}$$

(see Proposition 3.1 in [CSW]). Since for any $\epsilon > 0$ we have

$$|2\Re(\bar\partial^* g, \overline{g \llcorner_{(\bar\partial\delta^t)_\sharp}})| \le \frac{1}{\epsilon}\|\bar\partial^* g\|^2_{(t)} + \epsilon t^2 \|\frac{g^\nu}{\delta}\|^2_{(t)},$$

choosing ϵ small, we have from Corollary 4.6 that

$$\begin{aligned}\|\bar\partial g\|^2_{(t)} + \|\bar\partial^* g\|^2_{(t)} &\ge C_t(\|g\|^2_{(t)} + \|\frac{g^\nu}{\delta}\|^2_{(t)}) \\ &\ge C_t\|g\|^2_{(t)}.\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2 in [CSW], and we omit the details. When $\Omega \subset\subset \mathbb{C}^n$, the Sobolev regularity for $\bar\partial^* N$ and the Bergman projection have been obtained earlier in [**?**] (see also [**?**]).

Remarks: It is still not known if Theorems ***2.1 or 2.2 hold for pseudoconvex domains with C^1 or Lipschitz boundary.

5.1. **The $\bar\partial$-Neumann problem on domains in $\mathbb{C}P^n$.** We want to study regularity and solvability of $\bar\partial$ on a complex manifold and not only a bounded domain in $\mathbb{C}^n$. First of all, observe that if we consider all $\mathbb{C}^n$ (as a complex manifold), because it is unbounded we can have a L^2 theory as before. However $\mathbb{C}^n$ can be covered by balls of radius R, for R which tends to infinity. So it is possible to have a L^2 theory locally on each Ω in $\mathbb{C}^n$ sufficiently large. Then, if $f \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ and $\bar\partial f = 0$, for $q \ge 1$, there is $u \in L^2_{(p,q-1)}(\Omega, \mathrm{loc})$ solution to $\bar\partial u = f$ in $\mathbb{C}^n$. Moreover, when $f \in C^\infty_{(p,q)}(\Omega, \mathrm{loc})$, $\bar\partial f = 0$ and Ω is pseudoconvex, then there is $u \in C^\infty_{(p,q-1)}(\Omega, \mathrm{loc})$ solution to $\bar\partial u = f$ (and in particular Ω is a domain of holomorphy).

We racall that X is a *Stein manifold* if and only if it is a complex submanifold of $\mathbb{C}^N$, for some N. This is the same as to have a strictly plurisubharmonic exhaustion function φ. Then, on a Stein manifold existence of $\bar\partial$-Neumann operator is guaranteed by a result of Boas and Straube [BS2].

If $X = \mathbb{C}P^n$, then it is not Stein. In fact, on a compact complex manifold there aren't no constant strictly plurisubhramonic exhaustion functions. We are interested to study this particular case.

Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain, i.e. for any $p \in b\Omega$ there is $U \ni p$ such that $\Omega \cap U$ is pseudoconvex. Let ρ be a C^2 defining function for Ω. For any $(0,q)$-form f we have

$$\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2 = \|\overline\nabla f\|^2 + (R_{0,q} f, f) + \int_{b\Omega} \langle i\partial\bar\partial f, f\rangle d\sigma.$$

Here, $R_{0,q}$ is the curvature form. In $\mathbb{C}P^n$, this coincides with the Ricci curvature, so

$$R_{0,q} = (n+1)q > 0$$

Then we have

$$\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 > (n+1)q\|f\|^2.$$

When f is a (n,q)-form, $R_{n,q} \equiv 0$, then

$$\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2 = \|\overline{\nabla} f\|^2 + \int_{b\Omega} \langle i\partial\bar{\partial} f, f\rangle d\sigma.$$

If we set $\delta(z) = \operatorname{dist}(z, b\Omega)$ we have $i\partial\bar{\partial}(-\log\delta) \geq c\omega$, where $c > 0$, ω the Fubini-Study metric of $\mathbb{C}P^n$. Then Ω is Stein.

As in the case of bounded domain in $\mathbb{C}^n$, we want to start from a weighted L^2 theory. Set then $\varphi_t = -t\log\delta$, $t > 0$. Then $i\partial\bar{\partial}\varphi_t \geq Ct\omega$, where $c > 0, t > 0$. In particular we can study the L^2 weighted space with weight φ_t (in this case $e^{-\varphi_t} = \delta^t$) and we obtain, for all $t > 0$ the following estimate

$$\|\bar{\partial} f\|^2_{\varphi_t} + \|\bar{\partial}^*_t f\|^2_{\varphi_t} \geq \frac{1}{Ct}\|f\|^2_{\varphi_t}.$$

We have the following result.

Lemma 5.3. *Let $\Omega \subset\subset \mathbb{C}P^n$ a bounded doamin with C^2 boundary. Then there is a $0 < t_0 \leq 1$ such that $-\delta^{t_0}$ is plurisubharmonic, i.e. $i\partial\bar{\partial}(-\delta^{t_0}) \geq 0$. Furthermore, for any $0 < t < t_0$ we have*

$$i\partial\bar{\partial}(-\delta^t) \geq Ct\delta^t \left(\omega + \frac{\partial\delta \wedge \bar{\partial}\delta}{\delta^2}\right). \tag{5.5}$$

Theorem 5.4. *Let $\Omega \subset\subset \mathbb{C}P^n$ a bounded doamin with C^2 boundary. Then there is a bounded operator $N_{p,q} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q)}(\Omega)$ for any $q \geq 1$, such that:*
(1) $\Box N = N\Box = I$ on $\operatorname{Dom}(\Box)$;
(2) $N, \bar{\partial}N, \bar{\partial}^* N$ and $B = I - \bar{\partial}^* N\bar{\partial}$, the Bergman projection*, are all bounded from $W^s(\Omega)$ to itself, for any $0 < s < t_0/2$.*

Proof. We already know that

$$\int_\Omega \delta^t \left(|\bar{\partial} f|^2 + |\bar{\partial}^* f|^2\right) dV \geq C\int_\Omega \delta^t |f|^2 dV.$$

By the estimate (5.5) we get

$$\int_\Omega \delta^{t_0}|f|^2 dV \simeq \|f\|^2_{W^{-t_0/2}(\Omega)}.$$

Applying this two fact with $\bar{\partial}Nf$ and $\bar{\partial}^* Nf$ instead of f we obtain the desired regularities. □

6. Tangential Cauchy-Riemann equations and the Bochner Martinelli kernel

7. CR manifolds and tangential Cauchy-Riemann equations

7.1. **L^2 theory for $\Box_b$ on strongly pseudoconvex CR manifolds.** As an example of CR manifolds, we may consider real orientable hypersurfaces in $\mathbb{C}^n$. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with defining function ρ. Its boundary $M = b\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$ with condition $|\nabla \rho(z)| \neq 0$, $z \in M$, is a CR manifold of dimension $2n-1$. Its CR structure is generated by

$$L_j := \sum_{h=1}^{n} a_{hj} \frac{\partial}{\partial z_h}, \qquad j = 1, \dots, n-1,$$

purely tangential, i.e. $L_j \rho = 0$, $j = 1, \dots, n-1$. The system $overlineL_j \qquad 1 \leq j \leq n-1$ are the *tangential Cauchy-Riemann equations* on M.

If f is a holomoprhic function on Ω and $f \in C^1(\overline{\Omega})$, then f satisfies

$$\overline{L}_j f = 0, \qquad \text{on } M, \quad j = 1, \cdots, n-1 \tag{7.1}$$

Any function satisfying (5.1) is called a CR function.

For $n = 2$, let $M = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_2 = |z|^2\}$, is the boundary of the unbounded ball. If we parametrize M by (z_1, t) and $z_2 = t + i|z|^2$, we have

$$\bar{L} = \frac{\partial}{\partial \bar{z}_1} - i z_1 \frac{\partial}{\partial t}.$$

This is the *Lewy operator*. In [LE] he proved that the tangential Cauchy-Riemann equation $\overline{L}u = f$ is not solvable for a large class of functions f.

When $n > 2$, we have the system

$$\overline{L}_j u = f_j, \qquad j = 1, \dots, n-1$$

with the compatibility conditions

$$\overline{L}_j f_h - \overline{L}_h f_j = \overline{L}_j \overline{L}_h u - \overline{L}_h \overline{L}_j u = [\overline{L}_j, \overline{L}_h] u = c^{\ell}_{jh} \overline{L}_\ell u = c^{\ell}_{jh} f_\ell,$$

where $[\overline{L}_j, \overline{L}_h] = c^{\ell}_{jh} \overline{L}_\ell$, which is given by condition (iii) in the definition of CR manifolds. Defining $\bar{\partial}_b = \bar{\partial}|_M$ the previous system can be rewritten as

$$\begin{cases} \bar{\partial}_b u = f \\ \bar{\partial}_b f = 0 \end{cases}.$$

More precisely, if we denote $\bar{\partial}_b = d|_{T^{0,1}(M)}$, then the sequence

$$\dots \longrightarrow \Lambda^{0,q-1}(M) \overset{\bar{\partial}_b}{\to} \Lambda^{0,q}(M) \overset{\bar{\partial}_b}{\to} \Lambda^{0,q+1}(M) \overset{\bar{\partial}_b}{\to} \dots$$

is a complex, i.e. $\bar{\partial}_b^2 = 0$. $\bar{\partial}_b$ is the *tangential Cauchy-Riemann operator.*

We may equip $\mathbb{C}TM$ by the induced metric such that $T^{1,0}(M) \perp T^{0,1}(M)$, and define ϑ_b as the adjoint of $\bar{\partial}_b$. So we have the *$\bar{\partial}_b$-Laplacian*

$$\Box_b := \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b : \Lambda^{0,q}(M) \to \Lambda^{0,q}(M).$$

It is well know that $\Box_b$ is not elliptic. If $N^*(M)$ denotes the dual of $T^{1,0}(M)\oplus T^{0,1}(M)$ in $\mathbb{C}T(M)$, we have

$$\mathbb{C}T_zM = T_z^{1,0}(M) \oplus T_z^{0,1}(M) \oplus N_z(M).$$

In particular, $\dim_{\mathbb{R}} N(M) = k$, it is spanned by a vector field T which can be chosen to be purely imaginary, i.e. $\overline{T} = -T$.

We give an important definition.

Definition 7.1. Let $(M, T^{1,0}(M))$ be a CR manifold of dimension $2n-1$. Let $T \in N(M)$ and τ its dual, i.e. $\tau(T) = 1$, $\tau(T^{1,0}(M) \oplus T^{0,1}(M)) = 0$. Then

$$\Theta_{z,\tau}(L, L') = \Theta_z(L, L')(\tau) := \langle [L, \overline{L}'], \tau\rangle_z,$$

for every $L, L' \in T^{1,0}(M)$ is the *Levi form* in the direction τ.
We say that the Levi form satisfies *condition* $Z(q)$ $(0 \le q \le n-1)$ in the direction τ, if $\Theta_{z,\tau}$ has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues.
We say that the Levi form satisfies *condition* $Y(q)$ if it satisfies condition $Z(q)$ for all directions τ.

Since there are only two directions for τ, it is easy to see that condition $Y(q)$ is equivalent to the following one:

there are $\max(n-q, q+1)$ *eigenvalues with the same sign*
or
there are $\min(n-q, q+1)$ *pairs of eigenvalues with opposite signs.*

Furthermore, condition $Z(q)$ is equivalent to the estimate

$$\|f\|^2_{1/2} = \int_M |f|^2 d\sigma \le C\left(\|\bar\partial_b f\|^2 + \|\theta_b f\|^2\right).$$

Theorem 7.2. *Let $(M, T^{1,0}(M))$ be a compact CR manifold of dimension $2n-1$. Suppose that M satisfies condition $Y(q)$ for some $0 \le q \le n-1$. Then we have*

$$\|f\|^2_{1/2} \le C\left(\|\bar\partial_b f\|^2 + \|\vartheta_b f\|^2 + \|f\|^2\right),$$

for every $f \in C^1_{(0,q)}(M)$. Furthermore, there is an operator $G_b : L^2_{(0,q)}(M) \to L^2_{(0,q)}(M)$ such that

(i) $\Box_b G_b = G_b \Box_b = I - H_b$, where $H_b : L^2_{(0,q)}(M) \to \mathcal{H}^b_{(0,q)} = Ker(\Box_b) = \{\alpha \in L^2_{(0,q)}(M) : \bar\partial_b\alpha = \vartheta_b\alpha = 0\}$ is the projection;
(ii) $\mathcal{H}^b_{(0,q)}$ is finite dimensional;
(iii) G_b is compact and $\|G_b\|_{s+1} \le C\|f\|_s$.

Proof. We give an idea of the proof only. Suppose that $\{L_1, \dots, L_n, \overline{L}_1, \dots, \overline{L}_n, T\}$ is an orthonormal basis for $\mathbb{C}TM$. For any $f \in C^1_{(0,q)}(M)$ we have

$$\|\bar\partial_b f\|^2 + \|\theta_b f\|^2 = \|\nabla'' f\|^2 + (c_{jh}Tf_j, f_h) + O\left(\|\overline{\nabla} f\|\|f\| + \|f\|^2\right),$$

where $\|\nabla'' f\|^2 = \sum_{j,h=1}^{n-1} \|\overline{L}_j f_h\|^2$ and

$$[L_j, \overline{L}_h] = c_{jh} T \mod T^{1,0}(M) \oplus T^{0,1}(M).$$

Now the $Y(q)$ condition assures that, if we put $L_j = X_j + iY_j$, then the set of vector fields $\{X_j, Y_j : 1 \leq j \leq n-1\}$ satisfies a finite type condition of type 2, i.e. $\{X_j, Y_j, [X_j, Y_j]\}$ generate all the tangent space of M. So we have

$$\|Xf\|^2 + \|f\|^2 = \|\nabla' f\|^2 + \|\nabla'' f\|^2 + \|f\|^2 \geq C\|f\|_{1/2}^2, \tag{7.2}$$

where $\|\nabla' f\|^2 = \sum_{j,h=1}^{n-1} \|L_j f_h\|^2$. Furthermore we may prove that

$$\|Xf\|^2 + |(Tf, f)| \leq C\left(\|\bar\partial_b f\|^2 + \|\vartheta_b f\|^2 + \|f\|^2\right). \tag{7.3}$$

This implies that

$$\|f\|_{\frac{1}{2}} \leq C\left(\|\bar\partial_b f\|^2 + \|\vartheta_b f\|^2 + \|f\|^2\right).$$

$\square$

7.2. L^2 **theory for** $\bar\partial_b$ **on boundary of pseudoconvex domains.** Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary $b\Omega$. We study the global L^2 existence theorem for $\bar\partial_b$ on $b\Omega$. Let f be a (p,q)-form with L^2 coefficients, where $0 \leq p \leq n$ and $1 \leq q < n$. We study the global solvability of

$$\bar\partial_b u = f \quad \text{in } b\Omega. \tag{7.4}$$

When $q < n-1$, it is easy to see that for equation (5.1) to be solvable, f must satisfy the condition

$$\bar\partial_b f = 0 \quad \text{in } b\Omega. \tag{7.5}$$

When $q = n-1$, condition (5.2) is void and the compatibility condition is given by

$$\int_{b\Omega} f \wedge h = 0, \qquad h \in L^2_{(n-p,0)} \cap \operatorname{Ker}(\bar\partial_b). \tag{7.6}$$

In fact, for all $1 \leq q \leq n-1$, f must satisfy the compatibility condition

$$\int_{b\Omega} f \wedge h = 0, \qquad h \in L^2_{(n-p,n-1-q)} \cap \operatorname{Ker}(\bar\partial_b). \tag{7.7}$$

Lemma 7.3. *For $1 \leq q < n-1$, if f satisfies (5.4), then f is $\bar\partial_b$-closed.*

Proof. From the assumption that $q \leq n-2$, we set $h = \bar\partial v$ for some smooth $v \in C^\infty_{(n-p,n-2-q)}(\overline{\Omega}) \cap \operatorname{Ker}(\bar\partial)$, then it is easy to see that

$$\int_{b\Omega} f \wedge h = \int_{b\Omega} f \wedge \bar\partial v = (-1)^q \int_{b\Omega} \bar\partial_b f \wedge v = 0.$$

Since v is any smooth form in Ω, we have that $\bar\partial_b f = 0$ on $b\Omega$. $\square$

We have the following L^2 existence theorem.

Theorem 7.4. *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$ *with smooth boundary* M*. For every* $\alpha \in W^s_{(p,q)}(M)$*, where* $0 \leq p \leq n,\ 1 \leq q \leq n-2$ *and* $s \geq 0$*, such that*

$$\bar\partial_b \alpha = 0 \qquad \text{on } M, \tag{7.8}$$

there exists $u \in W^s_{(p,q-1)}(M)$ *satisfying* $\bar\partial_b u = \alpha$ *on* M*.*

When $q = n-1$*,* $\alpha \in W^s_{(p,n-1)}(M)$ *and* α *satisfies*

$$\int_M \alpha \wedge \phi = 0, \quad \phi \in C^\infty_{(n-p,0)}(M) \cap Ker(\bar\partial_b), \tag{7.9}$$

there exists $u \in W^s_{(p,n-2)}(M)$ *satisfying* $\bar\partial_b u = \alpha$ *on* M*.*

Corollary 7.5. *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$ *with smooth boundary* M*. Then* $\bar\partial_b : L^2_{(p,q-1)}(M) \to L^2_{(p,q)}(M)$*,* $0 \leq p \leq n,\ 1 \leq q \leq n-1$*, has closed range in* L^2*.*

We have the following strong Hodge decomposition theorem for $\bar\partial_b$. Let

$$S_{(p,0)} : L^2_{(p,0)}(M) \to \mathrm{Ker}(\bar\partial_b)$$
$$\tilde S_{(p,n-1)} : L^2_{(p,0)}(M) \to \mathrm{Ker}(\bar\partial_b^*).$$

The projection $S_{(0,0)}$ is the Szegö projection operator.

Corollary 7.6. *Let* M *be the boundary of a smooth bounded pseudoconvex domain* Ω *in* $\mathbb{C}^n$*,* $n \geq 2$*. Then for any* $0 \leq p \leq n,\ 0 \leq q \leq n-1$*, there exists a linear operator* $G_b : L^2_{(p,q)}(M) \to L^2_{(p,q)}(M)$ *such that*

G_b *is bounded and* $R(G_b) \subset Dom(\Box_b)$*.*
For any $\alpha \in L^2_{(p,q)}(M)$*, we have*

$$\begin{aligned}
\alpha &= \overline\partial_b \overline\partial_b^* G_b \alpha \oplus \overline\partial_b^* \overline\partial_b G_b \alpha, &&\text{if} \quad 1 \leq q \leq n-2,\\
\alpha &= \overline\partial_b^* \overline\partial_b G_b \alpha \oplus S_{(p,0)}\alpha, &&\text{if} \quad q = 0,\\
\alpha &= \overline\partial_b \overline\partial_b^* G_b \alpha \oplus \tilde S_{(p,n-1)}\alpha, &&\text{if} \quad q = n-1.
\end{aligned}$$

If $\alpha \in L^2_{(p,q)}(M)$ *with* $\overline\partial_b \alpha = 0$*, where* $1 \leq q \leq n-2$ *or* $\alpha \in L^2_{(p,n-1)}(M)$ *with* $\tilde S_{(p,n-1)}\alpha = 0$*, then* $\alpha = \overline\partial_b \overline\partial_b^* G_b \alpha$*.*

The solution $u = \overline\partial_b^* G_b \alpha$ in (4) is called the canonical solution, i.e., the unique solution orthogonal to $\mathrm{Ker}(\overline\partial_b)$.

Theorem 7.7. *Let* D *be a bounded pseudoconvex Lipschitz domain in* $\mathbb{C}^n$ *with boundary* bD*. We assume that* D *has a Lipschitz defining function* ρ *which is plurisubharmonic in* D*. For every* $\alpha \in L^2_{(p,q)}(bD)$*, where* $0 \leq p \leq n,\ 1 \leq q < n-1$*, such that*

$$\bar\partial_b \alpha = 0,$$

there exists $u \in L^2_{(p,q-1)}(bD)$ *satisfying* $\bar\partial_b u = \alpha$ *in* bD*.*

Furthermore, there exists a constant C depending only on the diameter and the Lipschitz constant of D but is independent of α such that

$$\| u \|_{L^2_{(p,q-1)}(bD)} \leq C \| \alpha \|_{L^2_{(p,q)}(bD)} .$$

When $q = n-1$, we assume that D has a Lipschitz defining function ρ which is plurisubharmonic in a neighborhood of $\overline{D}$. For every $\alpha \in L^2_{(p,n-1)}(bD)$ such that α satisfies

$$\int_M \alpha \wedge \phi = 0, \quad \textit{for every } \phi \in C^\infty_{(n-p,0)}(\overline{D}) \cap Ker(\bar\partial),$$

the same conclusion holds.

Corollary 7.8. *Let D be a bounded Lipschitz domain with a defining function ρ which is plurisubharmonic in a neighborhood of $\overline{D}$. The $\bar\partial_b$ operator has closed range in $L^2_{(p,q)}(bD)$ for every $0 \leq p \leq n,\ 1 \leq q \leq n-1$.*

Since $\bar\partial_b$ is a closed, densely defined operator from one Hilbert space to another, it has a Hilbert space adjoint, denoted by $\bar\partial_b^*$. The $\bar\partial_b^*$ operator is also a closed, densely defined operator (see Riesz-Nagy [**?**]). Let $\mathrm{Ker}(\bar\partial_b)$ and $\mathrm{Ker}(\bar\partial_b^*)$ denote the kernels of $\bar\partial_b$ and $\bar\partial_b^*$ respectively. Then $\mathrm{Ker}(\bar\partial_b)$ and $\mathrm{Ker}(\bar\partial_b^*)$ are both closed subspaces. Let $R(\bar\partial_b)$ and $R(\bar\partial_b^*)$ denote the range of $\bar\partial_b$ and $\bar\partial_b^*$ respectively. We also have the following consequence from the Main Theorem. Let the space of harmonic forms $H_b^{p,q}(bD)$ be defined by $H_b^{p,q}(bD) = \{\alpha \in \mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial_b^*) \mid \bar\partial_b\alpha = \bar\partial_b^*\alpha = 0\}$.

Corollary 7.9. *Corollary 2 (Hodge theory for $\bar\partial_b$)Let D be a bounded Lipschitz domain with a defining function ρ which is plurisubharmonic in a neighborhood of $\overline{D}$. The harmonic forms*

$$H_b^{p,q}(bD) = \{0\} \qquad \textit{for } 0 \leq p \leq n,\ 1 \leq q < n-1.$$

We have the following strong Hodge decomposition:

$$\begin{aligned} L^2_{(p,0)}(bD) &= Ker(\bar\partial_b) \oplus R(\bar\partial_b^*), \quad 0 \leq p \leq n, \\ L^2_{(p,q)}(bD) &= R(\bar\partial_b) \oplus R(\bar\partial_b^*), \quad 0 \leq p \leq n,\ 1 \leq q < n-1, \\ L^2_{(p,n-1)}(bD) &= R(\bar\partial_b) \oplus Ker(\bar\partial_b^*), \quad 0 \leq p \leq n. \end{aligned}$$

In particular, our results can be applied to the boundary of a strongly pseudoconvex Lipschitz domain or the boundary of any convex Lipschitz domain. When D is a smooth strongly pseudoconvex CR manifold, the $\bar\partial_b$ complex has been studied in the work of Kohn [**?**], Kohn-Rossi [**?**], Folland-Stein [**?**] and Rothschild-Stein [**?**] for $1 \leq q < n-1$. Henkin [**?**] has derived the kernel solutions for $\bar\partial_b$ on strongly pseudoconvex boundaries in $\mathbb{C}^n$ for $1 \leq q \leq n-1$. There are also numerous other results on $\bar\partial_b$ on boundaries of smooth bounded strongly pseudoconvex domains. We refer the reader to the book by Boggess [**?**] or Chen-Shaw [**?**] and the references therein.

When D is the boundary of a pseudoconvex domain with smooth boundary, the L^2 and Sobolev estimates for $\bar\partial_b$ have been obtained by Shaw in

[**?**] for $1 \leq q < n-1$ and for $q = n-1$ in Boas-Shaw [**?**] (See also Kohn [**?**]). The proof of the L^2 existence for $\bar{\partial}_b$ in these papers depends on the (at least C^4) smoothness of the boundary. Our results are the first for the $\bar{\partial}_b$ complex on C^1 or Lipschitz boundaries.

The $\bar{\partial}_b$ operator corresponds to a system of first order differential equations with variable coefficients. Since it is not possible to define the multiplication of two distributions in general, one cannot simply define $\bar{\partial}_b$ on nonsmooth manifolds in the sense of distributions. However, $\bar{\partial}$ is always well defined on any domain D in $\mathbb{C}^n$, regardless of the smoothness of the boundary bD. When D is a bounded Lipschitz domain, the $\bar{\partial}_b$ complex can be defined via duality by integration by parts and $\bar{\partial}$, since Stokes' theorem still holds on any bounded Lipschitz domain. Our method can obviously be extended to the boundaries of Lipschitz domains in complex manifolds. For the sake of simplicity and clarity in presentation, we will only stay in $\mathbb{C}^n$.

The L^2 estimates and existence theorems of $\bar{\partial}_b$ have many applications. The closed range property is also closely related to the embeddability of CR structures. We note that $\bar{\partial}_b$ does not have closed range locally in the top degree case (when $q = n-1$) on any smooth strongly pseudoconvex boundaries, since it corresponds to the Lewy operator. Moreover, it does not always have closed range for abstract smooth strongly pseudoconvex CR structures. Burns [**?**] has showed that, for Rossi's example [**?**] of a nonembeddable strongly pseudoconvex CR structure of real dimension three, the $\bar{\partial}_b$ operator does not have closed range in either the L^2 or C^∞ sense.

7.3. **Nonisotropic Sobolev and Hölder spaces.** Let $(M, T^{1,0}(M))$ a CR manifolds of dimension $2n+k$ and let $\{L_1, \ldots, L_n, \overline{L}_1, \ldots, \overline{L}_n, T_1, \ldots, T_k\}$ an horthonormal frame for $\mathbb{C}TM$. Define

$$\|Xf\|^2 := \sum_{j=1}^{n} \left(\|L_j f\|^2 + \|\overline{L}_j f\|^2 \right).$$

The *nonisotropic Sobolev space* $W^1_*(M)$ is defined as the space of function $u \in L^2(M)$ such that the $W^1_*(M)$ norm

$$\|u\|^2_{W^1_*} := \|Xu\|^2 + \|u\|^2$$

is finite. The space $W^{1,p}_*(M)$ is defined in the same way, only substituting the L^2 by the L^p norm.

For any integer $m > 1$, the space $W^{m,p}_*$ is defined inductively as the space of function u such that $L_j u, \overline{L}_j u \in W^{m-1,p}_*(M)$, for any $j = 1, \ldots, n$. Thus, $u \in W^{m,p}_*(M)$ if and only if $u \in L^p(M)$ and $X_1 \cdots X_m u \in L^p(M)$ for any $X_j \in T^{1,0}(M) \oplus T^{0,1}(M)$, $1 \leq j \leq m$. The space $W^{-1,p}_*(M)$ is defined as the dual space of $W^{1,p'}_*(M)$, where $1/p + 1/p' = 1$ or, equivalently, as the space of function $f = \sum_{j=1}^{N} X_j u_j$, $X_j \in T^{1,0}(M) \oplus T^{0,1}(M)$, $u_j \in L^p(M)$.

The *nonisotropic Hölder space* $C^{\alpha}_{*}(M)$, $0<\alpha<1$, is defined as follow: $u\in C^{\alpha}_{*}(M)$ if and only if

$$\sup_{\gamma(t)} \frac{|u(\gamma(t))-u(\gamma(0))|}{|t|^{\alpha}} \leq C < \infty,$$

where γ is any C^1 curve in M such that $\gamma'(t)\in T^{1,0}(M)\oplus T^{0,1}(M)$ and $|\gamma'(t)|\leq 1$. We define for any $m>1$ the space $C^{m,\alpha}_{*}(M)$ inductively as before.

As in the previous section, we have the following result.

Theorem 7.10. *Let $(M,T^{1,0}(M))$ be a compact CR manifold of dimension $2n+k$, $k>0$. Suppose that M satisfies condition $Y(q)$ for some $0\leq q\leq n$. Then, there is a compact operator $G_b : W^{-1,p}_{*}(M)\to W^{1,p}_{*}(M)$ such that*

$$\Box_b G_b = G_b\Box_b = I - H_b,$$

where $H_b : L^2_{(0,q)}(M)\to \mathrm{Ker}(\Box_b)$ is the projection as before. Furthermore, if $f\in C^{m,\alpha}_{*}(M)$ and $u=G_bf$ then $u\in\in C^{m+2,\alpha}_{*}(M)$, for any $m=-1,0,1,\ldots$, $0<\alpha<1$. Finally, when $f\in W^{m,p}_{*}(M)$ then $u\in W^{m+2,p}_{*}(M)$, $1<p<\infty$.

The main consequence of the previous theorem is the fact that the $\bar\partial_b$ problem is solvable in nonisotropic Sobolev and Hölder spaces, as stated by the following corollary.

Corollary 7.11. *Let $f\in L^2_{(0,q)}(M)$, such that $\bar\partial_b f=0$ and $f\perp\mathrm{Ker}(\Box_b)$. Then the function $u=\bar\partial^*_b G_b f$ is in $L^2_{(0,q+1)}$ and is a solution to $\bar\partial_b u=f$. Furthermore if $f\in C^{m,\alpha}_{*}(M)$ then the solution u is in $C^{m+1,\alpha}_{*}(M)$.*

What we stated in the previous results, may be written in terms of L^2 estimates. In fact, let

$$Q_b(f,f) := \|\bar\partial_b f\|^2 + \|\theta_b f\|^2 + \|f\|^2 = ((\Box_b+I)f,f)$$

denoting the energy functional. Then

$$|(Tf,f)| + \|Xf\|^2 \leq CQ_b(f,f),$$

from which

$$\|f\|^2_{W^1_{*}(M)} \leq CQ_b(f,f) = C((\Box_b+I)f,f)$$

for any $f\perp\mathrm{Ker}(\Box_b)$. Then

$$\|f\|_{W^1_{*}(M)} \leq C\|\Box_b f\|_{W^{-1}_{*}(M)}. \tag{7.10}$$

In the same way for estimates in the other nonisotropic spaces.

7.4. **Campanato Space.** We want to define another important class of function spaces, the so called *Campanato's spaces.* Let $0 < \alpha < 1$, $1 \le p \le \infty$ and $x_0 \in M$. Let μ a measure on M and let $B_\rho(x_0)$ the ball of radius ρ and center in x_0. We say that a function f is in the space $\mathcal{L}_*^{p,\alpha}(x_0)$ if and only if there is $\rho_0 > 0$ such that $f \in L^p(B_{\rho_0}(x_0))$ and there is a constant $C > 0$ such that

$$\left(\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)} |f - f|_{B_\rho(x_0)}|^p\right)^{1/p} \le C\rho^\alpha, \qquad \forall\ 0 < \rho < \rho_0.$$

This is the pointwise Hölder spaces in the L^p sense (for $p = \infty$, this is the pointwise nonisotropic Hölder spaces).

To define the higher order spaces $\mathcal{L}_*^{p,m+\alpha}(x_0)$, $m \in \mathbb{N} \cup \{-1\}$, we need to proceed in the following way. We may choose coordinates $x = (z,t) = (x_1, y_1, \dots, x_n, y_n, t_1, \dots, t_k)$ called normal coordinates, such that

$$L_j = \frac{\partial}{\partial z_j}\Big|_{x_0}, \quad T_h = \frac{\partial}{\partial t_h}\Big|_{x_0}, \qquad 1 \le j \le n, 1 \le h \le k.$$

We say that a polynomial $P(z,t)$ in (z,t) is of *degree* m if

$$P(z,t) = \sum_{|I|+|J|+2|L| \le m} a_{IJL} z^I \bar{z}^J t^L,$$

where I, J, L are multiindices.

Under this assumption we say that $f \in \mathcal{L}_*^{p,m+\alpha}(x_0)$ if and only if f is a L^p function and there is $\rho_0 > 0$ and a polynomial of degree m, $P_m^{x_0}$, such that

$$\left(\frac{1}{\mu(B_\rho(x_0))}\int_{B_\rho(x_0)} |f - P_m^{x_0}|^p\right)^{1/p} \le C\rho^{m+\alpha}, \qquad \forall\ 0 < \rho < \rho_0.$$

The relation between this space and the ones introduced in the previous section is stated in the the following lemma.

Lemma 7.12 (Campanato Embedding)**.** *For any* $0 < \alpha < 1$ *and* $1 \le p \le \infty$ *we have*

$$\mathcal{L}_*^{p,m+\alpha}(M) = C_*^{m,\alpha}(M).$$

Before proving regularity of $\Box_b$ on Campanato spaces, we want first to scale the equation for the $\Box_b$ to the unit ball $B_1(x_0)$. For any $\epsilon > 0$, we set $\epsilon x = (\epsilon z, \epsilon^2 t)$, the dilation. Let $\overline{L}_j^\epsilon$ the operator defined by

$$\overline{L}_j^\epsilon(f(\epsilon x)) = \epsilon(\overline{L}_j f)(\epsilon x), \qquad x \in B_1(x_0).$$

Then we have

$$L_j^\epsilon \to L_j^0 = \frac{\partial}{\partial \bar{z}_j} + \sum_{\ell=1}^k a_j^\ell \frac{\partial}{\partial t_\ell}.$$

Now, the $\bar{\partial}_b^\epsilon$ and $\Box_b^\epsilon$ associated to the CR structure $T_\epsilon^{1,0}$ generated by L_j^ϵ are such that

$$\left.\begin{array}{r}\bar{\partial}_b^\epsilon u \to \bar{\partial}_b^0 u\\ \Box_b^\epsilon u \to \Box_b^0 u\end{array}\right\} \qquad \text{as } \epsilon \to 0.$$

So, if M satisfies condition $Y(q)$, we can restate all the results of the section for the scaled situation, i.e for the CR manifold M_ϵ which approaches M, obtain again the same kind of estimates. This leads us to the following result.

Proposition 7.13. *If* $f \in \mathcal{L}_*^{2,\alpha}(x_0)$ *and* $u \in W_*^1(B_{\rho_0}(x_0))$ *is a solution to* $\Box_b u = f$ *in* $B_{\rho_0}(x_0))$*, then* $u \in \mathcal{L}_*^{2,2+\alpha}(x_0)$*.*

Proof. We will sketch the idea of the proof.

(1) It sufficees to show that there exists a polynomial P of degree 2 in normal coordinates (z, t) such that

$$\left(\frac{1}{B_\rho(x_0)}\int_{B_\rho(x_0)} |u-P|^2\right)^{1/2} \le C\rho^{2+\alpha}, \qquad \forall\ 0<\rho<\rho_0.$$

Furthermore, we can discretize the estimates, i.e., it suffices to prove for $\rho = r^m$, $m = 0, 1, 2, \ldots$, we get

$$\left(\frac{1}{B_{r^m}(x_0)}\int_{B_{r^m}(x_0)} |u-P|^2\right)^{1/2} \le Cr^{m(2+\alpha)}, \qquad \forall\ 0<r<r_0.$$

(2) Given $\eta > 0$, if f is small, i.e. $\|f\|^2 < \delta$, for a given δ, then there is a function h such that $\Box_b^0 h = 0$ and

$$\left(\frac{1}{B_\rho(x_0)}\int_{B_\rho(x_0)} |u-h|^2\right)^{1/2} < \eta, \qquad \forall\ 0<\rho<\rho_0.$$

Then, h is of the form $h = P + O(3)$, where P is a polynomial of degree 2.

This is proved by the compactness arguments since we already know that $\Box_b$ is subelliptic in the interior. Thus we have the estimates for $m = 0$. The estimates for $m = 1, 2, \cdots$ are obtained by scaling and induction. $\Box$

7.5. **The Tangential Cauchy-Riemann Complex.** Let M be a hypersurface in a complex manifoldχ. The $\overline{\partial}$ complex on χ, when restricted to M, induces the tangential Cauchy-Riemann complex, or the $\overline{\partial}_b$ complex. In fact, the tangential Cauchy-Riemann complex can be formulated on any CR manifold. There are two different approaches in this setting. One way is to define the tangential Cauchy-Riemann complex intrinsically on any abstract CR manifold itself without referring to the ambient space. On the other hand, if the CR manifold is sitting in a complex manifold, the tangential Cauchy-Riemann complex can also be defined extrinsically via the ambient complex structure.

First, we assume that M is a smooth hypersurface in χ, and let r be a defining function for M. In some open neighborhood U of M, let $I^{p,q}$,

$0 \leq p, q \leq n$, be the ideal in $\Lambda^{p,q}(\chi)$ such that at each point $z \in U$ the fiber $I^{p,q}_z$ is generated by r and $\overline{\partial} r$, namely, each element in the fiber $I^{p,q}_z$ can be expressed in the form

$$rH_1 + \overline{\partial} r \wedge H_2,$$

where H_1 is a smooth (p,q)-form and H_2 is a smooth $(p, q-1)$-form. Denote by $\Lambda^{p,q}(\chi)|_M$ and $I^{p,q}|_M$ the restriction of $\Lambda^{p,q}(\chi)$ and $I^{p,q}$ respectively to M.

We define the quotient bundle

$$\Lambda^{p,q}(M) = \Lambda^{p,q}(\chi)|_M / I^{p,q}|_M.$$

We denote by $\mathcal{E}^{p,q}$ the space of smooth sections of $\Lambda^{p,q}(M)$ over M, i.e., $\mathcal{E}^{p,q}(M) = \Gamma(M, \Lambda^{p,q}(M))$. Let τ denote the following map

$$\tau : \Lambda^{p,q}(\chi) \to \Lambda^{p,q}(M), \tag{7.11}$$

where τ is obtained by first restricting a (p,q)-form ϕ in χ to M, then projecting the restriction to $\Lambda^{p,q}(M)$. One should note that $\Lambda^{p,q}(M)$ is not intrinsic to M, i.e., $\Lambda^{p,q}(M)$ is not a subspace of the exterior algebra generated by the complexified cotangent bundle of M. This is due to the fact that $\overline{\partial} r$ is not orthogonal to the cotangent bundle of M.

The tangential Cauchy-Riemann operator

$$\overline{\partial}_b : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$$

is now defined as follows: For any $\phi \in \mathcal{E}^{p,q}(M)$, pick a smooth (p,q)-form ϕ_1 in χ that satisfies $\tau\phi_1 = \phi$. Then, $\overline{\partial}_b \phi$ is defined to be $\tau\overline{\partial}\phi_1$ in $\mathcal{E}^{p,q+1}(M)$. If ϕ_2 is another (p,q)-form in χ such that $\tau\phi_2 = \phi$, then

$$\phi_1 - \phi_2 = rg + \overline{\partial} r \wedge h,$$

for some (p,q)-form g and $(p, q-1)$-form h. It follows that

$$\overline{\partial}(\phi_1 - \phi_2) = r\overline{\partial} g + \overline{\partial} r \wedge g - \overline{\partial} r \wedge \overline{\partial} h,$$

and hence,

$$\tau\overline{\partial}(\phi_1 - \phi_2) = 0.$$

Thus, the definition of $\overline{\partial}_b$ is independent of the choice of ϕ_1. Since $\overline{\partial}^2 = 0$, we have $\overline{\partial}_b^2 = 0$ and the following boundary complex

$$0 \;\to\; \mathcal{E}^{p,0}(M) \overset{\overline{\partial}_b}{\to} \longrightarrow \mathcal{E}^{p,1}(M) \overset{\overline{\partial}_b}{\to} \longrightarrow \;\cdots\; \overset{\overline{\partial}_b}{\to} \longrightarrow \mathcal{E}^{p,n-1}(M) \;\to\; 0.$$

For the intrinsic approach, we will assume that $(M, T^{1,0}(M))$ is an orientable CR manifold of real dimension $2n-1$ with $n \geq 2$. A real smooth manifold M is said to be orientable if there exists a nonvanishing top degree form on M.

We define the quotient bundle

$$E = \mathbb{C}T(M)/T^{0,1}(M).$$

Then E is a (holomorphic) vector bundle of complex dimension n. If M is a real hypersurface in a complex manifold χ, then E is isomorphic to $T^{1,0}(\chi)$ restricted to M.

We define the holomorphic p-forms on E by Ω^p. Define the vector bundle $C^{p,q}(M)$, $0 \le p \le n$, $0 \le q \le n-1$, by

$$C^{p,q}(M) = \Omega^p \otimes \Lambda^q T^{*0,1}(M) = C^q(M, \Omega^p).$$

It follows that $C^{p,q}(M)$ defined in this way is intrinsic to M. Denote by $\mathcal{C}^{p,q}$ the space of smooth sections of $C^{p,q}(M)$ over M, i.e., $\mathcal{C}^{p,q}(M) = \Gamma(M, C^{p,q}(M))$. We define the operator

$$\overline{\partial}_b : \mathcal{C}^{p,q}(M) \to \mathcal{C}^{p,q+1}(M)$$

as follows: If $\phi \in \mathcal{C}^{p,0}$, $\overline{\partial}_b\phi$ is defined by

$$\langle \overline{\partial}_b\phi, \overline{L}\rangle = \overline{L}\langle\phi\rangle$$

for any section $\overline{L}$ of $T^{0,1}(M)$. Then $\overline{\partial}_b$ is extended to $\mathcal{C}^{p,q}(M)$ for $q > 0$ as a derivation. Namely, if $\phi \in \mathcal{C}^{p,q}(M)$, we define

$$\begin{aligned}
&\langle \overline{\partial}_b\phi, (\overline{L}_1 \wedge \cdots \wedge \overline{L}_{q+1})\rangle \\
&\quad = \frac{1}{q+1}\Bigg\{\sum_{j=1}^{q+1}(-1)^{j+1}\overline{L}_j\langle\phi, (\overline{L}_1\wedge\cdots\wedge\widehat{\overline{L}}_j\wedge\cdots\wedge\overline{L}_{q+1})\rangle \\
&\qquad + \underset{i<j}{\longrightarrow}\sum(-1)^{i+j}\langle\phi, ([\overline{L}_i, \overline{L}_j]\wedge\overline{L}_1\wedge\cdots\wedge\widehat{\overline{L}}_i\wedge\cdots\wedge\widehat{\overline{L}}_j\wedge\cdots\wedge\overline{L}_{q+1})\rangle\Bigg\}.
\end{aligned}$$

Here by $\widehat{\overline{L}}$ we mean that the term $\overline{L}$ is omitted from the expression.

One should note how the integrability condition of the CR structure $T^{1,0}(M)$ comes into play in the definition of $\overline{\partial}_b$, and it is standard to see that the following sequence

$$0 \;\to\; \mathcal{C}^{p,0}(M) \overset{\overline{\partial}_b}{\to} \longrightarrow \mathcal{C}^{p,1}(M) \overset{\overline{\partial}_b}{\to} \longrightarrow \;\cdots\; \overset{\overline{\partial}_b}{\to} \longrightarrow \mathcal{C}^{p,n-1}(M) \;\to\; 0,$$

forms a complex, i.e., $\overline{\partial}_b^2 = 0$.

When the CR manifold $(M, T^{1,0}(M))$ is embedded as a smooth hypersurface in a complex manifold χ with the CR structure $T^{1,0}(M)$ induced from the ambient space, the tangential Cauchy-Riemann complex on M can be defined either extrinsically or intrinsically. These two complexes are different, but one can easily show that they are isomorphic. Thus, if the CR manifold is embedded, we shall not distinguish the extrinsic or intrinsic definitions of the tangential Cauchy-Riemann complex. The operator $\overline{\partial}_b$ is a first order differential operator, and one may consider the inhomogeneous $\overline{\partial}_b$ equation

$$\overline{\partial}_b u = f, \tag{7.12}$$

where f is a (p,q)-form on M. Equation (***) is overdetermined when $0 < q < n-1$. Since $\overline{\partial}_b^2 = 0$, for equation (***) to be solvable, it is necessary that

$$\overline{\partial}_b f = 0.$$

Let $L_1, \cdots, L_{n-1}$ be a local basis for smooth sections of $T^{1,0}(M)$ over some open subset $U \subset M$, so $\overline{L}_1, \cdots, \overline{L}_{n-1}$ is a local basis for $T^{0,1}(M)$ over U. Next we choose a local section T of $\mathbb{C}T(M)$ such that $L_1, \cdots, L_{n-1}, \overline{L}_1, \cdots, \overline{L}_{n-1}$ and T span $\mathbb{C}T(M)$ over U. We may assume that T is purely imaginary.

Definition 7.14. The Hermitian matrix $(c_{ij})_{i,j=1}^{n-1}$ defined by

$$[L_i, \overline{L}_j] = c_{ij} T, \quad \text{mod } (T^{1,0}(U) \oplus T^{0,1}(U)) \tag{7.13}$$

is called the Levi form associated with the given CR structure.

The Levi matrix (c_{ij}) clearly depends on the choices of $L_1, \cdots, L_{n-1}$ and T. However, the number of nonzero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choices of $L_1, \cdots, L_{n-1}$ and T. Hence, after changing T to $-T$, it makes sense to consider positive definiteness of the matrix (c_{ij}).

Definition 7.15. The CR structure is called pseudoconvex at $p \in M$ if the matrix $(c_{ij}(p))$ is positive semidefinite after an appropriate choice of T. It is called strictly pseudoconvex at $p \in M$ if the matrix $(c_{ij}(p))$ is positive definite. If the CR structure is (strictly) pseudoconvex at every point of M, then M is called a (strictly) pseudoconvex CR manifold. If the Levi form vanishes completely on an open set $U \subset M$, i.e., $c_{ij} = 0$ on U for $1 \le i, j \le n-1$, M is called Levi flat.

Theorem 7.16. *Let $D \subset \mathbb{C}^n$, $n \ge 2$, be a bounded domain with C^∞ boundary. Then D is (strictly) pseudoconvex if and only if $M = bD$ is a (strictly) pseudoconvex CR manifold.*

Proof. Let r be a C^∞ defining function for D, and let $p \in bD$. We may assume that $(\partial r/\partial z_n)(p) \neq 0$. Hence,

$$L_k = \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_n}, \quad \text{for } k = 1, \cdots, n-1,$$

form a local basis for the tangential type $(1,0)$ vector fields near p on the boundary. If $L = \sum_{j=1}^n a_j(\partial/\partial z_j)$ is a tangential type $(1,0)$ vector field near p, then we have $\sum_{j=1}^n a_j(\partial r/\partial z_j) = 0$ on bD and $L = (\partial r/\partial z_n)^{-1} \sum_{j=1}^{n-1} a_j L_j$

on bD. Hence, if we let $\eta = \partial r - \overline{\partial} r$, we obtain

$$\begin{aligned}\sum_{i,j=1}^{n-1} c_{ij} a_i \overline{a}_j &= \sum_{i,j=1}^{n-1} \langle \eta, [L_i, \overline{L}_j] \rangle a_i \overline{a}_j \\ &= \sum_{i,j=1}^{n-1} (L_i \langle \eta, \overline{L}_j \rangle - \overline{L}_j \langle \eta, L_i \rangle - 2 \langle d\eta, L_i \wedge \overline{L}_j \rangle) a_i \overline{a}_j \\ &= \sum_{i,j=1}^{n-1} 4 \langle \partial \overline{\partial} r, L_i \wedge \overline{L}_j \rangle a_i \overline{a}_j \\ &= 4 \left| \frac{\partial r}{\partial z_n} \right|^2 \langle \partial \overline{\partial} r, L \wedge \overline{L} \rangle \\ &= 4 \left| \frac{\partial r}{\partial z_n} \right|^2 \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \overline{z}_j} a_i \overline{a}_j,\end{aligned}$$

which gives the desired equivalence between these two definitions. This proves the theorem.

□

We note that, locally, a CR manifold in $\mathbb{C}^n$ is pseudoconvex if and only if it is the boundary of a smooth pseudoconvex domain from one side.

Lemma 7.17. *Any compact strongly pseudoconvex CR manifold $(M, T^{1,0}(M))$ is orientable.*

Proof. Locally, let $\eta, \omega_1, \cdots, \omega_{n-1}$ be the one forms dual to $T, L_1, \cdots, L_{n-1}$ which are defined as above. The vector field T is chosen so that the Levi form is positive definite. Then we consider the following $2n-1$ form

$$\eta \wedge \omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_{n-1} \wedge \overline{\omega}_{n-1}.$$

It is not hard to see that the $2n-1$ form generated by other bases will differ only by a positive function. Hence, a partition of unity argument will give the desired nowhere vanishing $2n-1$ form on M, and the lemma is proved.

□

7.6. **Folland-Stein-Tanaka formula for $\Box_b$.**

Let M be a strongly pseudoconvex CR manifold with the pseudohermitian Levi metric. The $\bar{\partial}_b$ complex

$$\bar{\partial}_b : \mathcal{C}^{0,q}(M) \to \mathcal{C}^{0,q+1}(M), \quad 0 \le q \le n-1.$$

The formal adjoint of $\bar{\partial}_b$ under the L^2 norm, denoted by ϑ_b, is defined by the requirement that

$$(\vartheta_b f, \bar{g}) = (f, \bar{\partial}_b \bar{g}) * * * 1$$

for compactly supported smooth forms.

Proposition 7.18. *Let* $e_1, \cdots, e_{n-1}$ *be an orthonormal frame field for* $T^{1,0}(M)$ *in an open neighborhood in* M *and let* $w_1, \cdots, w_{n-1}$ *be its dual. Then*

$$\bar{\partial}_b = \sum_{j=1}^{n-1} \bar{w}_j \wedge \nabla_{\bar{e}_j}. *** 2$$

Proposition 7.19. *Let* M *be a strongly pseudoconvex CR manifold with the pseudohermitian Levi metric. Let* $e_1, \cdots, e_{n-1}$ *be an orthonormal frame field for* $T^{1,0}(M)$ *in an open neighborhood in* M *and let* $w_1, \cdots, w_{n-1}$ *be its dual. Then*

$$\vartheta_b = -\sum_{j=1}^{n-1} \lrcorner(\bar{e}_j) \nabla_{e_j}. *** 3$$

Proof. Let ϕ be a $(0,q)$-form and ψ be a $(0,q+1)$-form on M. We have

$$\langle \bar{\partial}_b \phi, \psi \rangle = \langle \phi, \vartheta_b \psi \rangle + \delta' \beta *** 4$$

where β is a (0,1)-form defined by $\beta = \beta_j \bar{w}_j$ with $\beta(\bar{e}_j) = \langle \phi, e_j \lrcorner \bar{\psi} \rangle$ and $\delta' \beta$ is the function defined to be

$$\delta' \beta = \sum_j^{n-1} \nabla_{\bar{e}_j} \beta_j.$$

Let $\star\beta$ be a $(2n-2)$-form defined by

$$\star\beta = \sum_j^{n-1} \beta_j \bar{e}_j \lrcorner dV.$$

We first prove the following claim:

$$d(\star\beta) = \delta' \beta dV. *** 5$$

To see this, let $X = \sum_j \beta_j \bar{e}_j$ be the dual of β. Let A_X be the (1,1) tensor on M defined by

$$A_X = L_X - \nabla_X.$$

Since $\nabla dV = 0$, we have

$$L_X dV = A_X dV.$$

Let $\xi_1, \cdots, \xi_{2n-1}$ be a basis for $T_x M$. Since A_X is a derivative which maps every function into zero, we have

$$\begin{aligned} L_X dV(\xi_1, \cdots, \xi_{2n-1}) &= A_X dV(\xi_1, \cdots, \xi_{2n-1}) \\ &= A_X(dV(\xi_1, \cdots, \xi_{2n-1})) - \sum_j dV(\xi_1, \cdots, A_X \xi_j, \cdots, \xi_{2n-1}) \\ &= -\sum_j dV(\xi_1, \cdots, A_X \xi_j, \cdots, \xi_{2n-1}) \\ &= -\mathrm{Trace} A_X dV(\xi_1, \cdots, \xi_{2n-1}) \end{aligned}$$

Thus we have

$$L_X dV = -\mathrm{Trace} A_X dV. *** 6$$

Recall the torsion *** formula

$$A_X Y = L_X Y - \nabla_X Y = -\nabla_Y X - T(X, Y). *** 7$$

where T is the torsion tensor of ∇. Now we choose a special frame for $\mathbb{C}T_x(M)$ with $e_j = \xi_j + i\xi_{n+j-1}$, $\bar{e}_j$, where $j = 1, \cdots, n-1$ and $\xi_{2n-1} = \xi$, the basic vector field,. Then we have

$$T(e_j, \bar{e}_k) = -\omega(e_i, \bar{e}_k)\xi = \frac{1}{i}\delta_{jk}\nabla_\xi$$

and

$$T(Y, \xi) \in T^{1,0} + T^{0,1}$$

for all $Y \in T^{1,0} + T^{0,1}$. Thus the torsion term in (7) has no contribution when taking trace of A_X. Also note that $\nabla_X Y$ preserves types for X and Y in $T^{1,0} + T^{0,1}$ and $\nabla\xi = 0$. We have

$$\mathrm{Trace} A_X = \langle \nabla_{\bar{e}_j} X, \bar{e}_j \rangle = \delta' \beta$$

and

$$\mathrm{Trace} A_X dV = \delta' \beta dV. *** 8$$

For each vector field X, the divergence formula is defined to be

$$(\mathrm{div}) X dV = L_X dV = d(X \lrcorner dV), ***9$$

we have proved from (6)-(9) that

$$\delta' \beta dV = L_X dV = d(\star\beta).$$

Integrating with respect to the volume element dV in (4), we have proved the proposition.

□

We define the $\bar{\partial}_b$-Laplacian

$$\Box_b = \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b.$$

Theorem 7.20. *Folland-Stein-Tanaka formula for $\Box_b$ Let $(M, T^{1,0}(M)$ be a strongly pseudoconvex CR manifold and ξ ba a basic field. Let $e_1, \cdots, e_{n-1}$ be an orthonormal frame field for $T^{1,0}(M)$ in an open neighborhood in M and let $w_1, \cdots, w_{n-1}$ be its dual.*

$$\Box_b = -\underset{j}{\rightarrow}\sum \nabla^2_{e_j\bar{e}_j} + \frac{q}{i}\nabla_\xi - \underset{i,j}{\rightarrow}\sum \bar{w}^k \wedge \lrcorner(\bar{e}_j) K_{e_j\bar{e}_k} - \underset{i,j}{\rightarrow}\sum \bar{w}^k \wedge \lrcorner(\bar{e}_j) R_{e_j\bar{e}_k}, ***4$$

$$\Box_b = -\underset{j}{\rightarrow}\sum \nabla^2_{\bar{e}_j e_j} - \frac{(n-q-1)}{i}\nabla_\xi - \underset{i,j}{\rightarrow}\sum \bar{w}^k \wedge \lrcorner(\bar{e}_j) K_{e_j\bar{e}_k}, ***5$$

$$\begin{aligned}\Box_b = &-\frac{n-q-1}{n-1}\underset{j}{\rightarrow}\sum \nabla^2_{e_j\bar{e}_j} - \frac{q}{n-1}\underset{j}{\rightarrow}\sum \nabla^2_{\bar{e}_j e_j} \\ &- \underset{i,j}{\rightarrow}\sum \bar{w}^k \wedge \lrcorner(\bar{e}_j) K_{e_j\bar{e}_k} - \frac{n-q-1}{n-1}\underset{i,j}{\rightarrow}\sum \bar{w}^k \wedge \lrcorner(\bar{e}_j) R_{e_j\bar{e}_k}\end{aligned} \quad ***6$$

where

$$\nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$$

is the second covariant differential and

$$R_{XY} = -\nabla_X\nabla_Y + \nabla_Y\nabla_X - \nabla_{[X,Y]} = \nabla^2_{YX} - \nabla^2_{XY}$$

is the curvature tensor, both extended $\mathbb{C}$-linearly to (p,q)-forms, K is the curvature for the holomorphic vector bundle Ω^p.

Proof.

$$\begin{aligned}\vartheta_b\bar\partial_b &= -\underset{j}{\to}\sum \lrcorner(\bar e_j)\nabla_{e_j}(\underset{k}{\to}\sum \bar w^k \wedge \nabla_{\bar e_k})\\ &= -\underset{j,k}{\to}\sum \lrcorner(\bar e_j)\bar w^k \wedge \nabla_{e_j}\nabla_{\bar e_k}\\ &= -\underset{j}{\to}\sum \nabla_{e_j}\nabla_{\bar e_j} + \underset{j,k}{\to}\sum \bar w^k \wedge \lrcorner(\bar e_j)\nabla_{e_j}\nabla_{\bar e_k}.\end{aligned}$$

$$\bar\partial_b\vartheta_b = -\underset{k}{\to}\sum \bar w^k \wedge \nabla_{e_k}(\underset{j}{\to}\sum \lrcorner(\bar e_j)\nabla_{e_j}) = -\underset{j,k}{\to}\sum \bar w^k \wedge \lrcorner(\bar e_j)(\nabla_{\bar e_k}\nabla_{e_j})$$

since $\nabla_{\bar e_k}\lrcorner(e_j) = \lrcorner(e_j)\nabla_{\bar e_k}$.

From the Ricci's formula, we have

$$[\nabla_{e_j}, \nabla_{\bar e_k}] = -R_{e_j\bar e_k} - K_{e_j\bar e_k} - D_{T(e_i,\bar e_k)}$$

where T is the torsion tensor of ∇ and

$$T(e_j, \bar e_k) = -\omega(e_i, \bar e_k)\xi = \frac{1}{i}\delta_{jk}\nabla_\xi.$$

Thus we obtain

$$\begin{aligned}\Box_b &= -\underset{j}{\to}\sum \nabla_{e_j}\nabla_{\bar e_j} + \underset{j,k}{\to}\sum \bar w^k \wedge \lrcorner(\bar e_j)[\nabla_{e_j}, \nabla_{\bar e_k}]\\ &= -\underset{j}{\to}\sum \nabla_{e_j}\nabla_{e_j} + \frac{1}{i}q\nabla_\xi - \underset{i,j}{\to}\sum \bar w^k \wedge \lrcorner(\bar e_j)K_{e_j\bar e_k} - \underset{i,j}{\to}\sum \bar w^k \wedge \lrcorner(\bar e_j)R_{e_j\bar e_k}\end{aligned}$$

which proves (4). (5) follows from (4) and the Ricci formula and $\sum_j K_{e_j\bar e_j} = 0$. (6) follows from (4) and (5).

$\Box$

Pseudo-hermitian manifolds with constant coefficients

1. The Heisenberg group, the flat case.
2. The sphere in $\mathbb{C}^n$, positive curvature.
3. The hyperbolic case.

8. Applications of $\bar\partial$ and $\bar\partial_b$

There are many applications from the solutions of the Cauchy-Riemann equations and the tangential Cauchy-Riemann equations. We will give applications in the following areas.

8.1. **The Levi problem.** Let D be a pseudoconvex domain in $\mathbb{C}^n$ with $n \geq 2$. One of the major problems in complex analysis is to show that a pseudoconvex domain D is a domain of holomorphy. Near each boundary point $p \in bD$, one must find a holomorphic function $f(z)$ on D which cannot be continued holomorphically near p. This problem is called the Levi problem for D at p. It involves the construction of a holomorphic function with certain specific local properties. From a well-known results of Benhke and Stein, it suffices to show that a strongly pseudoconvex domain is a domain of holomorphy.

If the domain D is strongly pseudoconvex with C^∞ boundary bD and $p \in bD$, one can construct a *local* holomorphic function f in an open neighborhood U of p, such that f is holomorphic in $U \cap D$, $f \in C(\overline{D} \cap U \setminus \{p\})$ and $f(z) \to \infty$ as $z \in D$ approaches p. In fact f can be easily obtained as follows: let r be a strictly plurisubharmonic defining function for D and we assume that $p = 0$. Let

$$F(z) = -2\sum_{i=1}^{n} \frac{\partial r}{\partial z_i}(0) z_i - \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial z_j}(0) z_i z_j.$$

$F(z)$ is holomorphic, and it is called the Levi polynomial of r at 0. Using Taylor's expansion at 0, there exists a sufficiently small neighborhood U of 0 and $C > 0$ such that for any $z \in \overline{D} \cap U$,

$$\mathrm{Re} F(z) = -r(z) + \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + O(|z|^3) \geq C|z|^2.$$

Thus, $F(z) \neq 0$ when $z \in \overline{D} \cap U \setminus \{0\}$. Setting

$$f = \frac{1}{F^N}$$

for some positive integer N, it is easily seen that f is locally a holomorphic function which cannot be extended holomorphically across 0. We can require that f is not locally L^2 near 0 by choosing N large.

Global holomorphic functions cannot be obtained simply by employing smooth cut-off functions to patch together the local holomorphic data, since the cut-off functions are no longer holomorphic. Let χ be a cut-off function such that $\chi \in C_0^\infty(U)$ and $\chi = 1$ in a neighborhood of 0. We note that χf is not holomorphic in D. However, if χf can be corrected by solving a $\bar{\partial}$-equation, then the Levi problem will be solved.

Let us consider the (0,1)-form g defined by

$$g = \bar{\partial}(\chi f) = (\bar{\partial}\chi) f.$$

This form g can obviously be extended smoothly up to the boundary. It is easy to see that g is a $\bar{\partial}$-closed form in D and $g \in C^\infty_{(0,1)}(\overline{D})$. If we can find a solution $u \in C^\infty(\overline{D})$ such that

$$\bar{\partial} u = g \quad \text{in} \quad D, \tag{8.1}$$

then we define for $z \in D$,

$$h(z) = \chi(z)f(z) - u(z).$$

It follows that h is holomorphic in D, $h \in C^\infty(\overline{D} \setminus \{0\})$ and h is singular at 0. Thus one can solve the Levi problem for strongly pseudoconvex domains provided one can solve equation (6.1) with solutions smooth up to the boundary. From Kohn's solution (or Hörmander's L^2 solution), we can take $u = \bar\partial^* N \bar\partial(\chi f)$ and $u \in C^\infty(\overline{D})$ and $h = \chi f - u$ is a holomorphic function with singularity at 0. Notice that $h = \chi f - \bar\partial^* N \bar\partial(\chi f) = Pf$ where P is the Bergman projection on D.

This gives that any strictly pseudoconvex domain is a domain of holomorphy. From a well known results of Benhke and Stein, any pseudoconvex domain is a domain of holomorphy since it can be exhausted by strictly pseudoconvex domains. This gives solution of the Levi problem. Notice that Kohn's method works even for pseudoconvex domains in complex manifolds.

8.2. **Regularity for the Bergman projection.** Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, we consider the Bergman projection

$$P : L^2(D) \to L^2(D) \cap \mathrm{Ker}(\bar\partial). \tag{8.2}$$

From the Hörmander's L^2 theory for $\bar\partial$, we have the strong Hodge decompostion for $\Box$ for $L^2_{(0,1)}(D)$. For any $f \in L^2_{(0,1)}(D)$, we have

$$f = \Box N f$$

where N is the $\bar\partial$-Neumann operator.

This gives that $\Box = \bar\partial^* \bar\partial$ on functions has closed range also. Thus we have for any $f \in L^2(D)$, Notice that $\mathrm{Ker}(\Box)$ is the same as $\mathrm{Ker}(\bar\partial)$ for functions. We have

$$f = \Box N_0 f \oplus Pf.$$

Since

$$\bar\partial f = \bar\partial \Box N_0 f$$

and

$$N_1 \bar\partial f = N_1(\bar\partial\bar\partial^*\bar\partial)N_0 f = N_1(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\bar\partial N_0 f,$$

we have

$$N_1 \bar\partial f = \bar\partial N_0 f$$

Thus $\Box N_0 = \bar\partial^* \bar\partial N_0 = \bar\partial^* N_1 \bar\partial$, this give the Kohn's formula for the Bergman projection :

$$Pf = f - \bar\partial N_1 \bar\partial f. \qquad f \in L^2(D). \tag{8.3}$$

Similarly, we have that

$$N_0 f = (I - P) N_0 f = \bar\partial^* N_1 \bar\partial N_0 f = \bar\partial^* N_1^2 \bar\partial f. \tag{8.4}$$

Theorem 8.1 (Kohn)**.** *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then the Bergman projection $P : C^\infty(\overline{D}) \to C^\infty(\overline{D})$.*

Theorem 8.2 (Fefferman)**.** *Let D_1 and D_2 be two bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary. Suppose that f is a biholomoprhic map from D_1 to D_2. Then f extends smoothly to the boundary bD_1.*

Definition 8.3 (Condition R)**.** Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then D is said to satisfy condition R if the Bergman projection $P : C^\infty(\overline{D}) \to C^\infty(\overline{D})$.

Theorem 8.4 (Bell-Ligocka)**.** *Let D_1 and D_2 be two bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary. Suppose that f is a biholomoprhic map from D_1 to D_2 and suppose that one of the domains satisfies condition R. Then f extends smoothly to the boundary bD_1.*

It was already known to Poincaré that the ball in $\mathbb{C}^2$ is not biholomorphic to the bidisc.

Corollary 8.5 (Bell-Ligocka)**.** *Let D_1 be a bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary and D_2 be a bounded weakly pseudoconvex domain with smooth boundary. Then D_1 and D_2 are not biholomorphic.*

We remark that condition R does not hold on Diederich-Fornaess worm domains (see Barrett [**?**] and Christ [**?**]).

8.3. **The Newlander-Nirenberg theorem.**

Definition 8.6. Let M be a real $(2n+k)$-dimensional manifold. M is a *CR manifold* if there is a subbundle $T^{1,0}(M)$ of $\mathbb{C}TM$, the so called *CR structure*, such that:

(*i*) $\dim_{\mathbb{C}} T^{1,0}_z(M) = n$, for any $z \in M$;
(*ii*) $T^{1,0}_z(M) \cap T^{0,1}_z(M) = (0)$, for any $z \in M$, where $T^{0,1}_z(M) = \overline{T^{1,0}_z(M)}$;
(*iii*) $[L, L'] \in \Gamma^\infty(T^{1,0}(M))$, for every $L, L' \in \Gamma^\infty(T^{1,0}(M))$.

When $k = 0$ in the previous definiton, M is an almost complex manifold.

Theorem 8.7. *(Newlander-Nirenberg) Let $(M, T^{1,0})$ be an almost complex manifold of class $C^k(U) \cap C^{k,\alpha}(x_0)$ in a neighborhood U of a point x_0 in M where $k \geq 1$ and $0 < \alpha < 1$. Assume that the almost complex structure $T^{1,0}$ is (formally) integrable. Then there exist a neighborhood $V \subset U$ of x_0 and holomorphic coordinates of class $C^k(V) \cap C^{k+1,\alpha}(x_0)$ in V which embed V into $\mathbb{C}^n$.*

Proof. The proof is exactly as in [**?**] combined with the pointwise interior Hölder regularity for elliptic operators.

Let $L_1, \cdots, L_n$ be a local basis for smooth sections of $T^{1,0}(M)$. Let $x_1, \cdots, x_{2n}$ be the real coordinates for M and we write $z_j = x_j + ix_{n+j}$.

We can using compatibility condition, after a quadratic change of coordinates, assume that

$$L_i = \frac{\partial}{\partial z_i} + \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial \bar{z}_j}, \quad i = 1, \cdots, n,$$

where the a_{ij}'s are $C^{1,\alpha}$ functions and $a_{ij}(0) = 0$ for all $i, j = 1, \cdots, n$. At the origin, L_i is the constant coefficient operator $\partial/\partial z_i$. Let $L_i^\epsilon = \frac{\partial}{\partial z_i} + \sum_{j=1}^n a_{ij}(\epsilon x) \frac{\partial}{\partial \bar{z}_j}, \quad i = 1, \cdots, n$, where $\epsilon > 0$ is small. Then $T_\epsilon^{0,1} = \langle L_1^\epsilon, \cdots, L_n^\epsilon \rangle$ defines an almost complex structure that is integrable for each $\epsilon < \epsilon_0$ for some sufficiently small $\epsilon_0 > 0$.

Let $\bar{\partial}_\epsilon$ denote the Cauchy-Riemann complex associated with the almost complex structure $T_\epsilon^{0,1}$ equipped with a Hermitian metric. Then the existence and regularity theory developed for $\bar{\partial}$ in the previous section on any complex manifold can be applied to M with $\bar{\partial}$ substituted by $\bar{\partial}_\epsilon$. Let $\phi = \sum_{i=1}^n |z_i|^2 = |x|^2$, ϕ is a strictly plurisubharmonic function near 0. Choosing ϵ_0 sufficiently small, we may assume that $\Omega = \{x \in \mathbb{R}^{2n} \mid |x|^2 < 1\} = B_1$ and Ω is strongly pseudoconvex with respect to the almost complex structure $T_\epsilon^{0,1}(M)$. Using the L^2 existence results for $\bar{\partial}$ (see [**?**] or [**?**]), the $\bar{\partial}$-Neumann operator N_ϵ exists on Ω and there exists a solution $u_i^\epsilon = \bar{\partial}_\epsilon^* N_\epsilon \bar{\partial}_\epsilon z_i$ on Ω such that $\bar{\partial}_\epsilon u_i^\epsilon = \bar{\partial}_\epsilon z_i$ and

$$\| u_i^\epsilon \|_\Omega \leq C \| \bar{\partial}_\epsilon z_i \|_\Omega \tag{8.5}$$

where C can be chosen uniformly for $\epsilon < \epsilon_0$. Since

$$\overline{L}_i^\epsilon z_j = \bar{a}_{ij}(\epsilon x),$$

we have

$$D^\alpha \bar{\partial}_\epsilon z_i = O(\epsilon)$$

for any $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_{2n})^{\alpha_{2n}}$, where the α_i's are nonnegative integers with $|\alpha| \leq k$. It is easy to check that

$$|\bar{\partial}_\epsilon z_i|_{C^{k,\alpha}(0)} \to 0 \quad \text{if} \quad \epsilon \to 0.$$

Let $\zeta_i^\epsilon = z_i - u_i^\epsilon$. From the interior Hölder regularity for N_ϵ and (**), we have

$$|u_i^\epsilon|_{C^{k+1,\alpha}(0)} \leq \leq C(|\bar{\partial}_\epsilon z_i|_{C^{k,\alpha}(0)} + \| u_i^\epsilon \|) \to 0 \quad \text{if } \epsilon \to 0.$$

We have that $\bar{\partial}_\epsilon \zeta_i^\epsilon = 0$ in Ω and also $d\zeta_i^\epsilon(0) = dz_i - du_i^\epsilon(0)$ are linearly independent if ϵ is sufficiently small. If we pull back ζ_i^ϵ to $\epsilon\Omega$ by setting $\zeta_i = \zeta_i^\epsilon(x/\epsilon)$, then we have that $\bar{\partial}\zeta_i = 0$ and $d\zeta_i$ are linearly independent in $\epsilon\Omega$ provided we choose ϵ sufficiently small. This proves the theorem.

□

Remarks (1) In the Newlander-Nirenberg Theorem above, we only need the assumption on Hölder condition $C^{k,\alpha}$ to be just at one point x_0. (2) If $(M, T^{1,0})$ is of real dimension 2, then we can relax the assumption by requiring only $k \geq 0$ since there is no compatibility condition to be satisfied. This is the result obtained in Bers and Chern (see [**?**, **?**]). Again, the assumption on Hölder condition C^α is only required at one point.

8.4. **Embedding strongly pseudoconvex compact CR manifolds.** In this section we first extend the embedding theorem of Boutet De Monvel [**?**] to CR manifolds of class $C^{3,\alpha}$.

Theorem 8.8 (Boutet De Monvel)**.** *Let $(M, T^{1,0})$ be a compact strongly pseudoconvex CR manifold with real dimension $2n+1$, $n \geq 2$, of class $C^{3,\alpha}$, where $0 < \alpha < 1$. Then there exist global CR functions $h_1, \cdots, h_N \in C^{2,\alpha}_*(M)$ such that $\bar\partial_b h_i = 0$ for every $i = 1, \cdots, N$ and $\Phi = (h_1, \cdots, h_N) : M \to \mathbb{C}^N$ is an embedding.*

Proof. The arguments are similar to the smooth case used in [**?**]. Since $T^{1,0}(M)$ is of class $C^{3,\alpha}$, we equip M with a metric which is of class $C^{3,\alpha}$. For each point $p \in M$, using a polynomial change of coordinates, we can choose C^3 coordinates $x = (z, t) = (z_1, \cdots, z_{n-1}, t)$ near a neighborhood U of $p = 0$ such that a basis for $T^{1,0}$ vector fields is given by

$$\overline{L}_j = \frac{\partial}{\partial \bar z_j} + \sum_{i=1}^{n-1} A_{ji} \frac{\partial}{\partial z_i} + (-\sqrt{-1} z_j + B_j) \frac{\partial}{\partial t}, \quad j = 1, \cdots, n-1$$

where both $A_{ji} = \mathcal{O}(|x|^2)$ and $B_j = \mathcal{O}(|x|^2)$. We choose

$$Z(z, t) = Z(x) = (z_1, \cdots, z_n),$$

where $z = (z_1, \cdots, z_{n-1})$ and $z_n = t + i|z|^2$. Then $Z : U \to \mathbb{C}^n$ Then we have

$$\overline{L}_j z_i = O(|x|^2), \quad j = 1, \cdots, n-1, \; i = 1, \cdots, n. \tag{8.6}$$

Thus the map Z gives an approximate embedding near p. Extending z_j to be a C^3 function ϕ_j on M, we have functions $\varphi_1, \cdots, \varphi_n \in C^3(M)$ such that $\varphi_j(p) = 0$, $d\varphi_1(p), \cdots, d\varphi_n(p)$ are linearly independent at p and $\varphi = (\varphi_1, \cdots, \varphi_n) : M \to \mathbb{C}^n$ is a C^3 diffeomorphism of a small neighborhood of p on M into $\mathbb{C}^n$ with $\varphi(p) = 0$ and $\varphi(M)$ is strongly pseudoconvex at the origin.

Let g be a polynomial $g(z, t) = -iz_n + z_n^2$. Then g satisfies $dg(p) \neq 0$ and $\overline{\partial}_b g(p) = O(|x|^2)$. Choosing U small, we have

$$\mathrm{Re} g = |z|^2 + t^2 - (|z|^2)^2 \geq c|x|^2, \tag{8.7}$$

where c is a positive constant. Extend g to M such that $g \in C^3(M)$ and g satisfies $\mathrm{Re} g > 0$ on $M \setminus p$.

Let η be a cut-off function such that $\eta = 1$ near a neighborhood $V \subset\subset U$ of p and η is supported in U. To construct CR embedding functions, we set $z_j^\lambda = \eta \varphi_j e^{-\lambda g}$ for sufficiently large $\lambda > 0$. Notice that Theorems 3.1, 3.2 and Corollary 3.3 hold for any metric of class C^3, not necessarily the pseudo-hermitian Levi metric. Let G_b be the inverse of $\Box_b$. Set

$$h^j = \mathcal{S}(z_j^\lambda) = z_j^\lambda - \vartheta_b G_b \bar\partial_b z_j^\lambda \quad \text{for} \quad j = 1, \cdots, n,$$

where $\mathcal{S}$ is the Szegö projection on M. We have $\overline{\partial}_b h^j = 0$ in M for $j = 1, \cdots, n$. Since z_j^λ is in $C^3(M) \subset C_*^{2,\alpha}(M)$, from Corollary 3.6 we have $h^j \in C_*^{2,\alpha}(M)$ for any $0 < \alpha < 1$.

Using (4.1) and (4.2), it follows that in a neighborhood U near p

$$\begin{aligned} |\overline{\partial}_b z_j^\lambda| &= |\bar{\partial}_b(\eta\varphi_j)e^{-\lambda g} - \lambda\eta\varphi_j e^{-\lambda g}(\overline{\partial}_b g)| \\ &\leq C(|x|^2 + \lambda|x|^3)e^{-\lambda|x|^2} \\ &\leq C(\lambda^{-1} + \lambda^{-\frac{1}{2}}) \underset{v>0}{\to} \sup(|v|^1 e^{-v} + |v|^{1+\frac{1}{2}} e^{-v}) \leq C\lambda^{-\frac{1}{2}} \to 0, \quad \lambda \to \infty. \end{aligned}$$

This gives that $\|\overline{\partial}_b z_j^\lambda\| \to 0$ as $\lambda \to \infty$. Since $\bar{\partial}_b z_j^\lambda \to 0$ in $C^{1,\alpha}(0)$ as $\lambda \to \infty$, we have $\vartheta_b G_b \bar{\partial}_b z_j^\lambda$ in $C_*^{2,\alpha}(0)$ can be made arbitrarily small as $\lambda \to \infty$. This gives $dh^1(0), \cdots, dh^n(0)$ are linearly independent for large λ. The map $\Phi = (h^1, \cdots, h^n)$ forms a local embedding of M into $\mathbb{C}^n$ by global $C_*^{2,\alpha}$ CR functions. From the same arguments as in [**?**], we also have that global CR functions separating points. From the compactness of M and a partition of unity, there exists a global embedding map consisting of CR functions. This proves the theorem. ☐

We remark that Boutet De Monvel 's theorem does not hold for strongly pseudoconvex CR manifold of real dimension 3.

We shall present in this section a compact real analytic three dimensional CR manifold which can not be globally CR embedded into $\mathbb{C}^n$ for any dimension n. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | \ |z_1|^2 + |z_2|^2 = 1\}$ be the boundary of the unit ball in $\mathbb{C}^2$, and let the induced CR structure $T^{1,0}(S^3)$ be generated by $L = \overline{z}_2(\partial/\partial z_1) - \overline{z}_1(\partial/\partial z_2)$. Thus, $(S^3, T^{1,0}(S^3))$ forms a compact strongly pseudoconvex CR manifold of real dimension three. For each $t \in \mathbb{R}$, $|t| < 1$, define a new CR structure $T_t^{1,0}(S^3)$ on S^3 by letting $T_t^{1,0}(S^3)$ be generated by the vector field $L_t = L + t\overline{L}$. If $t = 0$, $T_0^{1,0}(S^3)$ coincides with the induced standard CR structure $T^{1,0}(S^3)$. It is easily verified that for $|t| < 1$, $(S^3, T_t^{1,0}(S^3))$ is a compact real analytic strongly pseudoconvex CR manifold of real dimension three.

The next theorem shows that any L^2 integrable CR function f on S^3 with respect to the CR structure $(S^3, T_t^{1,0}(S^3))$ for $0 < |t| < 1$ must be even. Obviously, this implies that, for $0 < |t| < 1$, $(S^3, T_t^{1,0}(S^3))$ can not be globally CR embedded into any $\mathbb{C}^n$.

Theorem 8.9 (Rossi)**.** *Any L^2 integrable CR function $f(z)$ on S^3 with respect to the CR structure $T_t^{1,0}(S^3)$, $0 < |t| < 1$, is even, i.e., $f(z) = f(-z)$.*

Theorem 8.10 (Burns-Kohn)**.** *Any strongly pseudoconvex CR structure $T^{(0,1}(S^3)$ of real dimension 3 is embeddable if and only if $S_t; C^\infty(S^3) \to C^\infty(S^3)$. This is equivalent to that the range of $\bar{\partial}_b t$ is closed.*

8.5. **Nonexistence of Levi-flat hypersurfaces in $\mathbb{C}P^n$.**

Definition 8.11. Let M be a C^2 hypersurface in $\mathbb{C}P^n$. M is said to be *Levi flat* if for any C^2 defining function ρ for M, the Levi form $i\partial\bar{\partial}\rho|$ vanishes on $T^{1,0}(M)$

Definition 8.12. A Lipschitz hypersurface is a hypersurface which locally is the graph of a Lipschitz function. A Lipschitz (or C^1) hypersurface is said to be Levi-flat if it is locally foliated by complex manifolds of complex dimension $n-1$.

From the implicit function theorem, any C^1 hypersurface locally is the graph of some C^1 function. A C^2 hypersurface M is called Levi-flat if its Levi-form vanishes on M. Any C^k Levi-flat hypersurface, $k \geq 2$ is locally foliated by complex manifolds of complex dimension $n-1$. The foliation is of class C^k if the hypersurface is of class C^k, $k \geq 2$ (see Barrett-Fornaess [**?**]). The proof in [**?**] also gives that if a real C^1 hypersurface admits a continuous foliation by complex manifolds, then the foliation is actually C^1. Thus our definition is a natural generalization of Levi-flatness to Lipschitz or C^1 hypersurfaces.

Theorem 8.13. *There exist no Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ for $n \geq 3$.*

For example in $\mathbb{C}P^2$, let $M = \{[z_0, z_1, z_2] \in \mathbb{C}P^2 : |z_0| = |z_1|\}$. It is possible to show that $\mathbb{C}P^2 \backslash M = \Omega^+ \cup \Omega^-$, where both domains are pseudoconvex. The, M can be foliated (locally) by compact manifolds, i.e. $M \cap U = \bigcup \Sigma_t$, where Σ_t are complex manifolds.

Our proof follows the work by Siu [**?**]. Let M be a Lipschitz Levi flat hypersurface in $\mathbb{C}P^n$ such that $\mathbb{C}P^n \setminus M = \Omega^+ \cup \Omega^-$, where both domains are pseudoconvex. As before M is locally foliated by complex manifolds. Suppose that M is defined globally by a Lipschitz function ρ with $\nabla\rho \neq 0$ on M almost everywhere.

If we consider the curvature form Θ^N of the complex line bundle generated by complex normal $\partial\rho$, then $\Theta^N|_{\Sigma_t} > 0$. This is due to the fact that on the submanifold Σ_t, the curvature is no more than the curvature of the ambient manifold $\mathbb{C}P^n$. The complex line bundle generated by $\partial\rho$ is a quotient bundle, its curvature is more positive then that of $\mathbb{C}P^n$. If we can find a real-valued h such that

$$\Theta^N = i\partial_b\bar{\partial}_b h \qquad \text{on } M \tag{8.8}$$

with a continuous function h, then h must assume its maximum at some point p. But this implies that h is strictly plurisubharmonic in the interior of one of the leaves Σ_t, which is not possible by the maximum principle. This prove the nonexistence of Levi flat hypersurfaces.

Since the curvature form Θ^N is already known to be d-exact, we can reduced the problem to solving a continuous solution u such that

$$\bar{\partial}_b u = f, \tag{8.9}$$

when $f \in C^\alpha(M)$ with $\alpha > \frac{1}{2}$.

Now come back to regularity. To obtain the regularity for the solution, we need to extend first α to $\tilde{\alpha}$ in $\mathbb{C}P^n$, to get $\widetilde{\alpha} \in W^{1+\epsilon}(\mathbb{C}P^n)$. Next, we need to find $\bar{\partial} v_0^\pm = \bar{\partial}\widetilde{\alpha}$ on $\Omega^\pm$ such that $\alpha^\pm = \widetilde{\alpha} - v_0^\pm$ on $\Omega^\pm$. Then, we will have

$$\bar{\partial}_b \alpha^\pm = 0, \qquad \alpha^\pm|_{b\Omega} = 0.$$

If we set $v_0^\pm = \widetilde{\alpha} - \star\bar{\partial} N^\pm \star \bar{\partial}\widetilde{\alpha}$ we obtain what we request. Moreover, $v_0^\pm \wedge \bar{\partial}\rho = 0$ on $b\Omega^\pm$ and $v_0^\pm \in W^\epsilon(\Omega^\pm)$. So we can solve $\bar{\partial}\widetilde{u} = v_0^\pm$ on all $\mathbb{C}P^n$, to conclude that $\widetilde{u} \in W^{1+\epsilon}\mathbb{C}P^n$. By the Trace theorem, this leads to $u = \widetilde{u}|_M \in W^{1/2+\epsilon}(M)$.

Let (z', t) be the local coordinates on each leaf Σ_t. Notice that the tangential Cauchy-Riemann operator $\bar{\partial}_b$ is nothing but the Cauchy-Riemann equation $\bar{\partial}_b u = \bar{\partial}_{z'} u = \alpha$ on each leaf with $\alpha \in W^{1/2+\epsilon}(M)$. Since $u \in W^{1/2+\epsilon}(M)$, the restriction of u on each leaf Σ_t is a function with $W^s(\Sigma_t)$ for each t. Thus we can apply the interior regularity to obtain that $u(z', t)$ is Hölder continuous on each leaf. We also obtain that u is uniformly bounded on M. It remains to show that u is continuous in the transversal direction t. This amounts to a one-dimensional Sobolev embedding theorem. We modify an argument of finite difference methods in the Besov norms used by Hórmander to conclude that u actually is H older continuous.

9. **Open problems**

Theorem 9.1 (Boundary regularity for biholomorphic mappings)**.** *Let D_1 and D_2 be two bounded domains in $\mathbb{C}^n$ with smooth boundary. Suppose that f is a biholomoprhic map from D_1 to D_2. Then f extends smoothly to the boundary bD_1.*

Theorem 9.2 (Embedding strongly pseudoconvex CR structure of dimension ≥ 5)**.** *Let $(M, T^{1,0})$ be a strongly pseudoconvex CR manifold with real dimension $2n+1$, $n \geq 2$. Then there exist local CR functions $h_1, \cdots, h_n \in C_\infty(M)$ such that $\bar{\partial}_b h_i = 0$ for every $i = 1, \cdots, n$ and $\Phi = (h_1, \cdots, h_n) : M \to \mathbb{C}^n$ is an embedding.*

Theorem 9.3 (C^∞ regularity for $\bar{\partial}$ in $\mathbb{C}P^n$)**.** *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with C^∞ boundary. For any $f \in C^\infty_{(0,1)}(\overline{\Omega})$ with $\bar{\partial} f = 0$, there exists $u \in C^\infty(\overline{\Omega})$ such that $\bar{\partial} u = f$.*

When Ω is in $\mathbb{C}^n$ or a Stein manifold, this is proved by Kohn [**?**]. In fact, one would like to know if there exists a solution $u \in W^1(\Omega)$.

Theorem 9.4 (Bounded holomorphic functions on negatively curved manifolds)**.** *Let (M, g) be a complete simply connected Kähler manifold of complex dimension $n \geq 3$ with negative sectional curvature $-a^2 < k < -b^2$ with some nonzero a, b. Then there exist bounded holomorphic functions on M.*

10. **Appendix**

10.1. **Preliminaries for Lipschitz domains.** We first give some basic properties of Lipschitz domains in Chapter 0. These simple and useful facts do not seem to have been systematically treated.

A bounded domain $\Omega \subset\subset \mathbb{R}^n$ is called Lipschitz if locally the boundary $b\Omega$ is the graph of a Lipschitz function. Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function which satisfies the Lipschitz condition

$$|\psi(y') - \psi(x')| \leq M|y' - x'|, \quad \text{for all } y',\ x' \in \mathbb{R}^{n-1}. \tag{10.1}$$

A bounded domain Ω is called Lipschitz if near every boundary point $p \in bD$, there exists a neighborhood U of p such that after a rotation,

$$\Omega \cap U = \{(x', x_n) \in U \mid x_n > \psi(x')\}$$

for some Lipschitz function ψ. The smallest M in which (3.1) holds will be called the bound of the Lipschitz constant. By choosing finitely many balls $\{U_i\}$ covering $b\Omega$, the Lipschitz constant for a Lipschitz domain is the smallest M such that the Lipschitz constant is bounded by M in every ball U_i. A Lipschitz function is differentiable almost everywhere (See Evans-Gariepy [**?**] for a proof of this fact).

Let Ω be a bounded Lipschitz domain. A Lipschitz function ρ is called a *global* defining function for Ω if $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfies $\rho < 0$ in D, $\rho > 0$ outside $\overline{D}$ and

$$C_1 < |d\rho| < C_2 \quad \text{a.e. on } bD, \tag{10.2}$$

where $C_1,\ C_2$ are positive constants.

Lemma 10.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then D has a global Lipschitz defining function ρ.*

Proof. We cover bD by finitely many boundary coordinate patches U_i where $i = 1, \cdots, K$. Let r_i be a local defining function on U_i which is locally a Lipschitz graph. Let $\phi_i \in C_0^\infty(U_i)$ be a partition of unity such that $\sum_i \phi_i = 1$ in a neighborhood of bD. We define $\rho = \sum_i \phi_i r_i$. Then ρ is a defining function for D. To see that ρ satisfies (0.2), it is easy to see that $|\nabla\rho| \leq C_2$ a.e. on bD. Since D is Lipschitz, for each $p \in bD$ such that the tangent plane of p for bD exists, one can find ξ such that $\langle \nabla r_i, \xi\rangle_p \geq C_0 > 0$ for every i such that $p \in U_i$. Also we have $\nabla\rho = \sum_i \phi_i \nabla r_i$ on bD if ∇r_i exists. Thus

$$\langle \nabla\rho, \xi\rangle_p = \langle \sum_i \phi_i \nabla r_i, \xi\rangle_p \geq \sum_i \phi_i(p) C_0 = C_0.$$

This proves that ρ satisfies (3.2). $\square$

Lemma 10.2. *If ρ is any global Lipschitz defining function for D and r is a local defining function in a neighborhood U of a boundary point $p \in bD$*

such that $r(x) = x_n - \psi(x_1, \cdots, x_{n-1})$ *for some Lipschitz function* ψ*, then there exists a positive function* $h(x) \in L^\infty(\overline{D} \cap U) \cap C(D \cap U)$ *such that*

$$\rho(x) = h(x)r(x), \quad x \in \overline{D} \cap U, \tag{10.3}$$

$$d\rho(x) = h(x)dr(x), \quad a.e.\ x \in bD \cap U \tag{10.4}$$

where

$$\tilde{C}_1 \leq h(x) \leq \tilde{C}_2, \quad a.e.\ on\ bD$$

for some positive constants $\tilde{C}_1$ *and* $\tilde{C}_2$.

Proof. Since $\nabla r(x) = (-\nabla\psi, 1)$, we have

$$C_1 \leq |dr(x)| \leq C_2 \quad \text{a.e. on } U \cap bD.$$

Define

$$h(x) = \frac{\rho(x)}{r(x)}, \quad x \in U \cap D.$$

Then $h(x)$ is Lipschitz on D and $h(x) > 0$ in $U \cap D$. Let p be any point in $U \cap bD$ such that both $d\rho$ and dr exist at p. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{R}^n$ such that for any unit vector $\xi \in \Gamma$, $-\langle \nabla r, \xi\rangle_p \geq C_0 > 0$. This implies that

$$-\langle \frac{\nabla\rho}{|\nabla\rho|}, \xi\rangle_p \geq \frac{C_0}{|\nabla r|} > 0, \quad p \in U \cap bD. \tag{10.5}$$

Since r and ρ are differentiable at p, we have for any $x \in \{\xi + \{p\}\} \cap U$,

$$\frac{\rho(x)}{r(x)} = \frac{\rho(x) - r(p)}{r(x) - \rho(p)} \to \frac{D_\xi\rho(p)}{D_\xi r(p)}. \tag{10.6}$$

Thus $h(x)$ has nontangential boundary value a.e. on $U \cap bD$. Also from (3.5) and (3.6), there exist $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ such that

$$\tilde{C}_1 \leq h(x) \leq \tilde{C}_2, \quad \text{a.e. for } x \in U \cap bD.$$

(3.4) follows from the definition of differentiation.

$\square$

Lemma 10.3. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$*. There exists an exhaustion* $\{\Omega_\nu\}$ *of* Ω *such that*
$\{\Omega_\nu\}$ *is an increasing sequence of relatively compact subsets of* Ω *and*

$$\bigcup_\nu \Omega_\nu = \Omega.$$

Each Ω_ν *has a* C^∞ *defining function* η_ν*, i.e.,* $\Omega_\nu = \{x \in \mathbb{R}^n; \rho_\nu(x) < 0\}$. *There exist positive constants* c_1, c_2 *such that* $c_1 \leq |\nabla\eta_\nu| \leq c_2$ *on* $\partial\Omega_\nu$*, where* c_1, c_2 *are independent of* ν.

Proof. Using Lemma 0.1, there exists a global Lipschitz defining function ρ for Ω satisfying (0.2) and ρ is obtained by a partition of unity of defining functions r_i which is a Lipschitz graph. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi \geq 0, \int \chi dV = 1,$ $\chi(x)$ depends only on $|x|$ and vanishes when $|x| > 1$.

We define $\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$. Let $\delta_\nu \searrow 0$ and we define $\Omega_{\delta_\nu} = \{x \in \Omega|\ \rho(x) < -\delta_\nu\}$. Then Ω_{δ_ν} is a sequence of relatively compact open subsets of Ω with union equal to Ω. Each ρ_{ϵ_ν} is well defined if $0 < \epsilon_\nu < \delta_{\nu+1}$ for $x \in \Omega_{\delta_{\nu+1}}$. Letting $c_2 = \sup_\Omega |\nabla\rho|$, then for ϵ_ν sufficiently small, we have

$$\rho(x) < \rho_{\epsilon_\nu(x)} < \rho(x) + c_2\epsilon_\nu$$

on $\Omega_{\delta_{\nu+1}}$. For each ν we choose

$$\epsilon_\nu = \frac{1}{2c_2}(\delta_{\nu-1} - \delta_\nu)$$

and $t_\nu \in (\delta_{\nu+1}, \delta_\nu)$. We define

$$\Omega_\nu = \{x \in \mathbb{R}^n|\ \rho_{\varepsilon_\nu} < -t_\nu\}.$$

Since $\rho(x) < \rho_{\epsilon_\nu}(x) < -t_\nu < -\delta_{\nu+1}$, we have that $\Omega_{\delta_{\nu+1}} \supset \Omega_\nu$. Also if $x \in \Omega_{\delta_{\nu-1}}$, then $\rho_{\epsilon_\nu}(x) < \rho(x) + c_2\epsilon_\nu < -\delta_\nu < -t_\nu$. Thus we have $\Omega_{\delta_{\nu+1}} \supset \Omega_\nu \supset \Omega_{\delta_{\nu-1}}$ and (1) is satisfied. Each Ω_ν is defined by $\eta_\nu = \rho_{\varepsilon_\nu} + t_\nu$. That the subdomain Ω_ν has smooth boundary will follow from (3).

To prove (3), it is easy to see that $|\nabla\eta_\nu| \le c_2$ in bD_ν. To show that $|\nabla\eta_\nu|$ is uniformly bounded from below, we note bD satisfies the uniform interior cone property. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{R}^n$ such that for any unit vector $\xi \in \Gamma + \{p\}$, $-\langle\nabla\rho, \xi\rangle_p > C_0$ a.e. in $U \cap bD$, where C_0 is a positive constant independent of p if U is sufficiently small. There exist a finite covering $\{V_\mu\}_{1\le\mu\le K}$ of $\partial\Omega$, a finite set of unit vectors $\{\xi_\mu\}_{1\le\mu\le K}$ and $c_1 > 0$ such that the inner product $\langle\nabla\rho, \xi_\mu\rangle \ge c_1 > 0$ a.e. for $x \in V_\mu$, $1 \le \mu \le K$. Since this is preserved by convolution, (3) is proved. This proves Lemma 0.3. □

10.2. **Preliminaries for Lipschitz domains.** We first give some basic definitions and properties of Lipschitz domains.

A bounded domain $\Omega \subset\subset \mathbb{R}^n$ is called Lipschitz if locally the boundary $b\Omega$ is the graph of a Lipschitz function. Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function which satisfies the Lipschitz condition

$$|\psi(y') - \psi(x')| \le M|y' - x'|, \quad \text{for all } y',\ x' \in \mathbb{R}^{n-1}. \tag{10.7}$$

A bounded domain Ω is called Lipschitz if near every boundary point $p \in bD$, there exists a neighborhood U of p such that after a rotation,

$$\Omega \cap U = \{(x', x_n) \in U \mid x_n > \psi(x')\}$$

for some Lipschitz function ψ. The smallest M in which (3.1) holds will be called the bound of the Lipschitz constant. By choosing finitely many balls $\{U_i\}$ covering $b\Omega$, the Lipschitz constant for a Lipschitz domain is the smallest M such that the Lipschitz constant is bounded by M in every ball U_i. A Lipschitz function is differentiable almost everywhere (See Evans-Gariepy [**?**] for a proof of this fact).

Let Ω be a bounded Lipschitz domain. A Lipschitz function ρ is called a *global* defining function for Ω if $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfies $\rho < 0$ in D, $\rho > 0$ outside $\overline{D}$ and

$$C_1 < |d\rho| < C_2 \quad \text{a.e. on } bD, \tag{10.8}$$

where $C_1,\ C_2$ are positive constants.

Lemma 10.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then D has a global Lipschitz defining function ρ.*

Proof. We cover bD by finitely many boundary coordinate patches U_i where $i = 1, \cdots, K$. Let r_i be a local defining function on U_i which is locally a Lipschitz graph. Let $\phi_i \in C_0^\infty(U_i)$ be a partition of unity such that $\sum_i \phi_i = 1$ in a neighborhood of bD. We define $\rho = \sum_i \phi_i r_i$. Then ρ is a defining function for D. To see that ρ satisfies (3.2), it is easy to see that $|\nabla\rho| \le C_2$ a.e. on bD. Since D is Lipschitz, for each $p \in bD$ such that the tangent plane of p for bD exists, one can find ξ such that $\langle \nabla r_i, \xi\rangle_p \ge C_0 > 0$ for every i such that $p \in U_i$. Also we have $\nabla\rho = \sum_i \phi_i \nabla r_i$ on bD if ∇r_i exists. Thus

$$\langle \nabla\rho, \xi\rangle_p = \langle \sum_i \phi_i \nabla r_i, \xi\rangle_p \ge \sum_i \phi_i(p) C_0 = C_0.$$

This proves that ρ satisfies (3.2). $\square$

Lemma 10.5. *If ρ is any global Lipschitz defining function for D and r is a local defining function in a neighborhood U of a boundary point $p \in bD$ such that $r(x) = x_n - \psi(x_1, \cdots, x_{n-1})$ for some Lipschitz function ψ, then there exists a positive function $h(x) \in L^\infty(\overline{D} \cap U) \cap C(D \cap U)$ such that*

$$\rho(x) = h(x) r(x), \quad x \in \overline{D} \cap U, \tag{10.9}$$

$$d\rho(x) = h(x) dr(x), \quad a.e.\ x \in bD \cap U \tag{10.10}$$

where

$$\tilde{C}_1 \le h(x) \le \tilde{C}_2, \qquad a.e.\ on\ bD$$

for some positive constants $\tilde{C}_1$ and $\tilde{C}_2$.

Proof. Since $\nabla r(x) = (-\nabla\psi, 1)$, we have

$$C_1 \le |dr(x)| \le C_2 \quad \text{a.e. on } U \cap bD.$$

Define

$$h(x) = \frac{\rho(x)}{r(x)}, \quad x \in U \cap D.$$

Then $h(x)$ is Lipschitz on D and $h(x) > 0$ in $U \cap D$. Let p be any point in $U \cap bD$ such that both $d\rho$ and dr exist at p. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{R}^n$ such that for any unit vector $\xi \in \Gamma$, $-\langle \nabla r, \xi\rangle_p \ge C_0 > 0$. This implies that

$$-\langle \frac{\nabla\rho}{|\nabla\rho|}, \xi\rangle_p \ge \frac{C_0}{|\nabla r|} > 0, \quad p \in U \cap bD. \tag{10.11}$$

Since r and ρ are differentiable at p, we have for any $x \in \{\xi + \{p\}\} \cap U$,

$$\frac{\rho(x)}{r(x)} = \frac{\rho(x) - r(p)}{r(x) - \rho(p)} \to \frac{D_\xi \rho(p)}{D_\xi r(p)}. \tag{10.12}$$

Thus $h(x)$ has nontangential boundary value a.e. on $U \cap bD$. Also from (3.5) and (3.6), there exist $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ such that

$$\tilde{C}_1 \leq h(x) \leq \tilde{C}_2, \quad \text{a.e. for } x \in U \cap bD.$$

(3.4) follows from the definition of differentiation.

□

Lemma 10.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists an exhaustion $\{\Omega_\nu\}$ of Ω such that*
(1) *$\{\Omega_\nu\}$ is an increasing sequence of relatively compact subsets of Ω and $\bigcup\limits_\nu \Omega_\nu = \Omega$.*
(2) *Each Ω_ν has a C^∞ defining function η_ν, i.e., $\Omega_\nu = \{x \in \mathbb{R}^n; \rho_\nu(x) < 0\}$.*
(3) *There exist positive constants c_1, c_2 such that $c_1 \leq |\nabla \eta_\nu| \leq c_2$ on $\partial\Omega_\nu$, where c_1, c_2 are independent of ν.*

Proof. Using Lemma 3.1, there exists a global Lipschitz defining function ρ for Ω satisfying (3.2) and ρ is obtained by a partition of unity of defining functions r_i which is a Lipschitz graph. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi \geq 0, \int \chi dV = 1,\ \chi(x)$ depends only on $|x|$ and vanishes when $|x| > 1$.

We define $\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$. Let $\delta_\nu \searrow 0$ and we define $\Omega_{\delta_\nu} = \{x \in \Omega |\ \rho(x) < -\delta_\nu\}$. Then Ω_{δ_ν} is a sequence of relatively compact open subsets of Ω with union equal to Ω. Each ρ_{ϵ_ν} is well defined if $0 < \epsilon_\nu < \delta_{\nu+1}$ for $x \in \Omega_{\delta_{\nu+1}}$. Letting $c_2 = \sup\limits_\Omega |\nabla \rho|$, then for ϵ_ν sufficiently small, we have

$$\rho(x) < \rho_{\epsilon_\nu(x)} < \rho(x) + c_2\epsilon_\nu$$

on $\Omega_{\delta_{\nu+1}}$. For each ν we choose

$$\epsilon_\nu = \frac{1}{2c_2}(\delta_{\nu-1} - \delta_\nu)$$

and $t_\nu \in (\delta_{\nu+1}, \delta_\nu)$. We define

$$\Omega_\nu = \{x \in \mathbb{R}^n |\ \rho_{\varepsilon_\nu} < -t_\nu\}.$$

Since $\rho(x) < \rho_{\epsilon_\nu}(x) < -t_\nu < -\delta_{\nu+1}$, we have that $\Omega_{\delta_{\nu+1}} \supset \Omega_\nu$. Also if $x \in \Omega_{\delta_{\nu-1}}$, then $\rho_{\epsilon_\nu}(x) < \rho(x) + c_2\epsilon_\nu < -\delta_\nu < -t_\nu$. Thus we have $\Omega_{\delta_{\nu+1}} \supset \Omega_\nu \supset \Omega_{\delta_{\nu-1}}$ and (1) is satisfied. Each Ω_ν is defined by $\eta_\nu = \rho_{\varepsilon_\nu} + t_\nu$. That the subdomain Ω_ν has smooth boundary will follow from (3).

To prove (3), it is easy to see that $|\nabla \eta_\nu| \leq c_2$ in bD_ν. To show that $|\nabla \eta_\nu|$ is uniformly bounded from below, we note bD satisfies the uniform interior cone property. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{R}^n$ such that for any unit vector $\xi \in \Gamma + \{p\}$, $-\langle \nabla\rho, \xi \rangle_p > C_0$ a.e. in $U \cap bD$, where C_0 is a positive constant independent of p if U is sufficiently small. There exist a finite covering $\{V_\mu\}_{1 \leq \mu \leq K}$ of $\partial\Omega$, a finite set of unit vectors

$\{\xi_\mu\}_{1\leq\mu\leq K}$ and $c_1 > 0$ such that the inner product $\langle\nabla\rho, \xi_\mu\rangle \geq c_1 > 0$ a.e. for $x \in V_\mu$, $1 \leq \mu \leq K$. Since this is preserved by convolution, (3) is proved. This proves Lemma 0.3. $\square$

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